HYPERELLIPTIC DIOPHANTINE EQUATIONS FROM THE STUDY OF PARTITIONS

DORIN ANDRICA and GEORGE C. ŢURCAŞ

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We present two Diophantine equations that arise from some new results in the theory of partitions with equal sums. We link these to the problem of finding rational points on some hyperelliptic curves and we solve the latter, assisted by computer algebra packages, using a p-adic method pioneered by Chabauty and Coleman.

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1. INTRODUCTION AND MOTIVATION

For a positive integer $k \geq 2$ and an arbitrary positive integer n, in [2] and [1], the authors introduced the sequence $\{Q_k(n)\}_{n\geq 1}$,

$$Q_k(n) = \frac{1}{2\pi}\int_0^{2\pi}\prod_{s=1}^{n}(k-2+2\cos st)\,\mathrm{d}t. \tag{1}$$

An enumerative formula for $Q_k(n)$ is given by the number of ordered partitions of $[n] = \{1,\ldots,n\}$ into k disjoint sets $A_1,\ldots,A_k$ with the property that $\sigma(A_1) = \sigma(A_k)$, where $\sigma(A)$ denotes the sum of all elements in A. The case $k = 2$ corresponds to the number $S(n)$ of partitions of $[n]$ in two sets with equal sums. The sequence $\{S(n)\}_{n\geq 0}$ is indexed as A063865 in the Online Encyclopedia of Integer Sequences (OEIS) [10]. The asymptotic formula for $S(n)$ was conjectured by Andrica and Tomescu [3] in 2002 as

$$\lim_{\substack{n\to\infty \\ n\equiv 0 \text{ or } 3 \pmod 4}} \frac{S(n)}{\frac{2^n}{n\sqrt{n}}} = \sqrt{\frac{6}{\pi}}$$

and this was proved to be correct by Sullivan [13] in 2013.

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Clearly, $Q_k(n)$ is a monic polynomial of degree n in $k-2$. Moreover, in [2] is proved that

$$Q_k(n) = \sum_{d=0}^{n} N(d,n)(k-2)^{n-d}, \tag{2}$$

where for each $d = 0,\ldots,n$, the coefficient $N(d,n)$ is the number of ordered partitions of $[n]$ into 3 subsets A, B, C such that $|B| = d$ and $\sigma(A) = \sigma(C)$, where $|B|$ is the cardinality of B.

Therefore, $Q_k(n)$ has non-negative integer coefficients, and each coefficient has a combinatorial meaning in terms of partitions of the set $[n]$. A simple direct computation of the integral (1) shows that for $n = 3, 5, 7, 9$ and $k \geq 2$, we have

$$\begin{aligned}
Q_k(3) &= (k-2)^3 + 2;\\
Q_k(5) &= (k-2)^5 + 8(k-2)^2 + 6(k-2);\\
Q_k(7) &= (k-2)^7 + 18(k-2)^4 + 30(k-2)^3 + 18(k-2)^2 + 12(k-2) + 8;\\
Q_k(9) &= (k-2)^9 + 32(k-2)^6 + 82(k-2)^5 + 104(k-2)^4 + 130(k-2)^3 +\\
&\qquad 130(k-2)^3 + 136(k-2)^2 + 62(k-2).
\end{aligned}$$

The sequence $\{Q_k(3)\}_{k\geq 2}$ is indexed as A084380 in OEIS [10], where it is mentioned that it does not contain any perfect squares, i.e. the elliptic equation $X^3 + 2 = Y^2$ has no solutions in positive integers. Two different proofs for this result were given in [4]. The previous equation is linked to a Catalan-type conjecture related to Pillai's equation $X^U - Y^V = m$, with $X, Y, U, V \geq 2$ integers. The conjecture states that for any given integer m, there are finitely many perfect powers whose difference is m (see [14], Conjecture 1.6). For $m = 2$, it was computationally checked that the only solution involving perfect powers smaller than 10^{18} is $2 = 3^3 - 5^2$. The number of such solutions is linked to A076427 in OEIS.

