GENERALIZATIONS OF C3 MODULES AND C4 MODULES

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Let $\mathscr{A}$ be a class of right R-modules that is closed under isomorphisms, and let M be a right R-module. Then M is called $\mathscr{A}$-C3 if, whenever N and K are direct summands of M with $N \cap K = 0$ and $K \in \mathscr{A}$, then $N \oplus K$ is also a direct summand of M; M is called an $\mathscr{A}$-C4 module, if whenever $M = A \oplus B$ where A and B are submodules of M and $A \in \mathscr{A}$, then every monomorphism $f : A \to B$ splits. Some characterizations and properties of these classes of modules are investigated. As applications, some new characterizations of semisimple artinian rings, right V-rings, quasi-Frobenius rings and von Neumann regular rings are given.

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1. **INTRODUCTION**

Throughout, R is an associative ring with identity and all modules are unitary. Unless otherwise specified, $\mathscr{A}$ is a class of some right R-modules which is closed under isomorphisms. Recall that a right R-module M is called a *C*2 *module* [8] if every submodule K of M that is isomorphic to a direct summand of M is itself a direct summand of M; a right R-module M is called a *C*3 *module* [8, 2] if, whenever N and K are direct summands of M with $N \cap K = 0$, then $N \oplus K$ is also a direct summand of M. Clearly, C2 modules are C3 modules. In [4], Ding, Ibrahim, Yousif and Zhou generalized the concept of C3 modules to *C*4 *modules*. According to [4], a right R-module M is called a C4 module, if whenever $M = A \oplus B$ where A and B are submodules of M, then every monomorphism $f : A \to B$ splits. In this paper, we shall generalize the concepts of Ci modules ($i = 2, 3, 4$) to $\mathscr{A}$-Ci modules ($i = 2, 3, 4$), respectively, and give some interesting results on these modules. As applications, some new characterizations of semisimple artinian rings, right V-rings, quasi-Frobenius rings and von Neumann regular rings will be given.

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2. $\mathscr{A}$-C2 MODULES AND $\mathscr{A}$-C3 MODULES

Recall that a right R-module M is called *pseudo-injective* (resp., *pseudo FQ-injective, pseudo PQ-injective, pseudo QP-injective*) if every monomorphism from a submodule (resp., finitely generated submodule, principal submodule, M-cyclic submodule) of M to M extends to an endomorphism of M; a right R-module M is called *minimal quasi-injective* if every homomorphism from a minimal submodule of M to M extends to an endomorphism of M. These concepts can be found in [5, 13, 12, 14] and [10], respectively. Motivated by these concepts, we start this section with the following definitions.

Definition 2.1. Let $\mathscr{A}$ be a class of right R-modules, and let M and N be two right R-modules. Then M is called pseudo $\mathscr{A}$-N-injective if every monomorphism from a submodule $K \in \mathscr{A}$ of N to M extends to an homomorphism of N to M. M is called pseudo $\mathscr{A}$-injective if it is pseudo $\mathscr{A}$-M-injective.

Example 2.2. Let $\mathscr{A}$ be the class of all (resp., all finitely generated, all principal, all minimal, all M-cyclic) right R-modules. Then M is pseudo $\mathscr{A}$-injective if and only if it is pseudo-injective (resp., pseudo FQ-injective, pseudo PQ-injective, minimal quasi-injective, pseudo QP-injective).

Proposition 2.3. *Let $\mathscr{A}$ be a class of right R-modules, M, N be two right R-modules and N' be a submodule of N. If M is pseudo $\mathscr{A}$-N-injective, then*

(1) *Every direct summand of M is pseudo $\mathscr{A}$-N-injective.*
(2) *M is pseudo $\mathscr{A}$-N'-injective.*

Proof. (1) Let $M = M_1 \oplus M_2$. Then for every submodule $K \in \mathscr{A}$ of N and every monomorphism f of K to M_1, since M is pseudo $\mathscr{A}$-N-injective, f extends to a homomorphism of N to M. Which follows that f extends to a homomorphism of N to M_1 because M_1 is a direct summand of M.

(2) It is obvious. □

By Proposition 2.3, we have immediately the following corollary.

