ADJOINT PREENVELOPES AND ADJOINT PRECOVERS IN THE FUNCTOR CATEGORY

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Adjoint preenvelopes and adjoint precovers are defined in the category of functors by replacing the functor Hom with $\otimes$. We investigate the existence and basic properties of adjoint preenvelopes and adjoint precovers. The $\mathfrak{F}$-projective ($\mathfrak{F}$-injective, $\mathfrak{F}$-flat) functors introduced by Mao are characterized in terms of adjoint preenvelopes and adjoint precovers. We obtain relationships among adjoint preenvelopes, adjoint precovers, preenvelopes and precovers.

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1. INTRODUCTION

Given a ring R, the categories of covariant additive functors (mod-R, Ab) and contravariant additive functors ((mod-$R)^{op}$, Ab) from the category mod-R of finitely presented right R-modules to the category Ab of abelian groups have received widespread attention since the 1960s. In particular, they play initial roles in the study of the model theory of modules and the representation theory of artinian algebras [1, 2, 3, 4, 15]. Auslander [1] pointed out that if the category R-mod is abelian, then the full subcategory ((R-mod$)^{op}$, Ab$)^{\mathrm{fp}}$ of finitely presented contravariant functors is an abelian category. In some aspects, the category of covariant functors has several advantages over the category of contravariant functors. For instance, Auslander [2] proved that the subcategory (mod-R, Ab$)^{\mathrm{fp}}$ is an abelian category when R is any ring. Indeed, Freyd [8] showed that it is the free abelian category generated by R. Maybe one of the most profound applications of the functorial perspective was the construction of almost split sequence [4], which was based on the analysis of the finer aspects of the subcategory ((mod-$\Lambda)^{op}$, Ab$)^{\mathrm{fp}}$, where Λ is an artinian algebra [3].

Let $\mathcal{A}$ be an additive category, $\mathcal{C}$ a class of objects in $\mathcal{A}$. According to [7], a morphism $f : M \to N$ is a $\mathcal{C}$-preenvelope of M if $N \in \mathcal{C}$ and the sequence

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$(N, C) \to (M, C) \to 0$ is exact for any $C \in \mathcal{C}$, where (N, C) (resp. (M, C)) denotes the group of morphisms from N (resp. M) to C. Dually, the $\mathcal{C}$-precover is defined. The preenvelopes and precovers of objects have been studied by many authors, including that of functors, see [12, 18]. Based on the adjointness of the functors Hom and $\otimes$, Mao [14] introduced the concepts of adjoint preenvelopes and adjoint precovers of modules and some properties and relationships between adjoint preenvelopes, adjoint precovers and preenvelopes and precovers were studied. The current paper introduces the concepts of adjoint preenvelopes and adjoint precovers of functors. We also clarify the relationships between adjoint preenvelopes, adjoint precovers and preenvelopes and precovers. See Theorems 4.3 and 4.5.

$\mathfrak{F}$-projective ($\mathfrak{F}$-injective, $\mathfrak{F}$-flat) functors were introduced by Mao [13, Definition 3.1]. It was proved in [13] that these functors are closely related to the flat (pre)envelopes in ((mod-R)^{op}, Ab) or the FP-injective (pre)covers in (mod-R, Ab). In this paper, we give another description of these functors via the adjoint of preenvelopes and precovers. We obtain the following result.

THEOREM 1.1. *Let R be a ring. The following assertions hold.*

(1) *Let R be a right coherent ring and Q a finitely presented functor in ((mod-R)^{op}, Ab). Then Q is $\mathfrak{F}$-projective if and only if Q is a cokernel of a flat adjoint preenvelope in ((mod-R)^{op}, Ab).*

(2) *Let F be any functor in (mod-R, Ab). Then F^+ is $\mathfrak{F}$-flat in ((mod-R)^{op}, Ab) if and only if F is a kernel of an FP-injective adjoint precover in (mod-R, Ab).*

(3) *Let F be a finitely presented functor in (mod-R, Ab). Then F is $\mathfrak{F}$-injective if and only if F is a kernel of an injective adjoint precover in (mod-R, Ab)^{fp}.*

2. PRELIMINARIES

In this section, we recall some definitions and known facts needed in the sequel.

Throughout this paper, R is an associative ring with identity and all modules are unitary. For a ring R, we denote by M_R (resp. $_RM$) a right (resp. left) R-module, and by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ the character module of M, which is a left R-module.

The exactness in (mod-R, Ab) or ((mod-R)^{op}, Ab) is characterized by the exactness in Ab through evaluating. That is, a sequence of functors

$$F \xrightarrow{u} G \xrightarrow{v} H$$

in (mod-R, Ab) (resp. ((mod-$R)^{op}$, Ab)) is called exact provided that for every finitely presented right R-module M, the corresponding sequence of abelian groups

$$F(M) \xrightarrow{u_M} G(M) \xrightarrow{v_M} H(M)$$

is exact.

Let F be a functor in (mod-R, Ab). We denote by $F^+ \in$ ((mod-$R)^{op}$, Ab) the character functor of F, where $F^+(M) = (F(M))^+$ for any finitely presented right R-module M. If F, G are objects in (mod-R, Ab) or ((mod-$R)^{op}$, Ab), then (F, G) denotes the set of natural transformations from F to G.