Motivated by the property that the sequence $Q_k(3)$ does not contain any perfect squares, in [2] and [1], the authors suggested the following problem: study if the sequence $\{Q_k(n)\}_{k\geq 2}$ contains any $n-1$ powers for various values of n. These yield, for general n, very difficult Diophantine equations and it seems improbable to develop a general strategy for solving them. We will explain a method that can sometimes be successfully used to study the aforementioned problem when n is odd. To demonstrate its versatility, we discuss the problem above for $n = 5$ and 7. We obtain the following Diophantine equations:

$$X^5 + 8X^2 + 6X = Y^4; \tag{3}$$

(4) $$X^7 + 18X^4 + 30X^3 + 18X^2 + 12X + 8 = Y^6.$$

Using effective methods for identifying integral points on curves, we prove the following two theorems which give a negative answer to the problem described above.

THEOREM 1.1. *The only solutions in integers to the equation* $Y^4 = X^5 + 8X^2 + 6X$ *are given by the pairs* $(x, y) = (0, 0)$, $(-1, 1)$ *or* $(-1, -1)$.

Since we believe that the pairs above are all the rational solutions to (3), we formulated Conjecture 3.1 which asserts this. Our tools were more efficient when applied to the equation (4) for which we were able to find all the rational solutions.

THEOREM 1.2. *The only rational solutions of the equation* $X^7 + 18X^4 + 30X^3 + 18X^2 + 12X + 8 = Y^2$ *are given by* $(x, y) = (-1, 1)$ *or* $(-1, -1)$.

2. DIOPHANTINE EQUATIONS AND HYPERELLIPTIC CURVES

For any bivariate polynomial $f \in \mathbb{Z}[X, Y]$, let

$$C_f := \{(x, y) \in \overline{\mathbb{Q}}^2 : f(x, y) = 0\}$$

be an affine algebraic curve. The points of C_f with coordinates in $\mathbb{Q}$ are called rational and, in general, for any $S \subseteq \overline{\mathbb{Q}}$, we denote by $C_f(S) = C_f \cap S^2$. Curves can be classified by their genus, a non-negative integer associated to their projectivization. The genus is a geometric invariant. A classical result in number theory is the following theorem

THEOREM 2.1 (Siegel, 1929). *If* $f \in \mathbb{Z}[X, Y]$ *defines an irreducible curve* C_f *of genus* $g(C_f) > 0$, *then* $C_f(\mathbb{Z})$ *is finite.*

If additionally $g_f(C_f) \geq 2$, this result is superseded by the notorious Falting's theorem, which says that $C_f(\mathbb{Q})$ is also finite. Although both Siegels' and Faltings' theorems are milestones in number theory, they are "ineffective" results, meaning that their proof does not even allow one to control the size of the sets known to be finite. Therefore, they cannot be used to explicitly determine $C_f(\mathbb{Z})$ or $C_f(\mathbb{Q})$.

Effectively finding rational points on curves is an incredible difficult task and a very active topic of research. The toolbox for determining $C_f(\mathbb{Z})$ became a lot richer starting with the monumental work of Baker on linear forms in logarithms. As one of the first applications to his theory, Baker proved the following result.

Theorem 2.2 (Baker, 1969). *Let*

$$f(X,Y) = Y^2 - a_n X^n - a_{n-1} X^{n-1} - \cdots - a_0 \in \mathbb{Z}[X,Y].$$

Suppose that the polynomial $a_n X^n + \cdots + a_0$ *is irreducible in* $\mathbb{Z}[X]$, $a_n \neq 0$ *and* $n \geq 5$. *Let* $H = \max\{|a_0|, \ldots, |a_n|\}$. *Then, any integral point* $(x, y) \in C_f(\mathbb{Z})$ *satisfies* $\max(|x|, |y|) \leq \exp\exp\exp\{(n^{10n} H)^{n^2}\}$.

Bounds on such solutions have been improved by many authors, but they remain astronomical and often involve inexplicit constants.

For every smooth, projective and absolutely irreducible curve C of genus g defined over $\mathbb{Q}$, the Jacobian J_C is a g-dimensional abelian variety, *functorially* associated to C. Fixing a point $P_0 \in C(\mathbb{Q})$, the curve C can be identified as a subvariety of J_C via the Abel-Jacobi map $\iota : C \hookrightarrow J_C$ with base point P_0. The famous Mordell-Weil theorem gives that, as is the case for elliptic curves, the set of $\mathbb{Q}$ rational points of J_C has the structure of a finitely generated abelian group, i.e.

$$J_C(\mathbb{Q}) \equiv T \oplus \mathbb{Z}^r,$$

where T is a finite abelian group and r is a positive integer, called the *rank.*

A famous theorem due to Chabauty and Coleman [8] is the following.