Corollary 2.4. *Let $\mathscr{A}$ be a class of right R-modules. Then every direct summand of a pseudo $\mathscr{A}$-injective module is pseudo $\mathscr{A}$-injective.*

The concepts of C2 modules and C3 modules have been extended in several ways. For example, a module M is called $FC2$ (resp., $PC2$, *Min-C2*, *soc-C2*) if every finitely generated (resp., principle, minimal, semisimple) submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M; a module M is called $FC3$ (resp., $PC3$, *Min- C3*) if, whenever N and K are direct summands of M with $N \cap K = 0$ and N is finitely generated (resp., principle, minimal) then $N \oplus K$ is also a direct summand of M; a module M is called $GC2$ if every submodule of M that is isomorphic to M is itself a direct summand of M; a module M is called $GC3$ if, whenever N and K are direct summands of M with $N \cap K = 0$ and N is isomorphic to M, then $N \oplus K$ is also a direct summand of M. These concepts can be found in [13, 12, 10, 9, 11] and [1], respectively. We call a module M *soc-C3* if, whenever N and K are direct summands of M with $N \cap K = 0$ and N is semisimple, then $N \oplus K$ is also a direct summand of M. Note that our definition of soc-C3 modules is different from that defined in [1]. Now we extend these concepts as follows.

Definition 2.5. Let $\mathscr{A}$ be a class of right R-modules that is closed under isomorphisms, and let M be a right R-module. Then M is called $\mathscr{A}$-C2 if every submodule $K \in \mathscr{A}$ of M that is isomorphic to a direct summand of M is itself a direct summand of M. M is called $\mathscr{A}$-C3 if, whenever N and K are direct summands of M with $N \cap K = 0$ and $K \in \mathscr{A}$, then $N \oplus K$ is also a direct summand of M.

It is easy to see that pseudo injective $\Rightarrow C2 \Rightarrow C3$. In general, we have the following results.

THEOREM 2.6. *Let $\mathscr{A}$ be a class of right R-modules that is closed under isomorphisms, and let M be a right R-module. Consider the following conditions:*

(1) *M is pseudo $\mathscr{A}$-injective.*
(2) *M is $\mathscr{A}$-C2.*
(3) *M is $\mathscr{A}$-C3.*
Then, the following implications hold
(1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2). Let M_R be pseudo $\mathscr{A}$-injective with $S = End(M_R)$. If K is a submodule of M, $K \in \mathscr{A}$ and $K \cong eM$, where $e^2 = e \in S$, then eM is pseudo $\mathscr{A}$-M-injective by Proposition 2.3 and hence, K is also pseudo $\mathscr{A}$-M-injective, this follows that K is a direct summand of M because $K \in \mathscr{A}$. This proves (2).

(2) $\Rightarrow$ (3). Let N and K be direct summands of M with $N \cap K = 0$ and $K \in \mathscr{A}$. Write $N = eM$ and $K = fM$, where e, f are idempotents in S, then $eM \oplus fM = eM \oplus (1-e)fM$. Since $(1-e)fM \cong fM \in \mathscr{A}$, $(1-e)fM = hM$ for some $h^2 = h \in S$ by (2). Let $g = e + h - he$. Then $g^2 = g$ and $eM \oplus fM = gM$, as required. □

PROPOSITION 2.7. *Let $\mathscr{A}$ be a class of right R-modules that is closed under isomorphisms and direct summands, and let $M \in \mathscr{A}$ be a right R-module.*

Then M is a C2 module if and only if M is an $\mathscr{A}$-C2 module, M is a C3 module if and only if M is an $\mathscr{A}$-C3 module.

Proof. Obvious. □

Corollary 2.8. (1) *If M is a finitely generated module, then M is a C2 module if and only if it is a FC2 module, M is a C3 module if and only if it is a FC3 module.*

(2) *If M is a cyclic module, then M is a C2 module if and only if it is a PC2 module, M is a C3 module if and only if it is a PC3 module.*

(3) *If M is a finitely generated PFQ-injective (resp., cyclic PPQ-injective) module, then it is a C2 module.*

It is well known that C2 modules and C3 modules are inherited by direct summands [8, Proposition 1.30]. The next results show that $\mathscr{A}$-C2 modules and $\mathscr{A}$-C3 modules are also inherited by direct summands.

