ON p-CRITICAL MODULES AND THE GREEN CORRESPONDENCE

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Communicated by Sorin Dăscălescu

We define p-critical kG-modules, and prove that the Green correspondence induces a bijection between the isomorphism classes of indecomposable p-critical kG-modules and those of indecomposable p-critical kH-modules.

AMS 2020 Subject Classification: 20C05, 20C20.

Key words: p-critical, Green correspondence, isomorphism class, finite group.

1. INTRODUCTION

In the theory of finite group representation, the (absolutely) p-divisible module is introduced and used to study nilpotent elements in Green rings [1]. p-divisible modules are an interesting class of modules that contain all (relatively) projective modules and focus on the prime factor p of the order of finite groups. In this paper, we use p-divisible modules instead of projective modules to construct p-critical modules.

For a finite group G, we know that critical kG-modules and kG-modules with trivial Sylow restriction play an important role in the study of the group of endo-trivial kG-modules [3, 4, 9, 10]. In this paper, we extend them to p-critical modules. We also note that endo-trivial kG-modules can be generalized to p-endotrivial kG-modules [3], and that p-endotrivial kG-modules are special splitting trace kG-modules. Splitting trace kG-modules have important applications in the study of almost split sequences [5], and p-critical modules defined in this paper are p-endotrivial kG-modules.

The Green correspondence is of fundamental importance in finite group representation theory [2, 6]. In this paper, following the idea of the Green correspondence, we obtain some conclusions about the restriction and induction of p-critical modules (Theorem 3.4). In particular, we proof that the Green correspondence induces a bijection between the isomorphism classes of indecomposable p-critical kG-modules and those of indecomposable p-critical kH-modules (Theorem 3.7, Corollary 3.8). In addition, the structure of p-critical kG-modules is also obtained (Proposition 2.8).

MATH. REPORTS **25(75)** (2023), *1*, 63–71
doi: 10.59277/mrar.2023.25.75.1.63

In this paper, we fix a prime number p, a finite group G, and an algebraically closed field k of characteristic p. All modules are finitely generated, and the order of any group involved in p-divisible modules is divisible by p.

2. p-CRITICAL kG-MODULES

A kG-module M is called a *p-divisible* kG-module if the dimension of any direct summand of M is divisible by p.

Remark 2.1. (1) p-divisible kG-modules herein are based on absolute indecomposability. We note that k herein is an algebraically closed field of characteristic p, so any indecomposable kG-module is already absolutely indecomposable and any p-divisible kG-module in this paper is absolutely p-divisible (see [1]).

(2) The dual of a p-divisible kG-module, any G-conjugative module of a p-divisible kG-module [2, Example 10.10], the direct sum of p-divisible kG-modules, and the tensor product of a p-divisible kG-module and a kG-module, are p-divisible.

(3) p-divisible kG-modules form a large class: projective kG-modules, Q-projective kG-modules, where Q is any proper p-subgroup of G [2, Exercise 21.2(a), Exercise 23.1]; while the trivial kG-module k is not p-divisible. In addition, p-divisible kG-modules focus on the prime factor p of the order of a finite group G, so they might be closely related to arithmetic properties of G.

We give the following definition.

Definition 2.2. Let P be a Sylow p-subgroup of G, and let M be a kG-module. If $\mathrm{Res}^G_P(M) = k \oplus U$, where U is a p-divisible kP-module, then we say that M is a p-critical kG-module.

Remark 2.3. (1) The definition of p-critical kG-modules is independent of the Sylow p-subgroup P. Indeed, since any two Sylow p-subgroups of G are conjugate in G, and ${}^g(\mathrm{Res}^G_P(M)) \cong \mathrm{Res}^G_{{}^gP}(M)$, $g \in G$ [2, P.216], we see that the restriction of M to ${}^gP$ is the direct sum of k and a p-divisible kG-module.

(2) The trivial kG-module k is p-critical, while any p-divisible kG-module is not p-critical.

(3) Any indecomposable p-critical kG-module belongs to a full defect block of G (Lemma 3.1).

(4) With p-critical kG-modules we generalize critical kG-modules and kG-modules with trivial Sylow restriction, as follow.

