SINGULAR LIMITING RADIAL SOLUTIONS FOR 4-DIMENSIONAL ELLIPTIC PROBLEM INVOLVING EXPONENTIALLY DOMINATED NONLINEARITY

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We study the existence of solutions having singular limits for some four-dimensional semilinear elliptic problems involving exponential nonlinearity with nonlinear terms with Navier boundary condition. In particular, we extend the result of [2].

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1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we will use the method of domain decomposition to study the following problem

$$\left\{\begin{array}{ll} \Delta^2 u + \mathscr{Q}_\lambda(u) = \rho^4 |x|^{4\beta} f(|x|) e^u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1}$$

where $\Omega = B_1 \subset \mathbb{R}^4$ the unit ball centered at the origin, ρ is a parameter that tends to 0, β is a positive function defined in a neighborhood of 0 in $\mathbb{R}$, $f : [0, +\infty) \to \mathbb{R}$ is a smooth positive function satisfying $f(0) > 0$ and $\mathscr{Q}_\lambda$ is the nonlinear operator given by

$$\begin{aligned} \mathscr{Q}_\lambda(u) := \lambda\Big[(\Delta u)^2 + \Delta(|\nabla u|^2) + 2\nabla u \cdot \nabla(\Delta u)\Big] \\ + 2\lambda^2\Big[\Delta u\,|\nabla u|^2 + \nabla u \cdot \nabla(|\nabla u|^2)\Big] + \lambda^3 |\nabla u|^4. \end{aligned} \tag{2}$$

Using the following transformation

$$w := (\lambda \rho^4 e^u)^\lambda,$$

the function w satisfies the following equation

$$\Delta^2 w \;=\; V(x) w^{\frac{\lambda+1}{\lambda}} \text{ in } \Omega, \tag{3}$$

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with $V(x) = |x|^{4\beta} f(|x|)$. Problem (3) with $V \equiv 1$ has been studied by Ben Ayed, El Mehdi and Grossi in [5], since the exponent $p = \frac{\lambda+1}{\lambda}$ tends to infinity as λ tends to 0.

We denote by ε the smallest positive number satisfying

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}. \tag{4}$$

We will suppose in the following

$$(A_\beta) \quad \beta^{1+\frac{\delta}{2}}\varepsilon^{-\delta/(\beta+1)} \to 0 \quad \text{as} \quad \varepsilon \to 0, \text{ for any } \delta \in (0,1).$$

In particular, if we take $\beta = \mathcal{O}(\varepsilon^{2/3})$, then the condition (A_β) is satisfied. We also suppose that

$$(A_\lambda) \quad \lambda^{1+\frac{\delta}{2}}\varepsilon^{-\delta/(\beta+1)} \to 0 \quad \text{as} \quad \varepsilon \to 0, \text{ for any } \delta \in (0,1).$$

In particular, if we take $\lambda = \mathcal{O}(\varepsilon^{2/3})$, then the condition (A_λ) is satisfied.

Let G be the Green's function, solution of the problem

$$\begin{cases} \Delta^2 G &= 64\pi^2\delta_0 \quad \text{in } \Omega \\ \quad G &= \Delta G = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5}$$

and we denote by $H(x) = G(x) + 8\log r$ its regular part function. Here, $r = |x|$.

Our main result reads as follows.

THEOREM 1. *Let $\Omega = B_1$ be the unit ball in $\mathbb{R}^4$. Suppose that the assumptions (A_λ) and (A_β) are satisfied. Then there exist $\rho_0 > 0$, $\lambda_0 > 0$ and a family $\{u_{\rho,\lambda,\beta}\}_{\, 0<\rho<\rho_0, 0<\lambda<\lambda_0}$ of solutions of (1), such that*

$$\lim_{\substack{\rho \to 0 \\ \lambda \to 0}} u_{\rho,\, \lambda,\beta} = G \quad \text{in} \quad C^\infty_{loc}(B_1 \backslash \{0\}) \; .$$

In case $\lambda = 0$, we get the following problem

$$\begin{cases} \Delta^2 u = \rho^4 |x|^{4\beta} f(|x|) e^u & \text{in } \Omega \\ \Delta u \;\; = \;\; u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

The authors in [11] gave a sufficient condition for problem (6) to have a weak solution in Ω which is singular in 0 as ρ a small parameter satisfying the condition (A_β).

Problem (6) is a generalisation of

$$\begin{cases} \Delta^2 u &= \rho^4 e^u - 32\pi^2\beta\delta_0 & \text{in } \Omega \\ \Delta u &= u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

since, setting $v = u + \frac{1}{2}\beta G$, it is clear that u solves (7) if and only if v solves the following problem

$$\begin{cases} \Delta^2 v &= \rho^4 |x|^{4\beta} e^{-\frac{1}{2}\beta H} e^v \quad \text{in } \Omega \\ \Delta v &= v = 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{8}$$

Semilinear equations involving fourth order elliptic operator and exponential nonlinearity appear naturally in conformal geometry and, in particular, in the prescription of the so-called Q-curvature in four-dimensional Riemannian manifolds [8, 9]

$$Q_g = \frac{1}{12}\left(-\Delta_g S_g + S_g^2 - 3\left|\mathrm{Ric}_g\right|^2\right),$$

where Ric_g denotes the Ricci tensor and S_g is the scalar curvature of the metric g. Recall that the Q-curvature changes under a conformal change of metric

$$g_w = e^{2w}\, g,$$

according to

$$P_g\, w + 2\, Q_g = 2\, \tilde{Q}_{g_w}\, e^{4\,w}, \tag{9}$$

where

$$P_g := \Delta_g^2 + \delta\left(\frac{2}{3}\, S_g\, I - 2\,\mathrm{Ric}_g\right) d$$

is the Paneitz operator, which is an elliptic 4-th order partial differential operator [9] and which transforms according to

$$e^{4w}\, P_{e^{2w}g} = P_g,$$

under a conformal change of metric $g_w := e^{2w}\, g$.

There are two reasons that make this Q-curvature equation (9) attractive to study. The first consideration comes from the analytic point of view, namely that the generic singularities of the Q-curvature equation are isolated points. The second consideration comes from geometry: the Q-curvature prescribed by the Paneitz operator can be viewed as part of the integrand in the Chern-Gauss-Bonnet formula:

$$8\pi^2\chi(M) = \int_M \left(\frac{1}{4}|W_g|^2 + 2\tilde{Q}_{gw}\right)\mathrm{d}v,$$

where $\chi(M)$ is the Euler characteristic of M and W denotes the Weyl tensor. Note that $|W_g|^2 dv$ is a pointwise conformal invariant, thus the integration of $\tilde{Q}_{gw}$ is conformally invariant. Since the Q-curvature contains information about the Ricci tensor, it influences the geometry of the underlying manifold directly.

In the special case where the manifold is the Euclidean space and g is the Euclidean metric, the Paneitz operator is simply given by

$$P_{g_{eucl}} = \Delta^2,$$

in which case (9) can be written as

$$\Delta^2 w = \tilde{Q}_{gw}\, e^{4\,w},$$

the solutions of which give rise to the conformal metric $g_w = e^{2\,w}\, g_{eucl}$ whose Q-curvature is given by $\tilde{Q}_{gw}$. There is by now an extensive literature about this problem and we refer to [9] and [16] for references and recent developments.

In dimension two, the analogue of the Q-curvature is the Gauss curvature and the corresponding problem is

$$\begin{cases} -\Delta u \;=\; \rho^2 e^u - 4\pi \sum_{i=1}^{N} \beta_i \delta_{p_i} & \text{in } \mathcal{D} \\ u \;=\; 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{10}$$

where $\mathcal{D} \subset \mathbb{R}^2$ is a regular bounded domain, ρ is a parameter tending to $0, \Lambda := \{p_1, \cdots, p_N\} \subset \mathcal{D}$ is the set of singular sources and where δ_{p_i} denotes the Dirac mass at p_i.

Esposito in [13] has proved the existence of solutions to the problem (10) having a prescribed singular set S for the limits. To describe his result, we need to introduce some notation. Let $\Gamma(x,\ x')$ be the Green's function defined on $\mathcal{D} \times \mathcal{D}$, the solution of

$$\begin{cases} -\Delta\Gamma(x,\ x') \;=\; 8\pi\delta_{x=x'} & \text{in } \mathcal{D} \\ \Gamma(x,\ x') \;=\; 0 & \text{on } \partial\mathcal{D} \end{cases} \tag{11}$$

and let

$$h(x,\ x') = \Gamma(x,\ x') + 4 \log |x - x'|$$

be the regular part of Γ. Problem (10) is equivalent to solving for

$$v = u + \frac{1}{2} \sum_{i=1}^{N} \beta_i \Gamma(\cdot, p_i),$$

the equation

$$\begin{cases} -\Delta v \;=\; \rho^2 \prod_{i=1}^{N} |x - p_i|^{2\beta_i} e^{-\frac{1}{2}\Sigma_{i=1}^{N} \beta_i h(x,p_i)} e^v & \text{in } \mathcal{D} \\ v \;=\; 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{12}$$

For a given smooth function $f : \mathcal{D} \to (0,\ +\infty)$ consider the following "general model" problem

$$
(13) \qquad \begin{cases} -\Delta v \ = \ \rho^2 \prod_{i=1}^{N} |x - p_i|^{2\beta_i} f(x) e^v & \text{in } \mathcal{D} \\ v \ = \ 0 & \text{on } \partial\mathcal{D}, \end{cases}
$$

where $\Lambda = \{p_1, \cdots, p_N\} \subset \mathcal{D}$ and β_i are positive numbers. For $1 \leq s \leq N$ and $m \in \mathbb{N}$, we denote

$$
\begin{aligned}
\mathcal{F}(x_1, \cdots,\ x_m) = & \sum_{j=1}^{m} h(x_j,\ x_j) + \sum_{i \neq j} \Gamma(x_i,\ x_j) \\
& + 4 \sum_{i=1}^{s} \sum_{j=1}^{m} \beta_i \log(|x_j - p_i|) + 2 \sum_{j=1}^{m} \log(f(x_j)),
\end{aligned}
$$

which is well defined for $x_i \neq x_j$ when $i \neq j$. Let

$$
\mathcal{G}(x_1, \cdots\ ,\ x_m,\ w_1, \cdots\ ,\ w_s) = \sum_{j=1}^{m} \sum_{i=1}^{s} (1 + \beta_i) \Gamma(x_j,\ w_i)\ .
$$

$\mathcal{G}$ is well defined for $x_j \neq w_i$ with $x_j \in \mathcal{D}, w_i \in \mathcal{D}$. Esposito in [13] has proved the following.