Theorem 2.3. *Let C be a smooth, projective and absolutely irreducible curve of genus g over $\mathbb{Q}$, with Jacobian J. Assume that the rank r of the Mordell-Weil group $J_C(\mathbb{Q})$ is strictly less than g. Then, there is an algorithm for determining the set of rational points $C(\mathbb{Q})$. Moreover, if p is a prime of good reduction for C such that $p > 2g$, then*

$$\#C(\mathbb{Q}) \leq \overline{C}(\mathbb{F}_p) + 2g - 2.$$

Here, we denoted by $\overline{C}$ the curve obtained by reducing modulo p the coefficients of the equation defining C. An improvement due to Stoll [12], gives the sometimes smaller bound of $\#C(\mathbb{Q}) \leq \overline{C}(\mathbb{F}_p) + 2\,\mathrm{rank}(J_C(\mathbb{Q}))$ if C and p are as above.

Algebraic curves defined by equations of the type $Y^2 = f(X)$, where $f \in \mathbb{Q}[x]$ is a polynomial with distinct roots, are called hyperelliptic. Algorithms for computation in the Jacobian of such curves are described in [7]. These are implemented in the computer algebra package **Magma** [6]. Due to this computational convenience, when we are looking for integral solutions to the three Diophantine equations presented in the introduction, we make the passage to the problem of determining integral points on some hyperelliptic curves. We compute the Jacobian of the latter and apply the algorithm intrinsic in Theorem 2.3 to find all the rational points on such curves.

3. THE PROOF OF THEOREM 1.1

In the same paper, Andrica and Bagdasar conjecture that X^5+8X^2+6X is never a fourth power as X runs through the positive integers. To settle this, we are going to study points on the affine curve given by

$$Y^4 = X^5 + 8X^2 + 6X. \tag{5}$$

It is an exercise using the Riemann-Hurwitz formula to compute that the genus of this curve is equal to 6, therefore by results mentioned earlier we know that is has finitely many rational points. To find the set of all rational points turns out to be a notorious difficult task from which we choose to detach for now.

We start by proving the following easy result.

PROPOSITION 3.1. *If x, y are positive integers such that $y^4 = x^5+8x^2+6x$ then x is divisible by 6.*

Proof. Suppose $\gcd(x, 6) = 1$. Then, the numbers x and $x^4 + 8x + 6$ are coprime, and as their product is a fourth power, we can conclude that both numbers must be fourth powers. Hence, we obtain the equation $x^4+8x+6 = z^4$, for some positive integer z.

Since $z > x$ and $(z^2 - x^2)(z^2 + x^2) = 8x + 6$, we get that $8x + 6 \geq z^2 + x^2 \geq 1 + x^2$ implying that $1 \leq x \leq 9$. As x must be a fourth power, the only possibility left is $x = 1$ but then $y^4 = 15$, a contradiction.

Suppose for the sake of contradiction that x is not divisible by 3. The above implies that x is divisible by 2 and so is y. As $4 \leq v_2(y^4) = v_2(6x)$, we have that $v_2(x) \geq 3$. There exists positive integers a, b such that

$$\begin{cases} x = 2^3 a \text{ and} \\ y = 2b \end{cases}.$$

By substituting in the initial equation and dividing by 2^4, we obtain

$$b^4 = a \cdot (2^{11} \cdot a^4 + 2^5 \cdot a + 3).$$

As the terms of the product on the right hand side are, in this case, coprime we derive that both a and $2^{11} \cdot a^4 + 2^5 \cdot a + 3$ must be forth powers of positive integers. But the residue of $2^{11} \cdot a^4 + 2^5 \cdot a + 3$ when divided by 4 is always 3 and no fourth power has this property. The contradiction implies that 3 divides x.

All that is left is proving that 2 divides x as well. Suppose the contrary, namely x, y are odd and both divisible only by 3. Let c, d be positive integers such that

$$\begin{cases} x = 3c \text{ and} \\ y = 3d \end{cases}.$$

Again, by using the initial equation and dividing by 3^2 we obtain

$$3^2 d^4 = 3^3 \cdot c^5 + 8c^2 + 2c,$$

The last assumption implies that c is odd and by looking at the last equation modulo 8, it is easy to deduce that $c = 5 \pmod 8$. Analysing the equation modulo 16, we further find that $c = 5 \pmod{16}$. Now, we get a contradiction by looking modulo 32, since the left hand side of the equation above is 9 or 25 modulo 32, values that the right hand side never achieves when $c = 5 \pmod{16}$. This completes the proof of our proposition. □

Finally, we prove the next Theorem 1.1 giving a positive answer to the result conjectured by Andrica and Bagdasar [2].