Theorem 2.9. (1) *A direct summand of an $\mathscr{A}$-C2 module is again an $\mathscr{A}$-C2 module.*

(2) *A direct summand of an $\mathscr{A}$-C3 module is again an $\mathscr{A}$-C3 module.*

Proof. (1) Let M be an $\mathscr{A}$-C2 module and $N \subseteq^{\oplus} M$. We need to show that N is also $\mathscr{A}$-C2. Let $A \in \mathscr{A}$ be a submodule of N that is isomorphic to a direct summand of N. Since M is $\mathscr{A}$-C2, $A \subseteq^{\oplus} M$. Write $M = A \oplus M_1$. Then $N = M \cap N = (A \oplus M_1) \cap N = A \oplus (M_1 \cap N)$, as required.

(2) Let M be an $\mathscr{A}$-C3 module and $N \subseteq^{\oplus} M$. We prove that N is also $\mathscr{A}$-C3. Let A and B be two direct summands of N with $A \cap B = 0$ and $A \in \mathscr{A}$. Since M is $\mathscr{A}$-C3, $A \oplus B \subseteq^{\oplus} M$. Write $M = (A \oplus B) \oplus C$. Then $N = M \cap N = (A \oplus B \oplus C) \cap N = (A \oplus B) \oplus (C \cap N)$, as required. □

Corollary 2.10. (1) *A direct summand of a C2 (resp., GC2, PC2, FC2, Min-C2, soc-C2) module is again a C2 (resp., GC2, PC2, FC2, Min-C2, soc-C2) module.*

(2) *A direct summand of a C3 (resp., GC3, PC3, FC3, Min-C3, soc-C3) module is again a C3 (resp., GC3, PC3, FC3, Min-C3, soc-C3) module.*

The following theorem extends the results of [2, Proposition 2.2].

Theorem 2.11. *Let $\mathscr{A}$ be a class of right R-modules that is closed under isomorphisms, and let M be a right R-module. Consider the following conditions:*

(1) *M is an $\mathscr{A}$-C3 module.*

(2) *If $A \subseteq^{\oplus} M, B \subseteq^{\oplus} M, A \in \mathscr{A}$ and $A \cap B = 0$, then $M = A_1 \oplus B = A \oplus B_1$ for some submodules $A_1 \supseteq A$ and $B_1 \supseteq B$.*

(3) *If* $A \subseteq^{\oplus} M, B \subseteq^{\oplus} M, A \in \mathscr{A}$ *and* $A \cap B \subseteq^{\oplus} M$*, then* $A + B \subseteq^{\oplus} M$*.*

Then, the following implications hold

$(3) \Rightarrow (1) \Leftrightarrow (2)$.

Moreover, if $\mathscr{A}$ *is closed under direct summands, then the above three conditions are equivalent.*

Proof. $(1) \Rightarrow (2)$. Let $A \subseteq^{\oplus} M, B \subseteq^{\oplus} M, A \in \mathscr{A}$ and $A \cap B = 0$. Then by (1), $A \oplus B \subseteq^{\oplus} M$, and so $M = (A \oplus B) \oplus C$ for a submodule $C \subseteq M$. Write $A_1 = A \oplus C, B_1 = B \oplus C$. Then, we have $A_1 \supseteq A$, $B_1 \supseteq B$ and $M = A_1 \oplus B = A \oplus B_1$.

$(2) \Rightarrow (1)$. Let $A \subseteq^{\oplus} M, B \subseteq^{\oplus} M, A \in \mathscr{A}$ and $A \cap B = 0$. Then by (2), we have $M = A_1 \oplus B = A \oplus B_1$ for some submodules $A_1 \supseteq A$ and $B_1 \supseteq B$. Now $B_1 = B_1 \cap M = B_1 \cap (A_1 \oplus B) = B \oplus (A_1 \cap B_1)$, and so $M = A \oplus B_1 = A \oplus B \oplus (A_1 \cap B_1)$, as required.

$(3) \Rightarrow (1)$. It is clear.

