ON THE INDEPENDENT RAINBOW DOMINATION STABLE GRAPHS

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For a graph G and an integer $k \geq 2$, let $f : V(G) \to \mathcal{P}(\{1, 2, ..., k\})$ be a function. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have $\cup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$, then f is called a k-rainbow dominating function (or simply kRDF) of G. The weight of a kRDF f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is called the k-rainbow domination number of G, and is denoted by $\gamma_{rk}(G)$. An independent k-rainbow dominating function (IkRDF) is a kRDF f with the property that $\{v : f(v) \neq \emptyset\}$ is an independent set. The minimum weight of an IkRDF of G is called the independent k-rainbow domination number of G, and is denoted by $i_{rk}(G)$. A graph G is k-rainbow domination stable if the k-rainbow domination number of G remains unchanged under removal of any vertex. Likewise, a graph G is independent k-rainbow domination stable if the independent k-rainbow domination number of G remains unchanged under removal of any vertex. In this paper, we prove that determining whether a graph is k-rainbow domination stable or independent k-rainbow domination stable is NP-hard even when restricted to bipartite or planar graphs, thus answering a question posed in [11].

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1. INTRODUCTION

We refer to [10] for notation and terminology not given here. Let $G = (V, E)$ be a simple graph of order n with vertex set $V = V(G)$ and edge set $E = E(G)$. For a vertex $v \in V(G)$, let $N_G(v) = \{u | uv \in E(G)\}$ denotes the *open neighborhood* of v and $N_G[v] = N_G(v) \cup \{v\}$ denotes the *closed neighborhood* of v. For a set $S \subseteq V(G)$, we denote $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. The *degree* of a vertex v, $\deg_G(v)$, or just $\deg(v)$, in a graph G denotes the number of neighbors of v in G. We refer to $\Delta(G)$ and $\delta(G)$ as the *maximum degree* and the *minimum degree* among the vertices of G, respectively. A subset S of vertices is called an *independent* set if no pair of vertices of S are adjacent.

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A subset S of vertices of a graph G is a *dominating set* of G, if $N[S] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. We say that a vertex v is dominated by a set S if $v \in N[S]$.

Brešar et al. [3, 4] introduced the concept of rainbow domination in graphs. For a graph G, let $f : V(G) \to \mathcal{P}(\{1, 2, ..., k\})$ be a function. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have $\cup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$, then f is called a *k-rainbow dominating function* (or simply kRDF) of G.

The *weight*, $w(f)$, of f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is called the *k-rainbow domination number* of G, and is denoted by $\gamma_{rk}(G)$. Shao et al. [16] considered independent k-rainbow dominating functions as those k-rainbow dominating functions f with the property that $\{v : f(v) \neq \emptyset\}$ is an independent set.

An *independent k-rainbow dominating function* (IkRDF) is a kRDF f with the property that $\{v : f(v) \neq \emptyset\}$ is an independent set. The minimum weight of an IkRDF of G is called the *independent k-rainbow domination number* of G, and is denoted by $i_{rk}(G)$. The complexity of rainbow domination and independent rainbow domination in graphs are studied in [5] and [16], respectively.

Much have been written about the effect of the removal of a vertex on the domination number. This is a well-studied concept and is considered to several domination parameters. Bauer et al. [2] introduced the concept of domination stability in graphs. The domination stability $st_\gamma(G)$ of a graph G is the minimum number of vertices whose removal changes the domination number. A graph G is called *domination stable* if the domination number of G remains unchanged under removal of any vertex. The concept of stability has been considered for different types of domination, see for example, [6, 7, 9, 13, 15]. The complexity of whether a graph is domination stable is studied in [12], where the authors showed that determining whether a graph is domination stable is NP-hard even for bipartite graphs. The complexity of stability for some other variants of domination is considered by several authors, see for example [1, 7].

Li et al. [11] introduced the concept of 2-rainbow domination stability in graphs. A graph G is 2-*rainbow domination stable* if $\gamma_{r2}(G - v) = \gamma_{r2}(G)$ for any vertex v. They proved that determining whether a graph is 2-rainbow domination stable is NP-hard in general graphs. They also posed the following open problem.

Problem 1 (Li et al. [11]). Determine the computational complexity of the 2-rainbow domination stable graph problem on some special classes of graphs, such as planar graphs or bipartite graphs.

In this paper, we determine the complexity of two more generalized problems, namely k-rainbow domination stability and independent k-rainbow domination stability. For an integer $k \geq 2$, a graph G is *k-rainbow domination stable* if $\gamma_{rk}(G - v) = \gamma_{rk}(G)$ for any vertex v. A graph G is independent *k-rainbow domination stable* if $i_{rk}(G - v) = i_{rk}(G)$ for any vertex v.