THEOREM 2 ([13]). *Let $\mathcal{D} \subset \mathbb{R}^2$ be a smooth open set, f a smooth positive function and $\{\beta_1, \cdots,\ \beta_N\} \subset (0,\ +\infty) \backslash \mathbb{N}$ be a set of real numbers. We have the following.*

1. *Let $S = \{p_{j_1}, \cdots, p_{j_s}\} \subset \Lambda$. Then there exist $\rho_0 > 0$ small and a family $(v_\rho)_{0<\rho<\rho_0}$ of solutions for the problem (10) such that*

$$
v_\rho \to \sum_{i=1}^{s} (1 + \beta_{j_i}) \Gamma(\cdot, p_{j_i}),
$$

as $\rho \to 0$, in $C^{2,\alpha}_{loc}(\mathcal{D} \backslash S)$ for $\alpha \in (0, 1)$.

2. *Let $S = \{q_1, \cdots,\ q_m\} \subset \mathcal{D} \backslash \Lambda$ and $(q_1, \cdots,\ q_m)$ be a nondegenerate critical point of $\mathcal{F}$ such that $\Delta \log f(q_1) = \cdots = \Delta \log f(q_m) = 0$. Then there exist $\rho_0 > 0$ small and a family $(v_\rho)_{0<\rho<\rho_0}$ of solutions for the problem (10) such that*

$$
v_\rho \to \sum_{i=1}^{m} \Gamma(\cdot, q_i),
$$

as $\rho \to 0$, in $C^{2,\alpha}_{loc}(\mathcal{D} \backslash S)$ for $\alpha \in (0, 1)$.

3. *Let S be such that $S \cap \Lambda = \{p_{j_1}, \cdots, p_{j_s}\}$, $S \backslash \Lambda = \{q_1, \cdots, q_m\}$ and $(q_1, \cdots, q_m)$ a nondegenerate critical point of the function*

$$\mathcal{F}(q_1, \cdots, q_m) + \mathcal{G}(q_1, \cdots, q_m, p_{j_1}, \cdots, p_{j_s})$$

such that $\Delta \log f(q_1) = \cdots = \Delta \log f(q_m) = 0$, then there exist $\rho_0 > 0$ small and a family $(v_\rho)_{0<\rho<\rho_0}$ of solutions for the problem (10) such that

$$v_\rho \to \sum_{k=1}^{s} (1 + \beta_{j_k}) \Gamma(\cdot, p_{j_k}) + \sum_{i=1}^{m} \Gamma(\cdot, q_i),$$

as $\rho \to 0$, in $C^{2,\alpha}_{loc}(\mathcal{D} \backslash S)$ for $\alpha \in (0, 1)$.

In order to prove our result, we will use a matching argument inspired from [3].

2. ROTATIONALLY SYMMETRIC APPROXIMATE SOLUTIONS

Letting $\beta > 0$, we first describe the rotationally symmetric approximate solutions of

$$\Delta^2 u - \rho^4 |x|^{4\beta} \, e^u = 0 \tag{14}$$

in $\mathbb{R}^4$, which will be crucial in the construction of the approximate solution. Note that equation (14) is invariant under dilation but not under translation.

Given $\varepsilon > 0$, we define

$$u_\varepsilon(x) := 4 \log(1 + \varepsilon^2) - 4 \, \log(\varepsilon^2 + (|x|)^2),$$

which is a solution of

$$\Delta^2 u - \rho^4 \, e^u = 0, \tag{15}$$

when

$$\rho^4 = \frac{384 \, \varepsilon^4}{(1 + \varepsilon^2)^4}.$$

For $\tau > 0$, we remark that equation (15) is invariant under some dilation in the following sense: if u is solution of (15), then

$$\tau \mapsto u(\tau \, \cdot) + 4 \, \log \tau,$$

is also solution of (15). So, for $\beta > 0$ and $\tau > 0$ we define the function

$$u_{\varepsilon,\tau,\beta}(x) := \log \frac{(1 + \varepsilon^2)^4 \tau^4 (4\beta^2 + 8\beta + 6)(\beta + 1)^2}{6(\varepsilon^2 + \tau^2 |x|^{2(1+\beta)})^4}. \tag{16}$$

Easy computations show that $u_{\varepsilon,\tau,\beta}$ satisfies the equation
(17)
$$\Delta^2 u_{\varepsilon,\tau,\beta} - \rho^4 |x|^{4\beta}\, e^{u_{\varepsilon,\tau,\beta}} = -\frac{64\beta(\beta+2)(\beta+1)^2\tau^2\varepsilon^2|x|^{2(\beta-1)}}{(\varepsilon^2+\tau^2|x|^{2(1+\beta)})^4}\Big(\varepsilon^4 + \tau^4|x|^{4(1+\beta)}\Big)$$
in $\mathbb{R}^4$. We will use it as an approximate solution of (14). We notice that in dimension two the equation $\Delta u + \rho^2|x|^{2\beta}\, e^u = 0$ has an explicit solution on $\mathbb{R}^2$, see [13]. Here we do not have an explicit solution of (14) but we will construct a solution by perturbing the approximate solution given by (16).

We also define the following linear fourth order elliptic operator
$$L := \Delta^2 - \frac{384}{(1+|x|^2)^4},$$
which corresponds to the linearization of (15) about the solution $u_{1,1,0}$.

2.1. Construction of solutions without boundary conditions

For all $\varepsilon, \tau, \beta, \lambda > 0$, we set
$$R_{\varepsilon,\lambda,\beta} := \left(\frac{\tau}{\varepsilon}\right)^{\frac{1}{\beta+1}} r_{\varepsilon,\lambda,\beta},$$
where
$$r_{\varepsilon,\lambda,\beta} := \max(\sqrt{\lambda}, \sqrt{\beta}, \varepsilon^{\frac{1}{\beta+1}}). \tag{18}$$

Definition 1. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, we introduce the Hölder weighted spaces $\mathcal{C}^{k,\alpha}_\delta(\mathbb{R}^4)$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{loc}(\mathbb{R}^4)$ for which the following norm
$$\|w\|_{\mathcal{C}^{k,\alpha}_\delta(\mathbb{R}^4)} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1)} + \sup_{r\geqslant 1}\Big((1+r^2)^{-\delta/2}\,\|w(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1 - B_{1/2})}\Big)$$
is finite.

We also define
$$\mathcal{C}^{k,\alpha}_{rad,\delta}(\mathbb{R}^4) = \{f \in \mathcal{C}^{k,\alpha}_\delta(\mathbb{R}^4);\ \text{such that } f(x) = f(|x|), \forall\, x \in \mathbb{R}^4\}.$$

We recall the surjectivity result of L given in [3].

Proposition 1 ([3]). *Assume that $\delta > 0$ and $\delta \notin \mathbb{Z}$, then*
$$\begin{array}{rccc} L: & \mathcal{C}^{4,\alpha}_{rad,\delta}(\mathbb{R}^4) & \longrightarrow & \mathcal{C}^{0,\alpha}_{rad,\delta-4}(\mathbb{R}^4) \\ & w & \longmapsto & Lw \end{array}$$
is surjective.