Now suppose that $\mathscr{A}$ is closed under direct summands, we need to prove $(1) \Rightarrow (3)$. Since $A \cap B \subseteq^{\oplus} M$, $M = (A \cap B) \oplus K$ for some submodule K of M. So $A = (A \cap B) \oplus (A \cap K)$ and $B = (A \cap B) \oplus (B \cap K)$, and hence both $A \cap K$ and $B \cap K$ are direct summands of M because both A and B are direct summands of M. Clearly $(A \cap K) \cap (B \cap K) = 0$. Note that $\mathscr{A}$ is closed under direct summands, $A \cap K \in \mathscr{A}$. By (1), we have that $T =: (A \cap K) \oplus (B \cap K)$ is a direct summand of M. Again, since both T and $A \cap B$ are direct summands of M, and $(A \cap B) \cap T \subseteq (A \cap B) \cap K = 0$ as well as $A \cap B \in \mathscr{A}$, by (1), we have $(A \cap B) \oplus T$ is a direct summand of M. Thus, $A + B = [(A \cap B) \oplus (A \cap K)] + [(A \cap B) \oplus (B \cap K)] = (A \cap B) \oplus T$ is a direct summand of M. $\square$

Lemma 2.12 ([6, Lemma 2.6(1)(2)]). *Let* $M = A \oplus B, X \leq A$ *and* $f : X \to B$*. Then*

(1) $X \oplus B = \langle f \rangle \oplus B$*, where* $\langle f \rangle = \{x - f(x) \mid x \in X\}$*.*

(2) $\mathrm{Ker} f = \langle f \rangle \cap A$.

The following theorem extends the results of [2, Proposition 2.3, Corollary 2.4].

Theorem 2.13. *Let* $\mathscr{A}$ *be a class of right* R*-modules that is closed under isomorphisms. If* M *is an* $\mathscr{A}$*-C3 module,* $M = A \oplus B$ *for some submodules* A *and* B *where* $A \in \mathscr{A}$*, and* $f : A \to B$ *is an* R*-homomorphism, then*

(1) *If* f *is an* R*-monomorphism, then* $\mathrm{Im} f \subseteq^{\oplus} B$*.*

(2) *If* $\mathscr{A}$ *is closed under direct summands and* $\mathrm{Ker} f \subseteq^{\oplus} A$*, then* $\mathrm{Im} f \subseteq^{\oplus} B$*.*

Proof. (1) By Lemma 2.12(1), we have $M = \langle f \rangle \oplus B$. Since f is an R-monomorphism, by Lemma 2.12(2), $\langle f \rangle \cap A = 0$.

Since M is $\mathscr{A}$-C3, $\langle f \rangle \oplus A \subseteq^{\oplus} M$. Now we show that $\mathrm{Im} f \oplus A = \langle f \rangle \oplus A$. For, if $b \in \mathrm{Im} f$, then $b = f(a)$ for some $a \in A$, so $b = a - a + f(a) \in A + \langle f \rangle$, and hence $\mathrm{Im} f \oplus A = \langle f \rangle \oplus A$. Since $\langle f \rangle \oplus A \subseteq^{\oplus} M$, $\mathrm{Im} f \subseteq^{\oplus} M$, it implies that $\mathrm{Im} f \subseteq^{\oplus} B$.

(2) Let $f : A \to B$ be an R-homomorphism with $\mathrm{Ker} f \subseteq^{\oplus} A$. If $A = \mathrm{Ker} f \oplus A'$ for a submodule A' of A, then by hypothesis, $A' \in \mathscr{A}$, $M = A \oplus B = \mathrm{Ker} f \oplus A' \oplus B$, and the restriction map $f|_{A'} : A' \to B$ is a monomorphism. Since $A' \oplus B$ is an $\mathscr{A}$-C3 module by Theorem 2.9(2), we infer from (1) that $\mathrm{Im} f = \mathrm{Im}(f|_{A'}) \subseteq^{\oplus} B$. □

3. $\mathscr{A}$-C4 MODULES

Now, we extend the concept of C4 modules as following.

Definition 3.1. (1) Let $\mathscr{A}$ be a class of right R-modules that is closed under isomorphisms. A right R-module M is called an $\mathscr{A}$-C4 module, if whenever $M = A \oplus B$ where A and B are submodules of M and $A \in \mathscr{A}$, then every monomorphism $f : A \to B$ splits.