We show that determining whether a graph is k-rainbow domination stable or independent k-rainbow domination stable is NP-hard even when restricted to bipartite or planar graphs. The special case $k = 2$ for k-rainbow domination provides an answer to Problem 1.

For a function $f : V(G) \to \mathcal{P}(\{1, 2, ..., k\})$, and a vertex $v \in V(G)$ with $f(v) = \emptyset$, we say that v is k-rainbow dominated by f if $\cup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$.

2. PRELIMINARY

We define the *k-rainbow domination stability number* of a graph G, denoted by $st_{\gamma_{rk}}(G)$, as the minimum number of vertices whose removal changes the k-rainbow domination number of G. Likewise, the *independent k-rainbow domination stability number* of a graph G, denoted by $st_{i_{rk}}(G)$, is the minimum number of vertices whose removal changes the independent k-rainbow domination number of G.

Note that the k-rainbow domination-stability number and independent k-rainbow domination-stability number are defined for every graph G with $\gamma_{rk}(G) > 1$ and $i_{rk}(G) > 1$, respectively, since the removal of $|V(G)| - 1$ vertices of G results a K_1 with k-rainbow domination number and independent k-rainbow domination number equal to one. We also define $st_{\gamma_{rk}}(K_1) = st_{i_{rk}}(K_1) = 0$.

Observation 1. 1) A graph G is k-rainbow domination stable if and only if $st_{\gamma_{rk}}(G) > 1$.

2) A graph G is independent k-rainbow domination stable if and only if $st_{i_{rk}}(G) > 1$.

3. NP-HARDNESS RESULTS

The decision problem of k-rainbow domination stability number problem is stated in this paper as follows:

k-RAINBOW DOMINATION STABILITY NUMBER PROBLEM (krDSNP)
Instance: Graph $G = (V, E)$ and the k-rainbow domination number $\gamma_{rk}(G)$.
Question: Is $st_{\gamma_{rk}}(G) > 1$?

We use a transformation from 3-SAT, which was proven to be NP-complete in [8]. The problem 3-SAT is the problem of determining if there exists an interpretation that satisfies a given Boolean formula. The formula in 3-SAT is given in conjunctive normal form, where each clause contains three literals. We assume that the formula contains the instance of any literal u and its negation $\bar{u}$ (in the other case, all clauses containing the literal u are satisfied by the true assignment of u).

THEOREM 2. *krDSNP is NP-hard even for bipartite graphs.*

Proof. Let $\Phi = \{C, U\}$ be an instance in the 3-SAT Problem, that is, Φ is boolean formula in 3-conjunctive normal form. Let $U = \{u_1, u_2, \ldots, u_n\}$ be the set of literals and the $C = \{C_1, C_2, \ldots, C_m\}$ be the set of clauses. We construct the following graph G_Φ associated to Φ. For each literal u_i construct a graph G_i with vertex set $V(G_i) = \{u_i, a_i^1, a_i^2, ..., a_i^k, \bar{u}_i, d_i, e_i^1, e_i^2, ..., e_i^k, b_i\}$ and edge set $E(G_i) = \{u_ib_i, \bar{u}_id_i, u_ia_i^j, \bar{u}_ia_i^j, b_ie_i^j, d_ie_i^j : j = 1, 2, ..., k\}$. Figure 1 shows the graph G_i for $k = 3$.

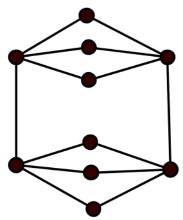

Figure 1 – The graph G_i for $k = 3$

For each clause C_j we add a clause vertex c_j, where vertex c_j is adjacent to the literal vertices that correspond to the three literals it contains. For example, if $C_j = (u_1 \vee \bar{u}_2 \vee u_3)$, then the clause vertex c_j is adjacent to the literal vertices u_1, $\bar{u}_2$ and u_3. Then add a star $K_{1,k}$ with center x and leaves $x_1, x_2, ..., x_k$, and join x to every clause vertex c_j, for $i = 1, 2, ..., m$. Hence x is of degree $m + k$. Clearly, we can see that G_Φ is a bipartite graph and it can be built in polynomial time.

Let f be a $\gamma_{kr}(G_\Phi)$-function. Clearly, $\sum_{u \in V(G_i)} |f(u)| \geq 2k$ for $i = 1, 2, ..., n$. Since $|f(x)| + |f(x_1)| + ... + |f(x_k)| \geq k$, we obtain that $\gamma_{rk}(G_\Phi) \geq 2nk + k$. On the other hand, assigning $\{1, 2, ..., k\}$ to x, u_i and d_i for $i = 1, 2, ..., n$, and $\emptyset$ to other vertices of G_Φ yields a krDF for G_Φ, implying that $\gamma_{rk}(G_\Phi) \leq (2n + 1)k$. We deduce that $\gamma_{rk}(G_\Phi) = (2n + 1)k$.