We say that a kG-module M is a kG-module with trivial Sylow restriction, if $\mathrm{Res}^G_P(M) = k \oplus (\text{projective})$, where P is a Sylow p-subgroup of G [3].

An indecomposable endo-trivial kG-module M is called a critical kG-module, if $\mathrm{Res}^G_H(M) = k \oplus (free)$, for any maximal subgroup H of G [4]. Note that the Sylow p-subgroup P in Definition 2.2 is a maximal p-subgroup of G, and any projective kP-module is a free kP-module as well [2, Proposition 21.1].

LEMMA 2.4. (1) *If M is a kG-module with trivial Sylow restriction, then M is a p-critical kG-module.*

(2) *If M is a critical kG-module, then M is a p-critical kG-module.*

Proof. (1) Let P be a Sylow p-subgroup of G. Following [3], we can set $\mathrm{Res}^G_P(M) = k \oplus (\text{projective})$, then by Remark 2.1 (3) we see that M is p-critical.

(2) Let H be a maximal subgroup of G such that H contains a Sylow p-subgroup P of G. Firstly, following [4], we see that $\mathrm{Res}^G_H(M) = k \oplus U$, where U is a free kH-module. Secondly,

$$\mathrm{Res}^G_P(M) = \mathrm{Res}^H_P(\mathrm{Res}^G_H(M)) = k \oplus \mathrm{Res}^H_P(U).$$

Thirdly, $\mathrm{Res}^H_P(U)$ is a free kP-module, and then $\mathrm{Res}^H_P(U)$ is a p-divisible kP-module (Remark 2.1 (3)).

Following the above, M is a p-critical kG-module. We are done. □

By a p-endotrivial kG-module M we mean a kG-module M such that $\mathrm{End}_k(M) = k \oplus U$, where U is a p-divisible kG-module; p-endotrivial kG-modules extend the notion of endo-trivial kG-modules [4].

Following [5], we say that a kG-module M is a splitting trace kG-module, if the trivial kG-module k is a direct summand of $\mathrm{End}_k(M)$. Obviously, any p-endotrivial kG-module(endo-trivial kG-module) is a splitting trace kG-module.

LEMMA 2.5. (1) *If M is a p-critical kG-module, then M is a p-endotrivial kG-module.*

(2) *If M is a p-critical kG-module, then M is a splitting trace kG-module.*

Proof. (1) Let P be a Sylow p-subgroup of G. Since M is a p-critical kG-module, we set $\mathrm{Res}^G_P(M) = k \oplus U$, where U is a p-divisible kP-module, and we have

$$\mathrm{Res}^G_P(\mathrm{End}(M)) \cong \mathrm{End}(\mathrm{Res}^G_P(M)) = \mathrm{End}(k \oplus U)$$
$$\cong k \oplus U \oplus U^* \oplus \mathrm{End}(U),$$

where $U \oplus U^* \oplus \mathrm{End}(U)$ is also p-divisible (Remark 2.1 (2)).

At the same time, since p does not divide $dim(M)$, we can set $\mathrm{End}(M) = k \oplus X$, [5, Corollary 4.7], where X is a kG-module. So $\mathrm{Res}^G_P(X)$ is a p-divisible kP-module, and X must be a p-divisible kG-module (Krull-Schmidt Theorem). That is, M is a p-endotrivial kG-module.

(2) follows from (1). □

Lemma 2.6. (1) *If M and N are p-critical kG-modules, then $M \otimes N$ is a p-critical kG-module.*

(2) *If M is a p-critical kG-module, and if N is a p-divisible kG-module, then $M \oplus N$ is a p-critical kG-module.*

Proof. (1) If P be a Sylow p-subgroup of G, then $\mathrm{Res}_P^G(M) = k \oplus U$ and $\mathrm{Res}_P^G(N) = k \oplus V$, where U and V are p-divisible kP-modules. Hence,

$$\mathrm{Res}_P^G(M \otimes N) \cong \mathrm{Res}_P^G(M) \otimes \mathrm{Res}_P^G(N) = (k \oplus U) \otimes (k \oplus V)$$
$$\cong k \oplus U \oplus V \oplus (U \otimes V),$$

it means that $M \otimes N$ is p-critical (Remark 2.1 (2)).