Remark. Naively applied, Baker's theorem tells us that if x, y are integers satisfying the equation $y^4 = x^5 + 8x^2 + 6x$, then $\max(|x|, y^2) \le \exp\exp\exp\{(5^{50} \cdot 8)^{5^2}\}$, which is astronomical and does not help much with our task. Instead, we show that a putative solution to this equation gives rise to a certain point on a projective hyperelliptic curve with special arithmetic properties. Assisted by the computer algebra package **Magma**, we determine the set of all rational points on the hyperelliptic curve and prove that our predicted point does not belong to it.

Proof of Theorem 1.1. Let $(x, y) \in \mathbb{Z}^2$ be a solution to our equation (3), i.e. $y^4 = x^5 + 8x^2 + 6x$. It is easy to see that $x = 0$ if and only if $y = 0$. Suppose that $y \neq 0$. If $x < 0$, the positivity of y^4 implies that $-2 < x < 0$ and as x is an integer, we obtain $x = -1$. Substituting in the original equation, this gives $y = -1$ and $y = 1$ as the only possibilities for $x < 0$.

We will prove that there are no integral solutions with $x > 0$. Suppose the contrary and let $(x, y) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ be such a solution. Since that $(x, -y)$ is also a solution to (3), we can assume without loosing generality that $y > 0$. We proved in Proposition 3.1 that x and hence y are divisible by 6. Moreover, in the proof of the same proposition, we saw that $v_2(x) \ge 3$. Let a, b be positive integers such that $x = 2^3 \cdot 3 \cdot a$ and $y = 2 \cdot 3 \cdot b$. Substituting into the initial equation and dividing both sides by $2^4 \cdot 3^2$, we get

$$(3 \cdot b^2)^2 = 2^{11} \cdot 3^3 \cdot a^5 + 2^7 \cdot a^2 + a.$$

We can now regard $(a, 3b^2)$ as a point on the affine model of the projective hyperelliptic curve

$$C^{\text{proj}} : Y^2 = 2^{11} \cdot 3^3 X^5 Z + 2^7 X^2 Z^4 + Z^6. \tag{6}$$

Notice that the ambient space is not the classical projective plane $\mathbb{P}^2$, but rather the weighted $\mathbb{P}^2_{(1,3,1)}$. The points of $\mathbb{P}^2_{(1,3,1)}$ over $\mathbb{Q}$ are the equivalence classes of triples $[X:Y:Z] \in \mathbb{Q}^3 \setminus \{[0:0:0]\}$, where two triples $[X_1:Y_1:Z_1]$ and $[X_2:Y_2:Z_2]$ are equivalent if there exists some $\lambda \in \mathbb{Q} \setminus \{0\}$, such that $[X_2:Y_2:Z_2] = [\lambda \cdot X_1 : \lambda^3 \cdot Y_1 : \lambda \cdot Z_1]$. A point with integral coordinates $(a, 3b^2)$ maps to the point with coordinates $[a:3b^2:1] \in \mathbb{P}^2_{(1,3,1)}$ on $C^{\text{proj}}(\mathbb{Q})$.

The later smooth projective curve has genus 2 and by Falting's theorem we know that

$$C^{\text{proj}}(\mathbb{Q}) = \{[X:Y:Z] \in \mathbb{P}^2_{(1,3,1)}(\mathbb{Q}) : Y^2 = 2^{11} \cdot 3^3 X^5 Z + 2^7 X^2 Z^4 + Z^6\}$$

is a finite set. We pursue the task of explicitly determining $C^{\text{proj}}(\mathbb{Q})$.

The Jacobian J_C is a 2 (equal to the genus) dimensional abelian variety that is "functorially associated" to C^{proj}.

We also know for $P_0 := [1:0:0] \in C^{\text{proj}}(\mathbb{Q})$, the Abel-Jacobi map associated to P_0 is the embedding $i : C^{\text{proj}}(\mathbb{Q}) \hookrightarrow J_C(\mathbb{Q})$ given by $P \mapsto [P - P_0]$, for every $P \in C^{\text{proj}}(\mathbb{Q})$. Using the computer algebra package **Magma** [6], we compute that

$$J_C(\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}.$$

Moreover, assisted by the same software, we show that the torsion subgroup T is generated by $i([0:0:1])$ and a generator for $J_C(\mathbb{Q})/T$ is $i([-1:-5760:24])$.