We set $\bar{B}_1^* = \bar{B}_1 - \{0\}$. Then, we define the subspace of radial functions in $\mathcal{C}_\delta^{k,\alpha}(\bar{B}_1^*)$ by

$$\mathcal{C}_{rad,\delta}^{k,\alpha}(\bar{B}_1^*) = \{f \in \mathcal{C}_\delta^{k,\alpha}(\mathbb{R}^4); \text{such that} f(x) = f(|x|), \forall\, x \in \bar{B}_1^*\}.$$

Our aim is the construction of a radial solution u of

$$\Delta^2 u + \mathscr{Q}_\lambda(u) - \rho^4 |x|^{4\beta}\ e^u = 0 \quad \text{in} \quad \bar{B}_{r_{\varepsilon,\lambda,\beta}}. \tag{19}$$

Thanks to the following transformation

$$v(x) = u\left((\frac{\varepsilon}{\tau})^{\frac{1}{\beta+1}} x\right) + 8\log\varepsilon - 4\log\left(\tau(1+\varepsilon^2)/2\right), \tag{20}$$

the equation (19) can be written as

$$\Delta^2 v + \mathscr{Q}_\lambda(v) - 24|x|^{4\beta} e^v = 0 \quad \text{in} \quad \bar{B}_{R_{\varepsilon,\lambda,\beta}}. \tag{21}$$

Now, we look for a solution of (21) of the form

$$v(x) = u_{1,1,\beta}(x) + h(x).$$

This amounts to solve

$$\begin{aligned} L\,h \;&=\; \tfrac{C_\beta |x|^{4\beta}}{(1+|x|^{2(\beta+1)})^4}(e^h - h - 1) + \frac{D_\beta |x|^{2(\beta-1)}}{(1+|x|^{2(\beta+1)})^4}(|x|^{4(\beta+1)} + 1) \\ &\quad - V_\beta(x) h - \mathscr{Q}_\lambda(u_1 + h) \end{aligned} \tag{22}$$

in $\bar{B}_{R_{\varepsilon,\lambda,\beta}}$, where $C_\beta = 64(4\beta^2 + 8\beta + 6)(\beta+1)^2, D_\beta = 64\beta(\beta+2)(\beta+1)^2$ and

$$V_\beta(x) = \frac{384}{(1+|x|^2)^4} - \frac{C_\beta |x|^{4\beta}}{(1+|x|^{2(\beta+1)})^4}. \tag{23}$$

Observe that, for $\beta > 0$ small enough, there exists $c > 0$ such that

$$|V_\beta(x)| \leq c\frac{1 + |\log|x||}{(1+|x|^2)^4}\beta. \tag{24}$$

We will need the following definition.

Definition 2. Given $\bar{r} \geqslant 1/2$, $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, the weighted space $\mathcal{C}_\delta^{k,\alpha}(B_{\bar{r}})$ is defined to be the space of functions $w \in \mathcal{C}^{k,\alpha}(B_{\bar{r}})$ endowed with the norm

$$\|w\|_{\mathcal{C}_\delta^{k,\alpha}(\bar{B}_{\bar{r}})} := \|w\|_{\mathcal{C}^{k,\alpha}(B_{1/2})} + \sup_{1/2 \leqslant r \leqslant \bar{r}} \left(r^{-\delta}\, \|w(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1 - B_{1/2})}\right).$$

For all $\sigma \geqslant 1$, we denote by

$$\mathscr{E}_\sigma : \mathcal{C}_\delta^{0,\alpha}(\bar{B}_\sigma) \longrightarrow \mathcal{C}_\delta^{0,\alpha}(\mathbb{R}^4),$$

the extension operator defined by

$$(25)\qquad \begin{cases} \mathscr{E}_\sigma(f)(x) & \equiv f(x) \qquad\qquad\qquad \text{for } |x| \le \sigma \\ \mathscr{E}_\sigma(f)(x) & = \chi\Big(\dfrac{|x|}{\sigma}\Big)\, f\Big(\sigma \dfrac{x}{|x|}\Big) \quad \text{for } |x| \ge \sigma, \end{cases}$$

where $t \longmapsto \chi(t)$ is a smooth nonnegative cutoff function identically equal to 1 for $t \leqslant 1$ and identically equal to 0 for $t \geqslant 2$. It is easy to check that there exists a constant $c = c(\delta) > 0$, independent of $\sigma \geqslant 1$, such that

$$(26)\qquad \|\mathscr{E}_\sigma(w)\|_{\mathcal{C}^{0,\alpha}_\delta(\mathbb{R}^4)} \leqslant c\, \|w\|_{\mathcal{C}^{0,\alpha}_\delta(\bar{B}_\sigma)}.$$

We fix

$$\delta \in (0,1)$$

and let $\mathscr{G}_\delta$ to be a right inverse of L assured by Proposition 1. Now, we use the result of Proposition 1 to rephrase the nonlinear equation (22) as a fixed point problem. Hence, to obtain a solution of (22), it is enough to find a fixed point h in a small ball of $\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)$ for the mapping

$$(27)\qquad h \mapsto \mathscr{N}(h) := \mathscr{G}_\delta \circ \mathscr{E}_{R_{\varepsilon,\lambda,\beta}} \circ \mathscr{R}(h),$$

where

$$(28)\qquad \begin{aligned} \mathscr{R}(h) \;=\; & \frac{C_\beta |x|^{4\beta}}{(1+|x|^{2(\beta+1)})^4}(e^h - h - 1) + \frac{D_\beta |x|^{2(\beta-1)}}{(1+|x|^{2(\beta+1)})^4}(|x|^{4(\beta+1)}+1) \\ & - V_\beta(x) h - \mathscr{Q}_\lambda(u_{1,1,\beta}+h). \end{aligned}$$

We have

$$\begin{aligned} \mathscr{R}(0) \;=\; & -\lambda\Big[(\Delta u_{1,1,\beta})^2 + \Delta(|\nabla u_{1,1,\beta}|^2) + 2\nabla u_{1,1,\beta}\cdot\nabla(\Delta u_{1,1,\beta})\Big] \\ & -2\lambda^2\Big[\Delta u_{1,1,\beta}\,|\nabla u_{1,1,\beta}|^2 + \nabla u_{1,1,\beta}\cdot\nabla(|\nabla u_{1,1,\beta}|^2)\Big] \\ & -\lambda^3|\nabla u_{1,1,\beta}|^4 + \frac{D_\beta|x|^{2(\beta-1)}}{(1+|x|^{2(\beta+1)})^4}(|x|^{4(\beta+1)}+1). \end{aligned}$$

Recall that

$$u_{1,1,\beta} = 4\log(2) - 4\,\log(1+r^{2(\beta+1)}) + \log((4\beta^2+8\beta+6)(\beta+1)^2) - \log(6).$$

Then

$$|\nabla u_{1,1,\beta}|^2 = 64(\beta+1)^2\frac{r^{4\beta+2}}{(1+r^{2(\beta+1)})^2}\;,\quad \Delta u_{1,1,\beta} = -16(1+\beta)\frac{(2+\beta)r^{2\beta}+r^{4\beta+2}}{(1+r^{2(\beta+1)})^2},$$

$$\Delta(|\nabla u_{1,1,\beta}|^2) = 512(1+\beta)^2\frac{(1+2\beta)(1+\beta)r^{4\beta} - (2+3\beta+\beta^2)r^{6\beta+2}}{(1+r^{2(\beta+1)})^4},$$

$$\nabla u_{1,1,\beta}\cdot\nabla\Delta u_{1,1,\beta}=256(1+\beta)^2\frac{\beta(2+\beta)r^{4\beta}-(3+2\beta+\beta^2)r^{6\beta+2}-r^{4+8\beta}}{(1+r^{2(\beta+1)})^4}$$

and

$$\nabla u_{1,1,\beta}\cdot\nabla|\nabla u_{1,1,\beta}|^2=-1024(1+\beta)^3\frac{(1+2\beta)r^{6\beta+2}-r^{8\beta+4}}{(1+r^{2(\beta+1)})^4}.$$

Hence

$$(1+r^2)^{2-\frac{\delta}{2}}\left|(\Delta u_{1,1,\beta})^2+\Delta(|\nabla u_{1,1,\beta}|^2)+2\nabla u_{1,1,\beta}\cdot\nabla(\Delta u_{1,1,\beta})\right|\leqslant c(1+r^2)^{-\frac{\delta}{2}},$$

$$(1+r^2)^{2-\frac{\delta}{2}}\left|\Delta u_{1,1,\beta}\,|\nabla u_{1,1,\beta}|^2+\nabla u_{1,1,\beta}\cdot\nabla(|\nabla u_{1,1,\beta}|^2)\right|\leqslant c(1+r^2)^{-\frac{\delta}{2}}$$

and

$$(1+r^2)^{2-\frac{\delta}{2}}\,|\nabla u_{1,1,\beta}|^4\leqslant c(1+r^2)^{-\frac{\delta}{2}},$$

then

$$\sup_{r\leqslant R_{\varepsilon,\lambda,\beta}}(1+r^2)^{2-\frac{\delta}{2}}\mathscr{Q}_\lambda(u_{1,1,\beta})\leqslant c\,\lambda.$$

Besides,

$$\sup_{r\leqslant R_{\varepsilon,\lambda,\beta}}(1+r^2)^{2-\frac{\delta}{2}}\frac{D_\beta|x|^{2(\beta-1)}}{(1+|x|^{2(\beta+1)})^4}(|x|^{4(\beta+1)}+1)\leqslant C\beta. \tag{29}$$

This implies that given $\kappa>0$, there exists $c_\kappa>0$ (which can depend only on κ) such that for $\delta\in(0,1)$ and $|x|=r$, we have

$$\sup_{r\leqslant R_{\varepsilon,\lambda,\beta}}(1+r^2)^{2-\frac{\delta}{2}}\,|\mathscr{R}(0)|\quad\leqslant c_\kappa(\beta+\lambda).$$

Therefore,

$$\|\mathscr{N}(0)\|_{\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)}\leqslant c_\kappa r^2_{\varepsilon,\lambda,\beta}. \tag{30}$$

Making use of Proposition 1 together with (26), we deduce that

$$\|h\|_{\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)}\leqslant 2c_\kappa\, r^2_{\varepsilon,\lambda,\beta}. \tag{31}$$