Assume that C has a satisfying truth assignment t. We show that

$$st_{\gamma_{rk}}(G_\Phi) > 1.$$

Let $v \in V(G_\Phi)$, and g be a $\gamma_{rk}(G_\Phi - v)$-function. Assume that $v \in V(G_i)$ for some integer i. Since each vertex of $\{a_i^j, e_i^j : j = 1, 2, ..., k\} - \{v\}$ is k-rainbow dominated by g, we find that $\sum_{u \in V(G_i)-\{v\}} |g(v)| \geq 2k$. Furthermore, $\sum_{u \in V(G_j)} |g(u)| \geq 2k$ for each $j \in \{1, 2, ..., n\} - \{i\}$ and $|g(x)| + |g(x_1)| + ... + |g(x_k)| \geq k$. Thus, $w(g) = \gamma_{rk}(G_\Phi - v) \geq (2n+1)k$.

We show that $\gamma_{rk}(G_\Phi - v) \leq (2n+1)k$. If $v = u_i$, then assigning $\{1, 2, ..., k\}$ to x, $\overline{u_i}$, b_i and u_j, d_j for each $j \in \{1, 2, ..., n\} - \{i\}$, and $\emptyset$ to other vertices of $G_\Phi - v$, yields a krDF for $G_\Phi - v$, and so $\gamma_{rk}(G_\Phi - v) \leq (2n+1)k$. Similarly, $\gamma_{rk}(G_\Phi - v) \leq (2n+1)k$ if $v \in \{\overline{u_i}, b_i, d_i\}$. If $v = a_i^j$, for some $j \in \{1, 2, ..., k\}$, then assigning $\{1, 2, ..., k\}$ to x, u_i and d_i for $i = 1, 2, ..., n$, and $\emptyset$ to other vertices of G_Φ yields a krDF for G_Φ, implying that $\gamma_{rk}(G_\Phi - v) \leq (2n+1)k$. Similarly $\gamma_{rk}(G_\Phi - v) \leq (2n+1)k$ if $v = e_i^j$, for some $j \in \{1, 2, ..., k\}$. We deduce that $\gamma_{rk}(G_\Phi - v) = (2n+1)k = \gamma_{rk}(G_\Phi)$.

Next assume that $v \in \{c_1, c_2, ..., c_m\}$. Since $\sum_{u \in V(G_j)} |g(u)| \geq 2k$ for each $j \in \{1, 2, ..., n\}$ and $|g(x)| + |g(x_1)| + ... + |g(x_k)| \geq k$, we have $w(g) = \gamma_{rk}(G_\Phi - v) \geq (2n+1)k$. On the other hand, assigning $\{1, 2, ..., k\}$ to x, u_i and d_i for $i = 1, 2, ..., n$, and $\emptyset$ to other vertices of $G_\Phi - v$ yields a krDF for $G_\Phi - v$, implying that $\gamma_{rk}(G_\Phi - v) \leq (2n+1)k$. Consequently, $\gamma_{rk}(G_\Phi - v) = (2n+1)k$. If $v \in \{x_1, x_2, ..., x_k\}$, then similarly we obtain that $\gamma_{rk}(G_\Phi - v) = (2n+1)k$. It remains to assume that $v = x$. Clearly $w(g) \geq (2n+1)k$, since $\sum_{u \in V(G_i)} |g(u)| \geq 2k$ for $i = 1, 2, ..., n$, and $|g(x_1)| + |g(x_2)| + ... + |g(x_k)| = k$.

We form a set D as follows. For each $i = 1, 2, ..., n$ if $t(u_i) = T$ then $u_i, d_i \in D$, and if $t(u_i) = F$ then $\overline{u_i}, b_i \in D$. Clearly $|D| = 2n$. We define a function h on $V(G_\Phi) - \{v\}$ by assigning $\{1, 2, ..., k\}$ to every vertex of D, $\{1\}$ to $x_1, x_2, ..., x_k$, and $\emptyset$ to each other vertex of $G_\Phi - v$. Since t is a truth assignment, h is a krDF for $G_\Phi - v$, implying that $\gamma_{rk}(G_\Phi - v) \leq (2n+1)k = \gamma_{rk}(G_\Phi)$, and so $\gamma_{rk}(G_\Phi - v) = \gamma_{rk}(G_\Phi)$. We conclude that $\gamma_{rk}(G_\Phi - v) = \gamma_{rk}(G_\Phi)$. Consequently, $st_{\gamma_{rk}}(G_\Phi) > 1$.