As the rank of the Jacobian is strictly less than the genus of our curve, we can apply the method of the algorithm mentioned in Theorem 2.3 to determine $C^{\text{proj}}(\mathbb{Q})$. A beautiful presentation of how the method works can be found in the expository article of McCallum and Poonen [9] and an algorithm suitable to our set-up is implemented in **Magma**. For any prime p of good reduction for C^{proj}, the closure of $J_C(\mathbb{Q})$ in $J_C(\mathbb{Q}_p)$ (under the p-adic topology) can be described as the locus where certain power series vanish. It turns out that, under a natural embedding, the image of C^{proj} in J_C meets this closure in a finite set, a set that must contain $C^{\text{proj}}(\mathbb{Q})$.

A description of the **Magma** implementation of the aforementioned algorithm can be found at

`https://magma.maths.usyd.edu.au/magma/handbook/text/1507.`

It takes as input the genus 2 hyperelliptic curve C^{proj} and $i([-1:-5760:24])$, the generator of the torsion-free part of its Jacobian. We remark that the procedure implemented in **Magma** combines the method of Chabauty-Coleman with the Mordell-Weil Sieve at primes $p \in \{5, 11, 29\}$ and, in a few seconds on a personal laptop, it outputs the full set of rational points

$$C^{\text{proj}}(\mathbb{Q}) = \{[0:0:1], [-1:5760:24], [1:0:0], [-1:-5760:24]\}.$$

As we cannot find a point of the form $[a:3b^2:1]$ for a, b positive integers, we get our contradiction and the proof of our theorem is complete. □

The reader might wonder why we did not apply Chabauty's method to the hyperelliptic curve defined by $Y^2 = X^5 + 8X^2 + 6X$. It turns out that, although it has genus 2, this curve has a rank 2 Jacobian and does not satisfy the hypothesis required by Theorem 2.3. We, therefore, had to work with local methods and to prove Proposition 3.1 in order to apply Chabauty's method to a different hyperelliptic curve.

As a result of extensive computations performed on a computer, we are confident in formulating the following conjecture.

CONJECTURE 3.1. *The only rational solutions of the equation* $Y^4 = X^5 + 8X^2 + 6X$ *are* $(x, y) = (0, 0)$, $(-1, 1)$ *and* $(-1, -1)$.

4. THE PROOF OF THEOREM 1.2

The equation $X^7 + 18X^4 + 30X^3 + 18X^2 + 12X + 8 = Y^6$ defines a curve of genus 15, therefore, by Faltings' theorem, the latter has finitely many rational points. The genus is a reflection of the complexity of the curve, hence trying to determine all the rational points on the given curve effectively is an extremely difficult task. A näive computer search for rational points returns $\{(-1, 1), (-1, -1)\}$ and we suspect these are all of them.

Let us consider the affine hyperelliptic curve

$$C : Y^2 = X^7 + 18X^4 + 30X^3 + 18X^2 + 12X + 8 \tag{7}$$

and its projective model

$$C^{\text{proj}} : Y^2 = X^7 \cdot Z + 18X^4 \cdot Z^4 + 30X^3 \cdot Z^5 + 18X^2 \cdot Z^6 + 12X \cdot Z^7 + 8 \cdot Z^8. \tag{8}$$

The latter is a smooth projective curve in the weighted projective plane $\mathbb{P}^2_{(1,4,1)}(\mathbb{Q})$. It is a genus 3, whose geometry is less complicated than the one of the original curve. We used **Magma**'s built-in *RankBounds* command and computed that the Mordell-Weil rank of its Jacobian is 1. As $1 < 3$, Theorem 2.3 tells us that there is an algorithm for determining the rational points on $C^{\text{proj}}(\mathbb{Q})$, and by letting $p = 7$, we obtain that

$$\#C^{\text{proj}}(\mathbb{Q}) \leq 12 + 2 \cdot 3 - 2 = 16.$$

If this bound would have been sharp, we could have tried to find 16 rational points on C^{proj} and conclude that this must be all of them. We cannot finish the problem in this manner, since as we will see, $\#C^{\text{proj}}(\mathbb{Q}) = 3$. We, therefore, have to run the Chabauty-Coleman algorithm, but unfortunately the implementation available in **Magma** can only handle curves of genus 2.