Now let h_1,h_2 in $B(0,2c_\kappa\, r^2_{\varepsilon,\lambda,\beta})$ of $\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)$ and for $\delta\in(0,1)$, then

$$|\mathscr{R}(h_2)-\mathscr{R}(h_1)|\leqslant\frac{C_\beta|x|^{4\beta}}{(1+|x|^{2(\beta+1)})^4}\left|e^{h_2}-e^{h_1}+h_1-h_2\right|$$
$$+|\mathscr{Q}_\lambda(u_{1,1,\beta}+h_2)-\mathscr{Q}_\lambda(u_{1,1,\beta}+h_1)|+\;V_\beta(x)\,|h_2-h_1|\,.$$

Furthermore,

- $$\sup_{r\leqslant R_{\varepsilon,\lambda,\beta}}r^{4-\delta}\,\frac{C_\beta|x|^{4\beta}}{(1+|x|^{2(\beta+1)})^4}\left|e^{h_2}-e^{h_1}+h_1-h_2\right|$$
$$\leqslant c\sup_{r\leqslant R_{\varepsilon,\lambda,\beta}}r^{-4-\delta-4\beta}|h_2-h_1||h_2+h_1|$$

$$\leqslant c_\kappa \sup_{r\leqslant R_{\varepsilon,\lambda,\beta}} r^2_{\varepsilon,\lambda,\beta} \|h_2 - h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\ \delta}(\mathbb{R}^4)}.$$

- $r^{4-\delta} \left|(\Delta(u_{1,1,\beta}+h_1))^2 - (\Delta(u_{1,1,\beta}+h_2))^2\right|$

$$= r^{4-\delta} \left|(\Delta(h_1-h_2))(\Delta(2u_{1,1,\beta}+h_1+h_2))\right|$$
$$\leqslant c_\kappa \left(1 + r^\delta\, r^2_{\varepsilon,\lambda,\beta}\right) \|h_2 - h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\ \delta}(\mathbb{R}^4)}.$$

- $r^{4-\delta} \left|\Delta|\nabla(u_{1,1,\beta}+h_2)|^2 - \Delta|\nabla(u_{1,1,\beta}+h_1)|^2\right|$

$$= r^{4-\delta} \left|\Delta(\nabla(h_1-h_2).\nabla(2u_{1,1,\beta}+h_1+h_2))\right|$$
$$\leqslant c_\kappa \left(1 + r^\delta\, r^2_{\varepsilon,\lambda,\beta}\right) \|h_2 - h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\ \delta}(\mathbb{R}^4)}.$$

- $r^{4-\delta}\ |\nabla(\Delta(u_{1,1,\beta}+h_2)).\nabla(u_{1,1,\beta}+h_2) - \nabla(\Delta(u_{1,1,\beta}+h_1)).\nabla(u_{1,1,\beta}+h_1)|$

$$= r^{4-\delta} \big|\nabla(\Delta(h_1-h_2)).\nabla(2u_{1,1,\beta}+h_1+h_2)$$
$$+ \nabla(h_2-h_1).\nabla(\Delta(2u_{1,1,\beta}+h_1+h_2))\big|$$
$$r^{4-\delta}\ |\nabla(\Delta(u_{1,1,\beta}+h_2)).\nabla(u_{1,1,\beta}+h_2) - \nabla(\Delta(u_{1,1,\beta}+h_1)).\nabla(u_{1,1,\beta}+h_1)|$$
$$\leqslant c_\kappa \left(1 + r^\delta\, r^2_{\varepsilon,\lambda,\beta}\right) \|h_2 - h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\ \delta}(\mathbb{R}^4)}.$$

- Since

$$|\nabla(u_{1,1,\beta}+h_1)|^2\Delta(u_{1,1,\beta}+h_1) - |\nabla(u_{1,1,\beta}+h_2)|^2\Delta(u_{1,1,\beta}+h_2)$$
$$= \Delta(h_1-h_2)\left[|\nabla(u_{1,1,\beta}+h_1)|^2 + |\nabla(u_{1,1,\beta}+h_2)|^2\right]$$
$$+ \Delta(2u_{1,1,\beta}+h_1+h_2)\left[|\nabla(u_{1,1,\beta}+h_1)|^2 - |\nabla(u_{1,1,\beta}+h_2)|^2\right],$$

then

$$r^{4-\delta}\Big|\,|\nabla(u_{1,1,\beta}+h_1)|^2\Delta(u_{1,1,\beta}+h_1) - |\nabla(u_{1,1,\beta}+h_2)|^2\Delta(u_{1,1,\beta}+h_2)\Big|$$
$$\leqslant c_\kappa \left(1 + r^\delta\, r^2_{\varepsilon,\lambda,\beta} + r^{2\delta}\, r^4_{\varepsilon,\lambda,\beta}\right) \|h_2 - h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\ \delta}(\mathbb{R}^4)}.$$

- Its easy to see that

$$\nabla(|\nabla(u_{1,1,\beta}+h_2)|^2)\nabla(u_{1,1,\beta}+h_2) - \nabla(|\nabla(u_{1,1,\beta}+h_1)|^2)\nabla(u_{1,1,\beta}+h_1)$$
$$= \nabla(h_2-h_1)\nabla\left(|\nabla(u_{1,1,\beta}+h_2)|^2 + |\nabla(u_{1,1,\beta}+h_1)|^2\right)$$
$$+ \nabla(2u_{1,1,\beta}+h_1+h_2)\nabla\left(|\nabla(u_{1,1,\beta}+h_2)|^2 - |\nabla(u_{1,1,\beta}+h_1)|^2\right),$$

hence

$$r^{4-\delta}\Big|\nabla(|\nabla(u_{1,1,\beta}+h_2)|^2)\nabla(u_{1,1,\beta}+h_2) - \nabla(|\nabla(u_{1,1,\beta}+h_1)|^2)\nabla(u_{1,1,\beta}+h_1)\Big|$$
$$\leqslant c_\kappa \left(1 + r^\delta\, r^2_{\varepsilon,\lambda,\beta} + r^{2\delta}\, r^4_{\varepsilon,\lambda,\beta}\right) \|h_2 - h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\ \delta}(\mathbb{R}^4)}.$$

• Finally, since

$$|\nabla(u_{1,1,\beta}+h_2)|^4-|\nabla(u_{1,1,\beta}+h_1)|^4 \\ =\nabla(h_2-h_1)\nabla(2u_{1,1,\beta}+h_2+h_1)\left(|\nabla(u_{1,1,\beta}+h_2)|^2+|\nabla(u_{1,1,\beta}+h_1)|^2\right),$$

then

$$r^{4-\delta}\Big||\nabla(u_{1,1,\beta}+h_2)|^4-|\nabla(u_{1,1,\beta}+h_1)|^4\Big| \\ \leqslant c_\kappa\left(1+r^{\delta}\,r^2_{\varepsilon,\lambda,\beta}+r^{2\delta}\,r^4_{\varepsilon,\lambda,\beta}+r^{3\delta}\,r^6_{\varepsilon,\lambda,\beta}\right)\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)}.$$

Which gives

(32)
$$\sup_{r\leqslant R_{\varepsilon,\lambda,\beta}} r^{4-\delta}|\mathscr{Q}_\lambda(u_{1,1,\beta}+h_2)-\mathscr{Q}_\lambda(u_{1,1,\beta}+h_1)|\leqslant c_\kappa r^2_{\varepsilon,\lambda,\beta}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)}.$$

Besides

$$\sup_{r\leqslant R_{\varepsilon,\lambda,\beta}} r^{4-\delta}\left(\frac{C_\beta|x|^{4\beta}}{(1+|x|^{2(\beta+1)})^4}\Big|e^{h_2}-e^{h_1}+h_1-h_2\Big|+V_\beta(x)\,|h_2-h_1|\right) \\ \leqslant c_\kappa r^2_{\varepsilon,\lambda,\beta}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)}. \tag{33}$$

Thanks to the conditions (A_β) and (A_λ), we deduce that

$$\sup_{r\leqslant R_{\varepsilon,\lambda,\beta}} r^{4-\delta}\,|\mathscr{R}(h_2)-\mathscr{R}(h_1)|\leqslant c_\kappa r^2_{\varepsilon,\lambda,\beta}\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)}.$$

Similarly, making use of Proposition 1 together with (26), we conclude that given $\kappa>0$, there exist $\bar{c}_\kappa>0$ (independent of ε and λ), λ_κ and ε_κ such that

$$\|\mathscr{N}(h_2)-\mathscr{N}(h_1)\|_{\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)}\leqslant \bar{c}_\kappa\,r^2_{\varepsilon,\lambda,\beta}\,\|h_2-h_1\|_{\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)}. \tag{34}$$

Reducing $\lambda_\kappa>0$ and $\varepsilon_\kappa>0$ if necessary, we can assume that, $\bar{c}_\kappa\,r^2_{\varepsilon,\lambda,\beta}\leqslant\frac{1}{2}$ for all $\lambda\in(0,\lambda_\kappa)$ and $\varepsilon\in(0,\varepsilon_\kappa)$. Then (34) and (31) are enough to show that $h\longmapsto\mathscr{N}(h)$ is a contraction from $\{h\in\mathcal{C}^{4,\alpha}_{rad,\delta}(\mathbb{R}^4):\|h\|_{\mathcal{C}^{4,\alpha}_{rad,\delta}(\mathbb{R}^4)}\leqslant 2\,c_\kappa\,r^2_{\varepsilon,\lambda,\beta}\}$ into itself and hence has a unique fixed point h in this set. This fixed point is solution of (27) in $\bar{B}_{R_{\varepsilon,\lambda,\beta}}$. We summarize this in the following proposition.