(2) A right R-module M is called a PC4 (resp., FC4, GC4, Min-C4, soc-C4, pro-C4) module if it is an $\mathscr{A}$-C4 module, where $\mathscr{A}$ is the class of all cyclic (resp., finitely generated, isomorphic to M, simple, semisimple, projective) right R-modules.

It is easy to see that Ci ⇒ FCi ⇒ PCi ⇒ Min-Ci; Ci ⇒ soc-Ci ⇒ Min-Ci and Ci ⇒ GCi, $i = 2, 3, 4$.

Lemma 3.2. *Let A, B, T be submodules of M, $A \cap B = 0$, $M = A \oplus T$, and $\pi : A \oplus T \to T$ be the natural projection. Then $A \oplus B = A \oplus \pi(B)$.*

Proof. For any $b \in B$, there exists $a \in A$ and $t \in T$ such that $b = a + t = a + \pi(b) \in A \oplus \pi(B)$, so $\pi(b) = t = -a + b \in A \oplus B$. This proves the result. □

Now, we give some characterizations of $\mathscr{A}$-C4 modules as follows.

Theorem 3.3. *Let $\mathscr{A}$ be a class of right R-modules that is closed under isomorphisms, and let M be a right R-module. Consider the following conditions:*

(1) *M is an $\mathscr{A}$-C4 module.*

(2) *If $M = A \oplus B$ where A and B are submodules of M and $A \in \mathscr{A}$, and $f : A \to B$ is a monomorphism, then $\mathrm{Im} f \subseteq^{\oplus} B$.*

(3) *If $B \cong A \subseteq^{\oplus} M, B \subseteq M, A \in \mathscr{A}$ and $A \cap B = 0$, then $A \oplus B \subseteq^{\oplus} M$.*

(4) *If* $B \cong A \subseteq^{\oplus} M, B \subseteq M, A \in \mathscr{A}$ *and* $A \cap B = 0$*, then* $B \subseteq^{\oplus} M$.

(5) *If* $M = A \oplus A' = B \oplus B', A \in \mathscr{A}$ *and* $A \cap B = A \cap B' = 0$*, then* $A \oplus B \subseteq^{\oplus} M$.

(6) *If* $B \subseteq M, A \subseteq^{\oplus} M, A \in \mathscr{A}, A \cong B$ *and* $A \cap B = 0$*, then* $A \oplus B \subseteq^{\oplus} M$.

(7) *If* $M = A \oplus B$ *for some submodules* A *and* B *where* $A \in \mathscr{A}$*, and* $f : A \to B$ *is an* R*-homomorphism such that* $\mathrm{Ker} f \subseteq^{\oplus} A$*, then* $\mathrm{Im} f \subseteq^{\oplus} B$.

Then, the following implications hold:

$(7) \Rightarrow (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6)$.

Moreover, if $\mathscr{A}$ *is closed under direct summands, then the above conditions are equivalent.*

Proof. (1) $\Leftrightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Let $B \overset{\sigma}{\cong} A \subseteq^{\oplus} M, B \subseteq M, A \in \mathscr{A}$ and $A \cap B = 0$. We need to prove that $A \oplus B \subseteq^{\oplus} M$. Write $M = A \oplus T$ for a submodule T of M, and let $\pi : A \oplus T \to T$ be the natural projection. Then by Lemma 3.2, $A \oplus B = A \oplus \pi(B)$. Since $A \cap B = 0$, $B \overset{\pi|_B}{\cong} \pi(B)$. Since $M = A \oplus T$ and $\pi|_B \circ \sigma^{-1} : A \to T$ is a monomorphism, by (2), we have that $\mathrm{Im}(\pi|_B \circ \sigma^{-1}) = \pi(B) \subseteq^{\oplus} T$. Let $T = \pi(B) \oplus C$. Then $M = A \oplus T = (A \oplus \pi(B)) \oplus C = (A \oplus B) \oplus C$, as required.

(3) $\Rightarrow$ (4). It is obvious.