We use instead a recently developed algorithm of Balakhrishnan *et. al.* [5], which successfully carries the Chabauty-Coleman algorithm for hyperelliptic curves of genus 3 whose Jacobian have Mordell-Weil rank 1. As we saw above, our curve C falls in this category. In order to succeed, this algorithm requires as input:

1. An hyperelliptic curve C of genus 3 over $\mathbb{Q}$ whose Jacobian J has rank 1. The curve C should be given by a model $y^2 = f(x)$ where $f \in \mathbb{Z}[x]$ has degree 7.

2. $p \geq 7$, a prime of good reduction which does not divide the leading coefficient of f.

3. A point $P \in C^{\text{proj}}(\mathbb{Q})$ whose image $[P - P_\infty]$ under the Abel-Jacobi map $\iota : C^{\text{proj}} \hookrightarrow J_{C^{\text{proj}}}$ with base point $P_\infty = [1 : 0 : 0]$ has infinite order.

4. A list L of known rational points on $C^{\text{proj}}(\mathbb{Q})$.

5. A positive integer which represents the chosen p-adic precision.

When the algorithm terminates, it returns as output, among other things, a list of all points in $C^{\text{proj}}(\mathbb{Q}) \setminus L$, modulo hyperelliptic involution. This means, the output is only going to show one of the points $[X : Y : 1]$ or $[X : -Y : 1]$ if they both are on $C^{\text{proj}}(\mathbb{Q}) \setminus L$. This algorithm is implemented in **Sage** [11] and can be downloaded from

`https://github.com/jbalakrishnan/WIN4`.

Proof of Theorem 1.2. A rational point (x, y) on the affine curve C given by (7) maps to the point $[x : y : 1] \in C^{\text{proj}}(\mathbb{Q})$ on the projective model given by the equation (8). We are therefore going to determine all the rational points on C^{proj}. It is easy to see that $P_\infty = [1 : 0 : 0]$ and $P_1 = [-1 : 1 : 1]$ are in $C^{\text{proj}}(\mathbb{Q})$. Write $\iota : C^{\text{proj}} \hookrightarrow J_C$ for the Abel-Jacobi map

$$P \mapsto [P - P_\infty].$$

Using **Magma**'s hyperelliptic curve package, we proved that

1. $J_C(\mathbb{Q})$ is a rank 1 free abelian group;

2. $\iota(P_1)$ is a point of infinite order on $J_C(\mathbb{Q})$;

3. The primes of bad reduction for C^{proj} are $2, 11159$ and 1863377.

Running the **Sage** implementation of the algorithm of Balakhrishnan *et. al.* with input C, prime 7, point P_1, an empty list of known points on C^{proj} and p-adic precision 10, we obtain

```
sage: load("chabautyg3r1.sage")
sage: R.<x> = PolynomialRing(Rationals())
sage: f = x^7 + 18*x^4 + 30*x^3 + 18*x^2 + 12*x + 8
sage: C = HyperellipticCurve(f)
sage: p = 7
sage: prec = 10
sage: P = C(-1,1)
sage: points,b,c,d = chabauty_test(C,p,P,[],prec)
sage: points
[(-1 : 1 : 1), (1 : 0 : 0)]
```

This confirms that the Chabauty-Coleman algorithm finished successfully and that $[-1:1:1], [1:0:0]$ are all the points on $C^{\text{proj}}(\mathbb{Q})$ modulo the hyperelliptic involution. Hence

$$C^{\text{proj}}(\mathbb{Q}) = \{P_\infty = [1:0:0], [-1:1:1], [-1:-1:1]\}.$$

□

Corollary 4.1. *When X runs through the set of positive integers, the expression $X^7+18X^4+30X^3+18X^2+12X+8$ is never a 6-th power, confirming the conjecture of Andrica and Bagdasar [1], [2].*

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Dorin Andrica
Babeş-Bolyai University
Cluj-Napoca, Romania
`dandrica@math.ubbcluj.ro`

George C. Ţurcaş
Simion Stoilow Institute of Mathematics
of the Romanian Academy
Bucharest, Romania
and
Babeş-Bolyai University
Cluj-Napoca, Romania
`george.turcas@math.ubbcluj.ro`