PROPOSITION 2. *Given $\delta\in(0,1)$ and $\kappa>0$, there exist $\varepsilon_\kappa>0$, $\lambda_\kappa>0$ and $\bar{c}_\kappa>0$ (depending on κ) such that for all $\lambda\in(0,\lambda_\kappa)$ and for $\varepsilon\in(0,\varepsilon_\kappa)$, there exists a unique solution $h_\beta\in\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)$ solution of (27) such that*

$$v(x)=u_{1,1,\beta}(x)+h_\beta(x)$$

solves (21) in $\bar{B}_{R_{\varepsilon,\lambda,\beta}}$. In addition

$$\|h_\beta\|_{\mathcal{C}^{4,\alpha}_{rad,\delta}(\mathbb{R}^4)}\leqslant 2c_\kappa r^2_{\varepsilon,\lambda,\beta}.$$

3. A LINEARIZED OPERATOR

We define the linear fourth-order elliptic operator L_β by

$$\mathrm{L}_\beta := \Delta^2 - \frac{C_\beta |x|^{4\beta}}{(1+|x|^{2(\beta+1)})^4},$$

which corresponds to the linearization of $\Delta^2 u - 24|x|^{4\beta}e^u = 0$ about the approximate solution $u_{1,1,\beta}$ defined above. This operator can be written as

$$\mathrm{L}_\beta := L + V_\beta(x),$$

where $V_\beta(x)$ is given by (23) satisfying the inequality (24). Using a perturbation argument one obtains the following.

PROPOSITION 3. *There exists $\beta_0 > 0$ such that for all $0 < \beta < \beta_0$ and for all $\delta > 0$, $\delta \notin \mathbb{N}$,*

$$\begin{array}{cccc} \mathrm{L}_\beta : & C^{4,\alpha}_{rad,\delta}(\mathbb{R}^4) & \longrightarrow & C^{0,\alpha}_{rad,\delta-4}(\mathbb{R}^4) \\ & w & \longmapsto & \mathrm{L}_\beta\, w \end{array}$$

is surjective. Moreover, if we denote by $\mathcal{G}_{\delta,\beta}$ a right inverse of L_β we have that

$$||\mathcal{G}_{\delta,\beta}\Phi - \mathscr{G}_\delta \Phi||_{C^{4,\alpha}_{\delta}(\mathbb{R}^4)} \le c_\kappa \beta ||\Phi||_{C^{0,\alpha}_{\delta-4}(\mathbb{R}^4)},$$

for every $\Phi \in C^{0,\alpha}_{rad,\delta-4}(\mathbb{R}^4)$.

We define $\bar{B}_1^* := \bar{B}_1 - \{0\}$. With this notation, we have the following.

Definition 3. Given $k \in \mathbb{R}, \alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we introduce the Hölder weighted space $C^{k,\alpha}_\nu(\bar{B}_1^*)$ as the space of functions $w \in C^{k,\alpha}_{loc}(\bar{B}_1^*)$ such that the norm

$$\|w\|_{C^{k,\alpha}_\nu(\bar{B}_1^*)} := \sup_{r\in(0,1)} \left(r^{-\nu} \|w(r\cdot)\|_{C^{k,\alpha}(\bar{B}_1 - B_{1/2})} \right)$$

is finite.

When $k \ge 2$, we denote by $[C^{k,\alpha}_\nu(\bar{B}_1^*)]_0$ the subspace of functions $w \in C^{k,\alpha}_\nu(\bar{B}_1^*)$ satisfying $w = \Delta w = 0$ on ∂B_1^*. We recall the analysis of the Bi-Laplace operator in weighted spaces performed in [3].

PROPOSITION 4 ([3]). *Assume that $\nu < 0$ and $\nu \notin \mathrm{Z}$, then*

$$\begin{array}{rcl} \Delta^2 \ : [C^{4,\alpha}_\nu(\bar{B}_1^*)]_0 & \to & C^{0,\alpha}_{\nu-4}(\bar{B}_1^*) \\ w & \mapsto & \Delta^2 w \end{array}$$

is surjective. Denote by $\tilde{\mathcal{G}}_\nu$ a right inverse of Δ^2.

Finally, we study the properties of interior and exterior Bi-harmonic extensions. Indeed, for a given real number γ, we define in B_1 the Bi-harmonic function $H^i_\gamma(x) = \gamma|x|^2$. This function satisfies $H^i_\gamma = \gamma$ on ∂B_1 and $\Delta H^i_\gamma = 8\gamma$ on ∂B_1. Similarly, for a given real number $\tilde{\gamma}$, we define in $\mathbb{R}^4 - B_1$ the Bi-harmonic function $H^e_{\tilde{\gamma}}(x) = \tilde{\gamma}|x|^{-2}$. This function satisfies $H^e_{\tilde{\gamma}} = \tilde{\gamma}$ on ∂B_1 and $\Delta H^e_{\tilde{\gamma}} = 0$ on ∂B_1.

4. THE NONLINEAR INTERIOR PROBLEM

We are interested in studying equations of the type

(35) $$\Delta^2 w + \mathscr{Q}_\lambda(w) - 24|x|^{4\beta} f\left((\varepsilon/\tau)^{1/(\beta+1)}|x|\right) e^w = 0$$

in $\bar{B}_{R_{\varepsilon,\lambda,\beta}}$.
Given a real number γ, we define

$$\mathbf{v} := u_{1,1,\beta} - \log(f(0)) + H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta}) + h_\beta,$$

then we look for a solution of (35) of the form $w = \mathbf{v} + v$ and using the fact that H^i_γ is biharmonic, this amounts to solve

(36)
$$\begin{aligned}
\mathrm{L}_\beta\, v = & \frac{C_\beta|x|^{4\beta}}{(1+|x|^{2(1+\beta)})^4} e^{H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})+h_\beta+v}\left(\frac{f\left((\varepsilon/\tau)^{1/(\beta+1)}|\cdot|\right)}{f(0)} - 1\right)\\
&+ \frac{C_\beta|x|^{4\beta}}{(1+|x|^{2(1+\beta)})^4} e^{h_\beta}(e^{H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})+v} - v - 1)\\
&+ \frac{C_\beta|x|^{4\beta}}{(1+|x|^{2(1+\beta)})^4}(e^{h_\beta} - 1)v\\
&+ \mathscr{Q}_\lambda\Big(u_{1,1,\beta} + h_\beta\Big) - \mathscr{Q}_\lambda\Big(u_{1,1,\beta} - \log(f(0)) + H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta}) + h_\beta + v\Big),
\end{aligned}$$

where $C_\beta = 64(4\beta^2 + 8\beta + 6)(\beta+1)^2$.

We fix

$$\delta \in (0,1).$$

By Proposition 3, to obtain a solution of (38) it is sufficient to find $v \in \mathcal{C}^{4,\alpha}_{rad,\delta}(\mathbb{R}^4)$, a solution of

(37) $$v = \mathcal{G}_{\delta,\beta} \circ \mathcal{E}_{R_{\varepsilon,\lambda,\beta}} \circ \mathscr{S}(v),$$

where

(38)
$$\begin{aligned}
\mathscr{S}(v) = & \frac{C_\beta|x|^{4\beta}}{(1+|x|^{2(1+\beta)})^4} e^{H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})+h_\beta+v}\left(\frac{f\left((\varepsilon/\tau)^{1/(\beta+1)}|\cdot|\right)}{f(0)} - 1\right)\\
&+ \frac{C_\beta|x|^{4\beta}}{(1+|x|^{2(1+\beta)})^4} e^{h_\beta}(e^{H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})+v} - v - 1) + \frac{C_\beta|x|^{4\beta}}{(1+|x|^{2(1+\beta)})^4}(e^{h_\beta} - 1)v\\
&+ \mathscr{Q}_\lambda\Big(u_{1,1,\beta} + h_\beta\Big) - \mathscr{Q}_\lambda\Big(u_{1,1,\beta} - \log(f(0)) + H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta}) + h_\beta + v\Big).
\end{aligned}$$

We denote by $\mathscr{N}(= \mathscr{N}_{\varepsilon,\lambda,\beta,\gamma})$ the nonlinear operator appearing on the right-hand side of equation (37). Given $\kappa > 0$ (whose value will be fixed later) and taking γ so that

(39) $$|\gamma| \le \kappa r^2_{\varepsilon,\lambda,\beta},$$

we have the following result.

LEMMA 1. *Given $\delta \in (0,1)$ and $\kappa > 0$, then there exist $\lambda_\kappa > 0$, $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\bar{c}_\kappa > 0$ (depending on κ) such that for all $\lambda \in (0, \lambda_\kappa)$ and $\varepsilon \in (0, \varepsilon_\kappa)$*

$$\|\mathscr{N}(0)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leqslant c_\kappa\, r^2_{\varepsilon,\lambda,\beta}. \tag{40}$$

Moreover,

$$\|\mathscr{N}(v_2) - \mathscr{N}(v_1)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leqslant \bar{c}_\kappa\, r^2_{\varepsilon,\lambda,\beta}\, \|v_2 - v_1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}, \tag{41}$$

provided that $v_1, v_2 \in \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$, satisfy $\|v_i\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leqslant 2\, c_\kappa\, r^2_{\varepsilon,\lambda,\beta}$, for $i = 1, 2$.