(2) Firstly, If P is a Sylow p-subgroup of G, then we have that $\mathrm{Res}_P^G(M) = k \oplus U$, where U is a p-divisible kP-module.

Secondly, $\mathrm{Res}_P^G(N)$ is p-divisible. Indeed, if $\mathrm{Res}_P^G(N)$ is not p-divisible, then $k|\mathrm{End}(\mathrm{Res}_P^G(N))$ [5, Corollary 4.7], and then

$$\mathrm{Ind}_P^G(k)|\mathrm{Ind}_P^G(\mathrm{Res}_P^G(\mathrm{End}(N))).$$

While by [6, Corollary 4.3.8],

$$\mathrm{Ind}_P^G(\mathrm{Res}_P^G(\mathrm{End}(N))) \cong \mathrm{Ind}_P^G(k) \otimes_k \mathrm{End}(N),$$

we see that $\mathrm{Ind}_P^G(k) \otimes \mathrm{End}(N)$ is p-divisible (Remark 2.1 (2)), and the direct summand $\mathrm{Ind}_P^G(k)$ is also p-divisible. It contradicts with the order of $\mathrm{Ind}_P^G(k)$.

Finally, the above conclusions mean that

$$\mathrm{Res}_P^G(M \oplus N) \cong \mathrm{Res}_P^G(M) \oplus \mathrm{Res}_P^G(N) = k \oplus U \oplus \mathrm{Res}_P^G(N) = k \oplus (p\text{-divisible}).$$

So $M \oplus N$ is p-critical. □

Lemma 2.7. (1) *If M is a p-critical kG-module, then M^* is a p-critical kG -module.*

(2) *If M is a p-critical kG-module, then for any $g \in G$, the conjugate kG-module ${}^gM$ is a p-critical kG-module.*

Proof. (1) Let P be a Sylow p-subgroup of G. We set $\mathrm{Res}_P^G(M) = k \oplus U$, where U is a p-divisible kP-module, and we see that

$$\mathrm{Res}_P^G(M^*) \cong (\mathrm{Res}_P^G(M))^* = (k \oplus U)^* \cong k \oplus U^*,$$

so M^* is a p-critical kG-module.

(2) In the case of [2, Example 10.10], if H is the group G herein, then we conclude that ${}^gM \cong M$, so ${}^gM$ is a p-critical kG-module, too. □

Proposition 2.8. *In the sense of kG-module isomorphism, any p-critical kG-module is the direct sum of an indecomposable p-critical kG-module and a p-divisible kG-module.*

Proof. If the indecomposable direct summand N of a p-critical kG-module M is not p-divisible, according to Krull-Schmidt Theorem,

$$\mathrm{Res}_P^G(N) = k \oplus (p\text{-divisible}),$$

where, N is a p-critical kG-module. Indeed, suppose that $\mathrm{Res}_P^G(N)$ is p-divisible, then N is also p-divisible, it is a contradiction.

Summing up the results above, in the sense of kG-module isomorphism, N is the unique indecomposable direct summand of M such that N is p-critical, and M is the direct sum of the unique indecomposable p-critical summand N and a p-divisible kG-module. □

3. THE GREEN CORRESPONDENCE FOR p-CRITICAL MODULES

LEMMA 3.1. *Let $G \geq H$ and M be a p-critical kG-module. If M is H-projective, then H contains a Sylow p-subgroup of G; in particular, the vertex of any indecomposable p-critical kG-module is a Sylow p-subgroup of G.*

Proof. If the Sylow p-subgroup Q of H is a proper p-subgroup of G, then M is a Q-projective kG-module, and then M is a p-divisible kG-module (Remark 2.1 (2)). It is a contradiction, so H contains a Sylow p-subgroup of G. Notice that if P is a Sylow p-subgroup of G, then M is P-projective [6, Proposition 11.3.5]. The above results conclude that the vertex of any indecomposable p-critical kG-module is a Sylow p-subgroup of G. We are done. □

PROPOSITION 3.2. *Let $G \geq H \geq P$, and let M be a kG-module. If P is a Sylow p-subgroup of G, then M is a p-critical kG-module if and only if $Res_H^G(M)$ is a p-critical kH-module.*

Proof. Proof of the necessity. Obviously, P is also a Sylow p-subgroup of H. If M is a p-critical kG-module, then $\mathrm{Res}_P^G(M) = k \oplus U$, where U is a p-divisible kP-module, and then

$$\mathrm{Res}_P^H(\mathrm{Res}_H^G(M)) = \mathrm{Res}_P^G(M) = k \oplus \mathrm{Res}_Q^P(U).$$

So $\mathrm{Res}_H^G(M)$ is a p-divisible kH-module.