(4) $\Rightarrow$ (5). Let $\pi : B \oplus B' \to B'$ be the natural projection. Then by Lemma 3.2, we have $A \oplus B = \pi(A) \oplus B$. Since $A \cap B = 0, \pi(A) \cong A \subseteq^{\oplus} M, A \in \mathscr{A}$ and $\pi(A) \cap A \subseteq B' \cap A = 0$, by (4), $\pi(A) \subseteq^{\oplus} M$, and so $\pi(A) \subseteq^{\oplus} B'$. write $B' = \pi(A) \oplus T$. Then $M = B \oplus B' = B \oplus (\pi(A) \oplus T) = (B \oplus \pi(A)) \oplus T = (A \oplus B) \oplus T$, and then $A \oplus B \subseteq^{\oplus} M$.

(5) $\Rightarrow$ (6). Write $M = A \oplus A'$, and let $\pi : A \oplus A' \to A'$ be the natural projection and $A \overset{f}{\cong} B$. Then by Lemma 2.12(1), we have $M = A \oplus A' = \langle \pi f \rangle \oplus A'$, where $\langle \pi f \rangle = \{a - \pi f(a) \mid a \in A\}$. Since $A \cap B = 0$, it is easy to see that the map πf is monic, and so $A \cap \langle \pi f \rangle = 0$ by Lemma 2.12(2). Thus, by (5), we have that $A \oplus B = A \oplus \pi(B) = A \oplus \langle \pi f \rangle \subseteq^{\oplus} M$.

(6) $\Rightarrow$ (2). Let $M = A \oplus B$ where A and B are submodules of M, $A \in \mathscr{A}$, and $f : A \to B$ be a monomorphism. We need to prove that $\mathrm{Im} f \subseteq^{\oplus} B$. By Lemma 2.12, we have $A \oplus B = \langle f \rangle \oplus B$ and $\langle f \rangle \cap A = 0$. Clearly, $A \cong \langle f \rangle$. So (6) implies that $A \oplus \langle f \rangle \subseteq^{\oplus} M$. Observing that $A \oplus \langle f \rangle = A \oplus \mathrm{Im} f$, we have that $\mathrm{Im} f \subseteq^{\oplus} B$.

(7) $\Rightarrow$ (2). It is obvious.

Now suppose that $\mathscr{A}$ is closed under direct summands, we need to prove (2) $\Rightarrow$ (7). Let $M = A \oplus B$ for some submodules A and B where $A \in \mathscr{A}$, and let $f : A \to B$ be an R-homomorphism with $\mathrm{Ker} f \subseteq^{\oplus} A$. Write $A = \mathrm{Ker} f \oplus C$. Then $M = A \oplus B = (\mathrm{Ker} f \oplus C) \oplus B = C \oplus (\mathrm{Ker} f \oplus B)$. Since $A \in \mathscr{A}$ and

$\mathscr{A}$ is closed under direct summands, $C \in \mathscr{A}$. Clearly, $f|_C$ is a monomorphism from C to $\mathrm{Ker}f \oplus B$. So, by (2), $\mathrm{Im}f = \mathrm{Im}(f|_C) \subseteq^{\oplus} (\mathrm{Ker}f \oplus B)$, and hence $\mathrm{Im}f \subseteq^{\oplus} B$, as required. □

Corollary 3.4. *If $\mathscr{A}$ is closed under isomorphisms and direct summands, M is an $\mathscr{A}$-C4 module and $M \in \mathscr{A}$, then M is a C4 module.*

Proof. It follows from Theorem 3.3(2). □

Corollary 3.5. *Every cyclic (resp., finitely generated, semisimple, projective) PC4 (resp., FC4, soc-C4, pro-C4) module is a C4 module.*

Recall that an R-module M is said to have the internal finite exchange property [7] if, for any direct summand X of M and any decomposition $M = \oplus_I M_\alpha$, where I is a finite index set, there exist submodules $M'_\alpha \subseteq M_\alpha$ such that $M = X \oplus (\oplus_I M'_\alpha)$.