Proof. The proof of the first estimate follows from the asymptotic behavior of H^i_γ. Indeed, letting c_κ be a constant depending only on κ (provided ε is chosen small enough) it follows from the expression of H^i_γ that

$$\|H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})\|_{C^{4,\alpha}_2(\bar{B}_{R_{\varepsilon,\lambda,\beta}})} \le c_\kappa R^{-2}_{\varepsilon,\lambda,\beta}|\gamma| \le c_\kappa \varepsilon^{2/(\beta+1)} \le c_\kappa r^2_{\varepsilon,\lambda,\beta}.$$

Let $\beta_0 > 0$, then for $\beta \in (0,\ \beta_0)$ and for $|x| \le R_{\varepsilon,\lambda,\beta}/2$, we have

$$|h_\beta(x)| \le r^{2+\delta}_{\varepsilon,\lambda,\beta}\varepsilon^{-\frac{\delta}{\beta+1}} \le \begin{cases} \lambda^{1+\frac{\delta}{2}}\varepsilon^{-\frac{\delta}{\beta+1}} & \longrightarrow 0 \quad \text{as } \varepsilon \text{ tends to } 0 \text{ using } A_\lambda \\ \beta^{1+\frac{\delta}{2}}\varepsilon^{-\frac{\delta}{\beta+1}} & \longrightarrow 0 \quad \text{as } \varepsilon \text{ tends to } 0 \text{ using } A_\beta \\ \varepsilon^{\frac{2}{\beta+1}} & \longrightarrow 0 \quad \text{as } \varepsilon \text{ tends to } 0, \end{cases}$$

provided ε is small enough, we then get

$$\left\|(1+|\cdot|^{2(\beta+1)})^{-4}|\cdot|^{4\beta}e^{h_\beta}(e^{H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})}-1)\right\|_{C^{0,\alpha}_{\delta-4}(\bar{B}_{R_{\varepsilon,\lambda,\beta}})} \le c_\kappa r^2_{\varepsilon,\lambda,\beta}$$

and

$$\left\|(1+|\cdot|^{2(\beta+1)})^{-4}|\cdot|^{4\beta}e^{H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})+h_\beta}\Big(\frac{f((\varepsilon/\tau)^{1/(\beta+1)}\cdot)}{f(0)}-1\Big)\right\|_{C^{0,\alpha}_{\delta-4}(\bar{B}_{R_{\varepsilon,\lambda,\beta}})} \tag{42}$$

$$\le c_\kappa \varepsilon^{1/(\beta+1)} \le c_\kappa r^2_{\varepsilon,\lambda,\beta}.$$

On the other hand, using the conditions (A_λ) and (A_β), we get also

$$\sup_{r\leqslant R_{\varepsilon,\lambda,\beta}} (1+r^2)^{2-\frac{\delta}{2}}\Big|\mathscr{Q}_\lambda\Big(u_{1,1,\beta}+h_\beta\Big) - \mathscr{Q}_\lambda\Big(u_{1,1,\beta}-\log(f(0))+H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})+h_\beta\Big)\Big| \leqslant c_\kappa r^2_{\varepsilon,\lambda,\beta}.$$

Making use of Proposition 1 together with (26), we get for $\delta \in (0,\ 1)$

$$\|\mathscr{N}(0)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leqslant c_\kappa\, r^2_{\varepsilon,\lambda,\beta}.$$

To derive the second estimate, let $v_i \in \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ satisfy $\|v_i\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leqslant 2\, c_\kappa\, r^2_{\varepsilon,\lambda,\beta}$, $i=1,2$, we have that

$$\Big\|(1+|\cdot|^{2(\beta+1)})^{-4}|\cdot|^{4\beta} e^{H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})+h_\beta} \times \Big(\frac{f((\varepsilon/\tau)^{1/(\beta+1)}\cdot)}{f(0)} - 1\Big)\,(e^{v_2}-e^{v_1})\,\Big\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(\bar B_{R_{\varepsilon,\lambda,\beta}})} \leqslant c_\kappa\, \varepsilon^{1/\beta+1}\, \|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)},$$

$$\Big\|(1+|\cdot|^{2(\beta+1)})^{-4}|\cdot|^{4\beta} e^{h_\beta}\Big(e^{H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})+v_1} - e^{H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta})+v_2} + (v_2-v_1)\Big)\Big\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(\bar B_{R_{\varepsilon,\lambda,\beta}})} \leqslant c_\kappa\, r^2_{\varepsilon,\lambda,\beta}\, \|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)},$$

$$\Big\|(1+|\cdot|^{2(\beta+1)})^{-4}|\cdot|^{4\beta}(e^{h_\beta}-1)\,(v_2-v_1)\Big\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(\bar B_{R_{\varepsilon,\lambda,\beta}})} \leqslant c_\kappa\, \beta\, \|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}$$

and

$$\Big\|\mathscr{Q}_\lambda\Big(u_{1,1,\beta} - \log(f(0)) + H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta}) + h_\beta + v_1\Big) - \mathscr{Q}_\lambda\Big(u_{1,1,\beta} - \log(f(0)) + H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta}) + h_\beta + v_2\Big)\Big\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(\bar B_{R_{\varepsilon,\lambda,\beta}})}$$

$$\leqslant c_\kappa\, r^2_{\varepsilon,\lambda,\beta}\, \Big\|v_2-v_1\Big\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}.$$

So,

$$\sup_{r\leqslant R_{\varepsilon,\lambda,\beta}} (1+r^2)^{2-\frac{\delta}{2}}\, |\mathscr{S}(v_2)-\mathscr{S}(v_1)| \leqslant c_\kappa r^2_{\varepsilon,\lambda,\beta}\|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_{rad,\,\delta}(\mathbb{R}^4)}.$$

Similarly, making use of Proposition 1 together with (26), we conclude that there exists $\bar c_\kappa > 0$ such that

$$\|\mathscr{N}(v_2)-\mathscr{N}(v_1)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leqslant \bar c_\kappa\, r^2_{\varepsilon,\lambda,\beta}\, \|v_2-v_1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}.$$

□

Reducing $\lambda_\kappa > 0$ and $\varepsilon_\kappa > 0$, if necessary, we can assume that $\bar c_\kappa\, r^2_{\varepsilon,\lambda,\beta} \leqslant \frac{1}{2}$, for all $\lambda \in (0,\lambda_\kappa)$ and $\varepsilon \in (0,\varepsilon_\kappa)$. Then (40) and (41) in Lemma 1 are enough to show that $v \longmapsto \mathscr{N}(v)$ is a contraction from

$$\Big\{v \in \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)\ :\ \|v\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \leqslant 2\, c_\kappa\, r^2_{\varepsilon,\lambda,\beta}\Big\}$$

into itself and hence has a unique fixed point $v = v(\varepsilon,\tau,\gamma,\cdot)$ in this set. This fixed point is a solution of (37) in $\mathbb{R}^4$. We summarize this in the following proposition.

Proposition 5. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$ (depending on κ) and $\beta_0 > 0$ such that for all $\varepsilon \in (0,\ \varepsilon_\kappa)$, for all $0 < \beta < \beta_0$ and for all τ in some fixed compact subset $[\tau_-,\ \tau_+] \subset (0,\ \infty)$, there exists a unique $v_\beta(= v_\beta(\varepsilon, \tau, \gamma, \cdot))$ solution of (37) such that*

$$\|v_\beta\|_{C^{4,\alpha}_\delta(\mathbb{R}^4)} \leq 2c_\kappa r^2_{\varepsilon,\lambda,\beta}.$$

As a conclusion,

$$\mathbf{v} + v_\beta(\varepsilon,\ \tau,\ \gamma, \cdot) = u_{1,1,\beta} + h_\beta - \log(f(0)) + H^i_\gamma(\cdot/R_{\varepsilon,\lambda,\beta}) + v_\beta(\varepsilon,\ \tau,\ \gamma,\ \cdot) \tag{43}$$

solves (35) in $\bar{B}_{R_{\varepsilon,\lambda,\beta}}$. Since the function v_β is being obtained as a fixed point for a contraction mapping, it depends smoothly on the parameter τ. Moreover, we claim that the mapping $\tau \to v_\beta(\varepsilon,\ \tau,\ \gamma,\ \cdot)|_{\bar{B}_{R_{\varepsilon,\lambda,\beta}}} \in C^{4,\alpha}(\bar{B}_{R_{\varepsilon,\lambda,\beta}})$ is compact. This follows from the fact that the equation we solve is semilinear and in (37) the right-hand side belongs to $C^{8,\alpha}(\bar{B}_{R_{\varepsilon,\lambda,\beta}})$.

5. THE NONLINEAR EXTERIOR PROBLEM

Let $\theta \in \mathbb{R}$ and $\tilde{\gamma} \in \mathbb{R}$ be close to 0. We define

$$\tilde{\mathbf{v}}(x) = (1 + \beta + \theta)G(x) + \chi(x)H^e_{\tilde{\gamma}}(x/r_{\varepsilon,\lambda,\beta}),$$

where χ is a cutoff function identically equal to 1 in $B_{1/4}$ and identically equal to 0 outside $B_{1/2}$. We would like to find a solution of the equation

$$\Delta^2 v + \mathscr{Q}_\lambda(v) - \rho^4|x|^{4\beta}f(|x|)e^v = 0, \tag{44}$$

in $\bar{B}_1 - B_{r_{\varepsilon,\lambda,\beta}}$ which is a perturbation of $\tilde{\mathbf{v}}$. Writing $v = \tilde{\mathbf{v}} + \tilde{v}$, this amounts to solving

$$\Delta^2 \tilde{v} = \rho^4|x|^{4\beta}f(|x|)e^{\tilde{\mathbf{v}}}e^{\tilde{v}} - \mathscr{Q}_\lambda(\tilde{\mathbf{v}} + \tilde{v}) - \Delta^2\tilde{\mathbf{v}}. \tag{45}$$

We need to define auxiliary weighted spaces.