Proof of the sufficiency. Since $\mathrm{Res}_P^G(M) = \mathrm{Res}_P^H(\mathrm{Res}_H^G(M))$, we obtain that if $\mathrm{Res}_H^G(M)$ is a p-critical kH-module, then $\mathrm{Res}_P^G(M) = k \oplus U$, where U is a p-divisible kP-module. So M is a p-critical kG-module. □

Proposition 3.3. *Let $G \geq H \geq N_G(P)$, where P is a Sylow p-subgroup of G. If M is an indecomposable p-critical kG-module, then the Green correspondent of M is a p-critical kH-module.*

Proof. Firstly, by Lemma 3.1 we see that P is a vertex of M. Secondly, by Proposition 3.2 we see that $\mathrm{Res}^G_H(M)$ is a p-critical kH-module. Thirdly, by Burry-Carlson-Puig Theorem [6, Theorem 11.6.9], in the sense of kH-module isomorphism, we conclude that there exists a unique indecomposable direct summand N of $\mathrm{Res}^G_H(M)$ such that P is a vertex of N, N herein is just the Green correspondent of M. Hence, with Proposition 2.8 and with Lemma 3.1 again, N must be the unique indecomposable direct summand of $\mathrm{Res}^G_H(M)$ with N being p-critical at the same time. We are done. □

Theorem 3.4. *Let $G \geq H$, and let M be a kG-module and N be a kH-module. If $M = Ind^G_H(N)$, then M is a p-critical kG-module if and only if N is a p-critical kH-module and H contains a Sylow p-subgroup P of G such that $p||P : P \cap {}^gH|$ for any $g \in G - H$.*

Proof. Proof of the sufficiency. If N is a p-critical kH-module and H contains a Sylow p-subgroup P of G such that $p||P : P\cap {}^gH|$ for any $g \in G-H$, then

$$\begin{aligned}\mathrm{Res}^G_P(M) &= \mathrm{Res}^G_P(\mathrm{Ind}^G_H(N)) \\ &\cong \bigoplus_{g\in[P\backslash G/H]} \mathrm{Ind}^P_{P\cap {}^gH}(\mathrm{Res}^{{}^gH}_{P\cap {}^gH}({}^gN)) \\ &= \mathrm{Res}^H_P(N) \oplus (\bigoplus_{1\neq g\in[P\backslash G/H]} \mathrm{Ind}^P_{P\cap {}^gH}(\mathrm{Res}^{{}^gH}_{P\cap {}^gH}({}^gN))). \qquad (1)\end{aligned}$$

Following the above, firstly, each $\mathrm{Ind}^P_{P\cap {}^gH}(\mathrm{Res}^{{}^gH}_{P\cap {}^gH}({}^gN)))$ is a $(P\cap {}^gH)$-projective kP-module, so it is p-divisible (Remark 2.1 (3)).

Secondly, $\mathrm{Res}^H_P(N) = k \oplus (p\text{-divisible})$. So we see (Remark 2.1 (2)) that $\mathrm{Res}^G_P(M) = k \oplus (p\text{-divisible})$, that is M is a p-critical kG-module.

Proof of the necessity. If M is p-critical, then H contains a Sylow p-subgroup P of G (Lemma 3.1), and p does not divide $dim(M)$. We notice that $dim(M) = |G : H|dim(N)$, so p does not divide $dim(N)$.

In addition, since $\mathrm{Res}^G_P(M) \cong k \oplus (p\text{-divisible})$, we conclude that, in (1), $\mathrm{Res}^H_P(N) = k \oplus (p\text{-divisible})$, and each $\mathrm{Ind}^P_{P\cap {}^gH}(\mathrm{Res}^{{}^gH}_{P\cap {}^gH}({}^gN))$ is a p-divisible kP-module. That is to say, N is a p-critical kH-module, and $P\cap {}^gH < P$, and so $p||P : P\cap {}^gH|$, for any $g \in G - H$.