Proposition 3.6. *If M is an $\mathscr{A}$-C3 module, then it is an $\mathscr{A}$-C4 module. Conversely, if $\mathscr{A}$ is closed under direct summands and M is an $\mathscr{A}$-C4 module with the internal finite exchange property, then it is an $\mathscr{A}$-C3 module.*

Proof. If M is an $\mathscr{A}$-C3 module, then it follows immediately from Theorem 2.13(1) that M is an $\mathscr{A}$-C4 module. Now assume that M is an $\mathscr{A}$-C4 module with the internal finite exchange property. Let A and B are direct summands of M with $A \cap B = 0$ and $A \in \mathscr{A}$. Write $M = A \oplus C = B \oplus D$. Then by the internal finite exchange property, there exists a submodule A' of A and a submodule C' of C such that $M = B \oplus A' \oplus C'$, and so, by modular law, we have $A = A' \oplus A''$ and $C = C' \oplus C''$, where $A'' = (B \oplus C') \cap A, C'' = (B \oplus A') \cap C$. It is easy to see that $A'' \in \mathscr{A}, M = A'' \oplus (A' \oplus C) = C' \oplus (B \oplus A'), A'' \cap C' = 0, A'' \cap (B \oplus A') = 0$, so we infer from Theorem 3.3(5) that $A \oplus B = A'' \oplus (B \oplus A') \subseteq^{\oplus} M$, and thus M is an $\mathscr{A}$-C3 module. □

Corollary 3.7. *Let M be a module with the internal finite exchange property, then it is a PC3 (resp., FC3, soc-C3, pro-C3) module if and only if it is a PC4 (resp., FC4, soc-C4, pro-C4) module.*

Remark 3.8. We remark that Min-C2 modules are nothing but the simple-direct-injective module defined in [3]. By [3, Proposition 2.1] and Theorem 3.3(2), a module M is Min-C2 if and only if it is Min-C3 if and only if it is Min-C4.

The following example shows that $\mathscr{A}$-Ci modules need not be Ci-modules for each $i = 2, 3, 4$, and Min-C4 modules need not be PC4.

Example 3.9. Let K be a field and R be the K-algebra consisting of all 3×3 matrices of the form $\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \alpha_4 & 0 \\ 0 & 0 & \alpha_5 \end{pmatrix}$, where $\alpha_i \in K$. Then by [3, Example 3.7], $e_{11}R \oplus E(e_{11}R)$ is a simple-direct-injective right R-module. But $e_{11}R$ is not injective, by Theorem 3.3(2), $e_{11}R \oplus E(e_{11}R)$ is not a PC4-module and hence it is not a C4-module. So, in general, $\mathscr{A}$-Ci modules need not be Ci-modules for each $i = 2, 3, 4$.

Proposition 3.10. (1) *A direct summand of an* $\mathscr{A}$*-C4 module is again an* $\mathscr{A}$*-C4 module.*

(2) *If* $M \oplus M$ *is an* $\mathscr{A}$*-C4 module, then* M *is an* $\mathscr{A}$*-C2 module.*

(3) *Let* M *be an* $\mathscr{A}$*-C4 module,* $A \subseteq^{\oplus} M, B \subseteq M, A \in \mathscr{A}$ *and* $A \cap B = 0$. *If there exists a monomorphism* $f : A \to B$*, then* A *is an* $\mathscr{A}$*-C2 module.*

Proof. (1) Let M be an $\mathscr{A}$-C4 module, $K \subseteq^{\oplus} M$ and write $M = K \oplus N$. Suppose $K = A \oplus B, A \in \mathscr{A}$ and $f : A \to B$ is a monomorphism. Then $M = A \oplus (B \oplus N), A \in \mathscr{A}$, and $f : A \to B \oplus N$ is a monomorphism. Since M is an $\mathscr{A}$-C4 module, $\mathrm{Im} f \subseteq^{\oplus} B \oplus N$, and so $\mathrm{Im} f \subseteq^{\oplus} B$. This follows that K is an $\mathscr{A}$-C4 module.

(2) Suppose that $M \oplus M$ is an $\mathscr{A}$-C4 module. Let $A \in \mathscr{A}$ and $A \overset{\sigma}{\cong} B \subseteq^{\oplus} M$. We need to prove that $A \subseteq^{\oplus} M$. Write $M = B \oplus C$ for a submodule C of M. Since $M \oplus M \cong B \oplus (M \oplus C)$ is an $\mathscr{A}$-C4 module and $B \in \mathscr{A}$ and $\iota\sigma^{-1} : B \to M \oplus C$ is monic, where $\iota : M \to M \oplus C$ is the natural injection, by Theorem 3.3(2), $\mathrm{Im}(\iota\sigma^{-1}) \subseteq^{\oplus} M \oplus C$, that is, $A \oplus 0 \subseteq^{\oplus} M \oplus C$, and so $A \subseteq^{\oplus} M$.