Definition 4. Given $\bar{r} \in (0, 1/2), k \in \mathbb{R}$ and $\nu \in \mathbb{R}$, we define the Hölder weighted space $C^{k,\alpha}_\nu(\bar{B}_1 - B_{\bar{r}})$ as the space of functions $w \in C^{k,\alpha}(\bar{B}_1 - B_{\bar{r}})$ endowed with the norm

$$\|w\|_{C^{k,\alpha}_\nu(\bar{B}_1 - B_{\bar{r}})} = \|w\|_{C^{k,\alpha}(\bar{B}_1 - B_{1/2})} + \sup_{\bar{r} \leq r < 1/2} r^{-\nu}\|w(r.)\|_{C^{k,\alpha}(\bar{B}_1 - B_{1/2})}.$$

For $\sigma \in (0, 1/2)$, we denote by

$$\tilde{\xi}_\sigma : C^{0,\alpha}_\nu(\bar{B}_1 - B_\sigma) \to C^{0,\alpha}_\nu(\bar{B}^*_1)$$

the extension operator defined by $\tilde{\xi}_\sigma(f) = f$ in $\bar{B}_1 - B_\sigma$,

$$\tilde{\xi}_\sigma(f)(x) = \tilde{\chi}(\frac{|x|}{\sigma}) f(\sigma \frac{x}{|x|}) \text{ in } B_\sigma - B_{\sigma/2}$$

and $\tilde{\xi}_\sigma(f) = 0$ in $B_{\sigma/2}$, where $t \mapsto \tilde{\chi}(t)$ is a cutoff function identically equal to 1 for $t \geq 1$ and identically equal to 0 for $t \leq 1/2$. It is easy to check that there exists a constant $c = c(\nu) > 0$ only depending on ν such that

$$\|\tilde{\xi}_\sigma(w)\|_{C^{0,\alpha}_\nu(\bar{B}^*_1)} \leq c \|w\|_{C^{0,\alpha}_\nu(\bar{B}_1 - B_\sigma)}. \tag{46}$$

Fix $\nu \in (-1, 0)$. Making use of Proposition 4, for solving equation (45) it suffices to find a solution $\tilde{v} \in C^{4,\alpha}_\nu(\bar{B}^*_1)$ of the following fixed point problem

(47)
$$\tilde{v} = \tilde{\mathcal{G}}_\nu \circ \tilde{\xi}_{r_{\varepsilon,\lambda,\beta}} \left(\rho^4 |x|^{4\beta} f(|x|) e^{\tilde{\mathbf{v}}} e^{\tilde{v}} - \mathscr{Q}_\lambda(\tilde{\mathbf{v}} + \tilde{v}) - \Delta^2 \tilde{\mathbf{v}}\right) = \tilde{\mathcal{G}}_\nu \circ \tilde{\xi}_{r_{\varepsilon,\lambda,\beta}} \circ \tilde{S}(\tilde{v}).$$

We denote by $\tilde{\mathcal{N}} (= \tilde{\mathcal{N}}_{\varepsilon,\lambda,\beta,\theta,\tilde{\gamma}})$ the nonlinear operator appearing on the right hand side of this equation.

Given $\kappa > 0$ (whose value will be fixed later on), suppose that the parameters θ and $\tilde{\gamma}$ satisfy

$$|\theta| \leq \kappa r^2_{\varepsilon,\lambda,\beta} \tag{48}$$

and

$$|\tilde{\gamma}| \leq \kappa r^2_{\varepsilon,\lambda,\beta}. \tag{49}$$

Then the following result holds.

Lemma 2. *Under the above assumptions, there exists a constant $c_\kappa > 0$ such that*

$$\|\tilde{\mathcal{N}}(0)\|_{C^{4,\alpha}_\nu(\bar{B}^*_1)} \leq c_\kappa r^2_{\varepsilon,\lambda,\beta}$$

and

$$\|\tilde{\mathcal{N}}(\tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}_1)\|_{C^{4,\alpha}_\nu(\bar{B}^*_1)} \leq c_\kappa r^2_{\varepsilon,\lambda,\beta} \|\tilde{v}_2 - \tilde{v}_1\|_{C^{4,\alpha}_\nu(\bar{B}^*_1)},$$

provided $\tilde{v}_1, \tilde{v}_2 \in C^{4,\alpha}_\nu(\bar{B}^*_1)$ and satisfy $\|\tilde{v}_i\|_{C^{4,\alpha}_\nu(\bar{B}^*_1)} \leq 2c_\kappa r^2_{\varepsilon,\lambda,\beta}$, for $i = 1, 2$.

Proof. In $B_{1/2} - B_{r_{\varepsilon,\lambda,\beta}}$, we have $\chi = 1$ and $\Delta^2 \tilde{\mathbf{v}} = 0$, thus

$$|\tilde{S}(0)| \leq c_\kappa (\varepsilon^4 r^{-4\beta - 8(1+\theta)} + \lambda).$$

In $\bar{B}_1 - B_{1/2}$, we have $|H^e_{\tilde{\gamma}}(x/r_{\varepsilon,\lambda,\beta})| \leq \kappa r^3_{\varepsilon,\lambda,\beta} r^{-1}$, thus

$$\begin{aligned} |\tilde{S}(0)| &\leq c_\kappa \Big(\varepsilon^4 |x|^{-4\beta - 8(1+\theta)} + |\mathscr{Q}_\lambda(\tilde{\mathbf{v}})| + [\Delta^2,\ \chi(x)] ||H^e_{\tilde{\gamma}}(x/r_{\varepsilon,\lambda,\beta})|\Big) \\ &\leq c_\kappa (\varepsilon^4 + r^{-1} r^3_{\varepsilon,\lambda,\beta} + \lambda). \end{aligned}$$

Here, we use the notation

$$[\Delta^2,\ \chi] w = 2\Delta\chi\Delta w + w\Delta^2\chi + 4\nabla\chi \cdot \nabla(\Delta w) + 4\nabla w \cdot \nabla(\Delta\chi) + 4\nabla^2\chi \cdot \nabla^2 w.$$

It follow that

$$||\tilde{S}(0)||_{C^{0,\alpha}_{\nu-4}(\bar{B}_1 - B_{r_{\varepsilon,\lambda,\beta}})} \leq c_\kappa r^2_{\varepsilon,\lambda,\beta}.$$

Then the proof of the first estimate follows from (46).

For the proof of the second estimate, letting $\tilde{v}_1, \tilde{v}_2 \in C^{4,\alpha}_\nu(\bar{B}^*_1)$ satisfying $\|\tilde{v}_i\|_{C^{4,\alpha}_\nu(\bar{B}^*_1)} \le 2c_\kappa r^2_{\varepsilon,\lambda,\beta}$ for $i = 1, 2$, we have

$$|\tilde{S}(\tilde{v}_2) - \tilde{S}(\tilde{v}_1)| \le c_\kappa \left|\rho^4|x|^{4\beta}|f(|x|)|e^{\tilde{\mathbf{v}}}(e^{\tilde{v}_2} - e^{\tilde{v}_1})| - \big(\mathscr{Q}_\lambda(\tilde{\mathbf{v}}+\tilde{v}_2) - \mathscr{Q}_\lambda(\tilde{\mathbf{v}}+\tilde{v}_1)\big)\right|.$$

This clearly implies

$$|\tilde{S}(\tilde{v}_2) - \tilde{S}(\tilde{v}_1)| \le c_\kappa(\varepsilon^4 r^{-4\beta-8(1+\theta)} + \lambda)|\tilde{v}_2 - \tilde{v}_1|.$$

For $\nu \in (-1, 0)$ and θ small enough, we get

$$\|\tilde{S}(\tilde{v}_2) - \tilde{S}(\tilde{v}_1)\|_{C^{0,\alpha}_{\nu-4}(\bar{B}_1 - B_{r_{\varepsilon,\lambda,\beta}})} \le c_\kappa r^2_{\varepsilon,\lambda,\beta}\|\tilde{v}_2 - \tilde{v}_1\|_{C^{4,\alpha}_\nu(\bar{B}^*_1)}.$$

Using also equation (46) we obtain the second estimate. □

Applying a fixed point theorem for contraction mappings we obtain the following result.

Proposition 6. *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$ and $\beta_0 > 0$ such that for all $\varepsilon \in (0,\ \varepsilon_\kappa)$, for all $\beta \in (0,\ \beta_0)$, for θ satisfying (48) and a boundary constant $\tilde{\gamma}$ satisfying (49), there exists a unique solution $\tilde{v}_\beta (= \tilde{v}_\beta(\varepsilon,\ \tau, \tilde{\gamma}, .))$ of* (47) *such that*

$$\|\tilde{v}_\beta\|_{C^{4,\alpha}_\nu(\bar{B}^*_1)} \le 2c_\kappa r^2_{\varepsilon,\lambda,\beta}.$$

As in the previous section, since the function $\tilde{v}_\beta$ is being obtained as a fixed point for a contraction mapping, it depends smoothly on the parameter θ. Again this follows from the fact that the equation we solve is semilinear and in (47) the right-hand side belongs to $C^{8,\alpha}(\bar{B}^*_1)$.