Assume now that C does not have a satisfying truth assignment. We consider the graph $G_\Phi - x$. Let h_1 be a $\gamma_{rk}(G_\Phi - x)$-function. Clearly $|h_1(x_1)| = |h_1(x_2)| = ... = |h_1(x_k)| = 1$, and $\sum_{u \in V(G_i)} |h_1(u)| \geq 2k$ for each $i = 1, 2, ..., n$, and thus $w(h_1) \geq (2n+1)k$. Assume that $w(h_1) = (2n+1)k$. Clearly there is no integer $i \in \{1, 2, ..., n\}$ such that $h_1(u_i) = h_1(\overline{u_i}) = \{1, 2, ..., k\}$, since then $e_i^1, ..., e_i^k$ are not k-rainbow dominated by h_1. Let $A = \{u_i : h_1(u_i) = \{1, 2, ..., k\}, i = 1, 2, ..., k\} \cup \{\overline{u_i} : h_1(\overline{u_i}) = \{1, 2, ..., k\}, i = 1, 2, ..., n\}$. Since h_1 is a k-rainbow dominating function for $G_\Phi - v$, any vertex of $\{c_1, c_2, ..., c_m\}$ is dominated by a vertex of A. Now we define an assignment $t_1 : U \to \{T, F\}$ by

$t_1(u_i) = T$ if $u_i \in A$ and $t_1(u_i) = F$ if $\overline{u_i} \in A$. Then t_1 is a truth assignment for C, a contradiction. Thus $w(h_1) > 4n + 2$. We conclude that $st_{\gamma_{rk}}(G_\Phi) = 1$, as desired. □

A natural graph to associate with the 3-SAT Problem is the bipartite graph $G_{\{C,U\}}$ that has $C \cup U$ as its vertex set and has an edge between the vertices u_i and c_j if c_j contains either u_i or $\bar{u}_i$. PLANAR 3-SAT is 3-SAT restricted to those instances $\{C, U\}$ for which $G_{\{C,U\}}$ is planar. It is well-know that the PLANAR 3-SAT Problem is NP-complete [14]. The same proof of Theorem 2 using a transformation from PLANAR 3-SAT Problem yields the following.

THEOREM 3. *krDSNP is NP-hard even for planar graphs.*

It is clear that if a $\gamma_{rk}(G)$-function f is an IkRDF, then $\gamma_{rk}(G) = i_{rk}(G)$.

LEMMA 4. *Let G_Φ be the graph defined in the proof of Theorem 2. Then $i_{rk}(G_\Phi) = \gamma_{rk}(G_\Phi) = i_{rk}(G_\Phi - v) = \gamma_{rk}(G_\Phi - v)$ for any vertex $v \in V(G_\Phi)$.*

Proof. Clearly $i_{rk}(G_\Phi) \geq \gamma_{rk}(G_\Phi) = (2n + 1)k$. On the other hand, the function assigning $\{1, 2, ..., k\}$ to x, u_i and d_i for $i = 1, 2, ..., n$, and $\emptyset$ to other vertices of G_Φ is an IkrDF for G_Φ, implying that $i_{rk}(G_\Phi) \leq (2n + 1)k$. We deduce that $i_{rk}(G_\Phi) = \gamma_{rk}(G_\Phi) = (2n + 1)k$.

Now let $v \in V(G_\Phi)$. Clearly $i_{rk}(G_\Phi - v) \geq \gamma_{rk}(G_\Phi - v) = (2n + 1)k$. On the other hand, each $\gamma_{rk}(G_\Phi - v)$-function defined in the proof of Theorem 2 is independent. Consequently, $i_{rk}(G_\Phi - v) = \gamma_{rk}(G_\Phi - v) = (2n + 1)k$. □

Consider the following decision problem associated to independent k-rainbow domination stability:

INDEPENDENT k-RAINBOW DOMINATION STABILITY NUMBER PROBLEM (IkrDSNP)
Instance: Graph $G = (V, E)$ and the independent k-rainbow domination number $i_{rk}(G)$.
Question: Is $i_{rk}(G - v) = i_{rk}(G)$ for every vertex $v \in V(G)$?

As a consequence of Theorems 2 and 3 and Lemma 4 we obtain the following.

THEOREM 5. *IkrDSNP is NP-hard even for bipartite or planar graphs.*

4. CONCLUDING REMARKS

Using a transformation of the 3-SAT problem, it is proven in [12] that determining whether a graph is domination stable is NP-hard even for bipartite graphs, it is proven in [1] that determining whether a graph is Roman domination stable is NP-hard even for bipartite graphs, and it is proven in [7] that determining whether a graph is subdivision domination stable is NP-hard even for bipartite graphs [7]. The same proofs using a transformation from the PLANAR 3-SAT Problem imply that the above three decision problems are NP-hard even for planar graphs.

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