(3) Since M is an $\mathscr{A}$-C4 module, we infer from Theorem 3.3(3) that $A \oplus A \cong A \oplus \mathrm{Im} f \subseteq^{\oplus} M$. By (1), $A \oplus A$ is an $\mathscr{A}$-C4 module. And so, by (2), A is an $\mathscr{A}$-C2 module. □

Theorem 3.11. *The following statements are equivalent for a ring* R*:*

(1) *Every* $A \in \mathscr{A}$ *is injective.*

(2) *Every right* R*-module is an* $\mathscr{A}$*-C4 module.*

Proof. (1) ⇒ (2). It follows from Theorem 3.3(2).

(2) ⇒ (1). Let $A \in \mathscr{A}$. Since $A \oplus E(A)$ is an $\mathscr{A}$-C4 module, by Theorem 3.3(2), $A \subseteq^{\oplus} E(A)$, and so $A = E(A)$ is injective. □

Recall that a ring R is semisimple artinian if and only if every cyclic module is injective, a ring R is a right V-ring if every simple right R-module is injective, a ring R is quasi-Frobenius if and only if every projective right

R-module is injective. Based on these facts, by Theorem 3.11, we have the following corollaries.

COROLLARY 3.12. (1) *A ring R is a semisimple artinian ring if and only if every right R-module is a PC4 module.*

(2) *[3, Proposition 4.1] A ring R is a right V-ring if and only if every right R-module is a simple-direct-injective module.*

(3) *A ring R is a quasi-Frobenius ring if and only if every right R-module is a pro-C4 module.*

COROLLARY 3.13. *A ring R is a right noetherian right V-ring if and only if every right R-module is a soc-C4 module.*

Proof. $\Rightarrow$. Since R is a right V-ring, every simple right R-module is injective. But R is right noetherian, every direct sum of injective R-modules is injective. And so, every semisimple right R-module is injective. Thus, by Theorem 3.11, we have that every right R-module is a soc-C4 module.

$\Leftarrow$. Since every right R-module is a soc-C4 module, by Theorem 3.11, we have that every semisimple right R-module is injective. Clearly, R is a right V-ring. Now let $K_1, K_2, ...$ be simple right R-modules. Then $\oplus_{i=1}^{\infty} K_i$ is injective, and so $\oplus_{i=1}^{\infty} K_i \subseteq^{\oplus} \oplus_{i=1}^{\infty} E(K_i)$. Observing that $\oplus_{i=1}^{\infty} K_i \subseteq^{ess} \oplus_{i=1}^{\infty} E(K_i)$, we have $\oplus_{i=1}^{\infty} K_i = \oplus_{i=1}^{\infty} E(K_i)$, and so $\oplus_{i=1}^{\infty} E(K_i)$ is injective. By [8, Theorem 7.48], R is a right noetherian ring. $\square$

We end this paper with a characterization of von Neumann regular rings in terms of C3 modules, PC3 modules and PC4 modules.

PROPOSITION 3.14. *The following statements are equivalent for a ring R:*

(1) *R is a von Neumann regular ring.*

(2) *Every finitely generated submodule of a projective right R-module is a C3 module.*

(3) *Every finitely generated submodule of a projective right R-module is a PC3 module.*

(4) *Every 2-generated submodule of a projective right R-module is a PC3 module.*

(5) *Every 2-generated submodule of a projective right R-module is a PC4 module.*

Proof. (1) $\Rightarrow$ (2). Since R is a regular ring, every finitely generated submodule of a projective right R-module is a direct summand, and so (2) holds.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). These implications are straightforward.

(5) $\Rightarrow$ (1). Let I be a principal right ideal. By (5), $I \oplus R$ is PC4. And so, by Theorem 3.3(2), $I \subseteq^{\oplus} R$, as required. □

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