6. THE NONLINEAR CAUCHY-DATA MATCHING

We gather the results of the previous sections, keeping the notation and applying the result of Section 4 as well as the results of Section 5.

Assume that $\tau \in [\tau_-,\ \tau_+] \subset (0,\ \infty)$ is given (the values of τ- and τ_+ will be fixed later) and consider some set of boundary data γ satisfying (39). Given $\kappa > 0$, according to the result of Proposition 5, there exist $\varepsilon_\kappa > 0$ such that, provided $\varepsilon \in (0,\ \varepsilon_\kappa)$, we can find in $B_{r_{\varepsilon,\lambda,\beta}}$ a solution of

$$\Delta^2 v + \mathscr{Q}_\lambda(v) - \rho^4|x|^{4\beta} f(|x|)e^v = 0, \tag{50}$$

which can be decomposed, by (20), as

$$v_{int}(x) = v_{\varepsilon,\tau,\beta}(x) + h_\beta(R_{\varepsilon,\lambda,\beta}x/r_{\varepsilon,\lambda,\beta}) - \log(f(0))$$
$$+H^i_\gamma(x/r_{\varepsilon,\lambda,\beta}) + v_\beta(\varepsilon, \tau, \gamma,\ R_{\varepsilon,\lambda,\beta}x/r_{\varepsilon,\lambda,\beta}),$$

where the function $v_\beta(= v_\beta(\varepsilon,\tau,\gamma,\ \cdot)) \in C^{4,\alpha}_{rad,\mu}(\mathbb{R}^4)$ satisfies

$$\|v_\beta\|_{C^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r^2_{\varepsilon,\lambda,\beta}. \tag{51}$$

Similarly, given any constant boundary data $\tilde\gamma$ satisfying (49) and a parameter θ in $\mathbb{R}$ satisfying (48), we can use the result of Proposition 6 to find a solution v_{ext} in $\bar B_1 - B_{r_{\varepsilon,\lambda,\beta}}$ (provided $\varepsilon \in (0,\ \varepsilon_k)$), of (50) which can be decomposed as

$$v_{ext}(x) = (1+\beta+\theta)G(x) + \chi(x)H^e_{\tilde\gamma}(x/r_{\varepsilon,\lambda,\beta}) + \tilde v_\beta(\varepsilon,\ \tau,\tilde\gamma,\ x),$$

where the function $\tilde v_\beta(= \tilde v_\beta(\varepsilon,\ \tau,\tilde\gamma,\ \cdot)) \in C^{4,\alpha}_\nu(\bar B_1^*)$ satisfies

$$\|\tilde v_\beta\|_{C^{4,\alpha}_\nu(\bar B_1^*)} \le 2c_\kappa r^2_{\varepsilon,\lambda,\beta}. \tag{52}$$

It remains to choose the parameters $\gamma, \tilde\gamma, \theta$ and τ in such a way that the function which is equal to v_{int} in $B_{r_{\varepsilon,\lambda,\beta}}$ and v_{ext} in $\bar B_1 - B_{r_{\varepsilon,\lambda,\beta}}$ is a smooth function. This amounts to finding these parameters so that

$$v_{int} = v_{ext},\ \partial_r v_{int} = \partial_r v_{ext},\ \Delta v_{int} = \Delta v_{ext} \quad \text{and} \quad \partial_r \Delta v_{int} = \partial_r \Delta v_{ext}, \tag{53}$$

near $\partial B_{r_{\varepsilon,\lambda,\beta}}$.

Assuming we have already done so, this provides for each ε and β small enough a function $v_{\varepsilon,\lambda,\beta} \in C^{4,\alpha}(\bar B_1)$ (which is obtained by patching together the functions v_{int} and v_{ext}) which is a solution of our equation, and elliptic regularity theory implies that this solution is in fact smooth. This will complete the proof of our result since, as ε tends to 0, the sequence of solutions we have obtained satisfies the required properties, namely, away from the 0 the sequence $v_{\varepsilon,\lambda,\beta}$ converges to G.

Before we proceed, the following remarks are due. First, it will be convenient to notice that the function $v_{\varepsilon,\tau,\beta}$ can be expanded as

$$v_{\varepsilon,\tau,\beta}(x) \ = \ -4\log\tau - 8(1+\beta)\log|x| + \mathcal{O}\left(\frac{\varepsilon^2\tau^{-2}}{|x|^{2(\beta+1)}}\right) \tag{54}$$

near $\partial B_{r_{\varepsilon,\lambda,\beta}}$. Similarly, we can write the function $(1+\beta+\theta)G(x)$ (which appear in the expression of v_{ext}) as

$$\begin{aligned}(1+\beta+\theta)G(x) &= -8(1+\beta+\theta)\log|x| + (1+\beta+\theta)H(x)\\ &= -8(1+\beta+\theta)\log|x| + H(0) + \mathcal{O}(r^2_{\varepsilon,\lambda,\beta})\end{aligned} \tag{55}$$

near $\partial B_{r_{\varepsilon,\lambda,\beta}}$. Then one gets

$$\begin{aligned}(v_{int} - v_{ext})(x) = \quad & -4\log\tau + 8\theta\log|x| + H^i_\gamma(x/r_{\varepsilon,\lambda,\beta})\\ & -H^e_{\tilde\gamma}(x/r_{\varepsilon,\lambda,\beta}) - H(0) - \log(f(0)) + \mathcal{O}(r^2_{\varepsilon,\lambda,\beta}).\end{aligned} \tag{56}$$

It will be convenient to solve instead of (53) the following set of equations

$$\begin{aligned}&(v_{int} - v_{ext})(r_{\varepsilon,\lambda,\beta}\cdot) = 0, && \Delta(v_{int} - v_{ext})(r_{\varepsilon,\lambda,\beta}\cdot) = 0,\\ &\partial_r(v_{int} - v_{ext})(r_{\varepsilon,\lambda,\beta}\cdot) = 0 \quad \text{and} && \partial_r\Delta(v_{int} - v_{ext})(r_{\varepsilon,\lambda,\beta}\cdot) = 0,\end{aligned} \tag{57}$$

on S^3.

Here we assume that our functions are defined on S^3 using simply the change of variables $x = r_{\varepsilon,\lambda,\beta} y$ to parameterize $\partial B_{r_{\varepsilon,\lambda,\beta}}$. Then the set of equations (57) yields the system

$$
\begin{cases}
-4\log\tau - H(0) - \log(f(0)) + \gamma - \tilde{\gamma} + 8\theta\log r_{\varepsilon,\lambda,\beta} + \mathcal{O}(r^2_{\varepsilon,\lambda,\beta}) & = 0 \\
8\theta + 2\gamma + 2\tilde{\gamma} + \mathcal{O}(r^2_{\varepsilon,\lambda,\beta}) & = 0 \\
16\theta + 8\gamma + \mathcal{O}(r^2_{\varepsilon,\lambda,\beta}) & = 0 \\
-32\theta + \mathcal{O}(r^2_{\varepsilon,\lambda,\beta}) & = 0.
\end{cases} \tag{58}
$$

Here and below the terms $\mathcal{O}(r^2_{\varepsilon,\lambda,\beta})$ depend nonlinearly on β, θ, γ and $\tilde{\gamma}$ but are bounded (in the appropriate norm) by a constant (independent of ε and β) times $r^2_{\varepsilon,\lambda,\beta}$. Let us comment briefly on how these equations are obtained. These equations simply come from (57) when expansions (54) and (55) are used, together with the expression of H^i_γ and $H^e_{\tilde{\gamma}}$ and also the estimates (51) and (52). This system can be readily simplified into

$$
\frac{1}{\log r_{\varepsilon,\lambda,\beta}}[4\log\tau + H(0) + \log(f(0))] = \mathcal{O}(r^2_{\varepsilon,\lambda,\beta}),\ \theta = \mathcal{O}(r^2_{\varepsilon,\lambda,\beta}),\ \gamma = \mathcal{O}(r^2_{\varepsilon,\lambda,\beta}) \tag{59}
$$

$$
\text{and } \tilde{\gamma} = \mathcal{O}(r^2_{\varepsilon,\lambda,\beta}). \tag{60}
$$

We are now in a position to define τ_- and τ_+ since, according to the above, as ε tends to 0 we expect that τ will converge to τ^* satisfying

$$
-4\log\tau^* = H(0) + \log(f(0))
$$

and hence it is enough to choose τ_- and τ_+ so that

$$
4\log\tau_- < -[H(0) + \log(f(0))] < 4\log\tau_+.
$$

If we define

$$
t = \frac{1}{\log r_{\varepsilon,\lambda,\beta}}[4\log\tau + H(0) + \log(f(0))],
$$

then our system (58) reads

$$
(t,\ \beta, \theta,\ \gamma, \tilde{\gamma}) = \mathcal{O}(r^2_{\varepsilon,\lambda,\beta}). \tag{61}
$$

The nonlinear term which appears on the right-hand side of (61) is continuous and compact. In addition, this nonlinear term sends the ball of radius $\kappa r^2_{\varepsilon,\lambda,\beta}$ into itself, provided κ is large enough. Applying Schauder's fixed point theorem in the ball of radius $\kappa r^2_{\varepsilon,\lambda,\beta}$ in the product space, (61) can then be solved and the proof of Theorem 1 follows at once.

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