Indeed, if $P\cap {}^gH = P$, where $g \in G - H$, then

$$\mathrm{Ind}^P_{P\cap {}^gH}(\mathrm{Res}^{{}^gH}_{P\cap {}^gH}({}^gN)) = \mathrm{Res}^{{}^gH}_P({}^gN),$$

while $\mathrm{Res}_P^{{}^gH}({}^gN) = k \oplus (p\text{-divisible})$ (Remark 2.1 (2)). It is a contradiction. We are done. □

Corollary 3.5. *Let $G \geq H$, and let M be a kG-module and N be a kH-module such that $M = Ind_H^G(N)$. If $H \geq N_G(P)$, where P is a Sylow p-subgroup of G, then M is a p-critical kG-module if and only if N is a p-critical kH-module.*

Proof. Proof of the necessity. It follows from Theorem 3.4.

Proof of the sufficiency. Suppose that for some $g \in G - H$, $P = P \cap {}^gH$, we have ${}^gH \geq P$. Notice that ${}^gH \geq {}^gP$, then we see that ${}^gP \geq {}^xP$ for some $x \in {}^gH$, and then we conclude that $g \in N_G(P)$, is a contradiction.

So $p||P : P \cap {}^gH|$ for any $g \in G - H$, and so the sufficiency follows from Theorem 3.4. □

We recall that a subgroup H of G is strongly p-embedded if $|H|$ is divisible by p but $|H \cap {}^gH|$ is not divisible by p, for any $g \in G - H$. Note that strongly p-embedded subgroups have important applications in the classification of finite simple groups, and such H exists whenever Sylow p-subgroups of G are trivial intersection (that is, T.I.) [7, 8].

Corollary 3.6. *Let $G \geq H$, and let M be a kG-module and N be a kH-module such that $M = Ind_H^G(N)$. If H is strongly p-embedded in G, then M is a p-critical kG-module if and only if N is a p-critical kH-module.*

Proof. Since the strongly p-embedded subgroup H always contains the normalizer $N_G(P)$ of a Sylow p-subgroup P of G [7], the result follows from Corollary 3.5. □

Theorem 3.7. *If $G \geq H \geq N_G(P)$, where P is a Sylow p-subgroup of G, then the Green correspondence induces a bijection between the isomorphism classes of indecomposable p-critical kG-modules and those of indecomposable p-critical kH-modules.*

Proof. Firstly, if M is an indecomposable p-critical kG-module, then the Green correspondent of M is an indecomposable p-critical kG-module (Proposition 3.3).

Secondly, if N is an indecomposable p-critical kH-module, then $\mathrm{Ind}_H^G(N)$ is a p-critical kG-module (Theorem 3.4).

Thirdly, by Burry-Carlson-Puig Theorem, Proposition 2.8, and Lemma 3.1, the unique indecomposable p-critical summand of $\mathrm{Ind}_H^G(N)$ is just the

Green correspondent of N, and this Green correspondent is a p-critical kG-module.

Summing up the results above, all of indecomposable p-critical modules are closed under the Green correspondence. At the same time, the Green correspondence induces a bijection between the isomorphism classes of indecomposable kG-modules and those of indecomposable kH-modules with the same vertex P [6, Theorem 11.6.4]. Hence, the Green correspondence herein induces a bijection between the isomorphism classes of indecomposable p-critical kG-modules and those of indecomposable p-critical kH-modules. □

COROLLARY 3.8. *Let $G \geq H$. If H is strongly p-embedded in G, then the Green correspondence induces a bijection between the isomorphism classes of indecomposable p-critical kG-modules and those of indecomposable p-critical kH-modules.*

Proof. The result follows from Corollary 3.6 and Theorem 3.7. □

Acknowledgments. The author thanks the referee for his (her) detailed comments and valuable suggestions. This paper was supported by College Mathematics Teaching Research and Development Center (CMC20210413), and Research Project of RUC Capital Development and Governance Institute, and RUC Gold Course Construction Project.

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Received April 19, 2019

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