According to [5, Theorem 1.4], the functor Υ : Mod-$R \to$ ((mod-$R)^{op}$, Ab) given by $M \mapsto (-, M)$ represents a full and faithful left exact functor. Similarly, the functor ϵ : R-Mod$\to$ (mod-R, Ab) given by $M \mapsto - \otimes M$ represents a full and faithful right exact functor.

Let $\mathcal{A}$ be an additive category. A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ is said to be pure-exact [5] provided that the sequence $0 \to (F, A) \to (F, B) \to (F, C) \to 0$ is exact for each finitely presented object F in $\mathcal{A}$. In this case, f is called a pure monomorphism and g a pure epimorphism. An object C is said to be pure-projective if every pure-exact sequence with third term C splits. Dually, an object A is said to be pure-injective if every pure-exact sequence with first term A splits.

A functor F in ((mod-$R)^{op}$, Ab) is called finitely presented [5] provided that $(F, -)$ commutes with direct limits, equivalently, there is an exact sequence $(-, M) \to (-, N) \to F \to 0$ with M and N finitely presented right R-modules. A functor G in ((mod-$R)^{op}$, Ab) is called representable if it is isomorphic to a functor of the form $(-, M)$, where M is a finitely presented right R-module. Yoneda's Lemma implies that a functor in ((mod-$R)^{op}$, Ab) is representable if and only if it is a finitely generated projective object in ((mod-$R)^{op}$, Ab). Thus, a functor G in ((mod-$R)^{op}$, Ab) is projective if and only if it is isomorphic to a direct summand of a direct sum of representable functors if and only if it is isomorphic to a functor of the form $(-, M)$, where M is a pure-projective right R-module [5, Lemma 3.1].

Finally, we recall the notion of tensor product of functors [8, 10]. Given $F \in$ ((mod-$R)^{op}$, Ab) and $G \in$ (mod-R, Ab). The tensor product of the functors F and G is defined to be an object $F \otimes G \in Ab$ such that the functor

$$F \otimes - : \text{(mod-}R\text{, Ab)} \to \text{Ab}$$

is the left adjoint of the functor

$$(F(\cdot), -) : \text{Ab} \to \text{(mod-}R\text{, Ab)},$$

where $(F(\cdot),-)(C)(D) = (F(D),C)$ for $C \in \mathrm{Ab}$ and $D \in (\text{mod-}R, \mathrm{Ab})$. Similarly, we can also define

$$-\otimes G : ((\text{mod-}R)^{op}, \mathrm{Ab}) \to \mathrm{Ab}$$

as the left adjoint of the functor

$$(G(\cdot),-) : \mathrm{Ab} \to ((\text{mod-}R)^{op}, \mathrm{Ab}).$$

In fact, the above notion can be justified by the following isomorphisms [19, Theorem 1]

$$(F \otimes G, C) \cong (F, (G(\cdot), C)) \cong (G, (F(\cdot), C)),$$

where C is any abelian group.

For unexplained concepts and notations, we refer the reader to [6], [7], [17], [20].

3. ADJOINT PREENVELOPES AND ADJOINT PRECOVERS OF FUNCTORS

This section is devoted to investigating the properties of adjoint preenvelopes and adjoint precovers of functors in (mod-R, Ab) and ((mod-R)^{op}, Ab). We start with the following definition.

Definition 3.1. (1) Let $\mathcal{C}$ be a class of functors in (mod-R, Ab). A morphism $f : A \to B$ in (mod-R, Ab) is called a $\mathcal{C}$-adjoint preenvelope of A if $B \in \mathcal{C}$ and the sequence of abelian groups $0 \to C^+ \otimes A \to C^+ \otimes B$ is exact for any functor $C \in \mathcal{C}$. $f : A \to B$ in (mod-R, Ab) is called a $\mathcal{C}$-adjoint precover of B if $A \in \mathcal{C}$ and the sequence of abelian groups $0 \to B^+ \otimes C \to A^+ \otimes C$ is exact for any functor $C \in \mathcal{C}$.

(2) A monomorphism $A \to B$ in (mod-R, Ab) is called weakly Z-pure if the sequence of abelian groups $0 \to Z \otimes A \to Z \otimes B$ is exact for any functor $Z \in ((\text{mod-}R)^{op}, \mathrm{Ab})$.

We have the following simple facts.

Remark 3.2. Let $\mathcal{C}$ be a class of functors in (mod-R, Ab).

(1) A monomorphism $f : A \to B$ in (mod-R, Ab) is a $\mathcal{C}$-adjoint preenvelope of A if and only if $B \in \mathcal{C}$ and $f : A \to B$ is weakly C^+-pure for any $C \in \mathcal{C}$.

(2) An epimorphism $f : A \to B$ in (mod-R, Ab) is a $\mathcal{C}$-adjoint precover of B if and only if $A \in \mathcal{C}$ and $f^+ : B^+ \to A^+$ is weakly C-pure for any $C \in \mathcal{C}$.

Proposition 3.3. *Let $\mathcal{C}$ be a class of functors in (mod-R, Ab).*

(1) *Any pure monomorphism $f : A \to B$ in (mod-R, Ab) with $B \in \mathcal{C}$ is a $\mathcal{C}$-adjoint preenvelope of A.*

(2) *Any pure epimorphism $f : A \to B$ in (mod-R, Ab) with $A \in \mathcal{C}$ is a $\mathcal{C}$-adjoint precover of B.*

Proof. (1) For any functor $C \in \mathcal{C}$, $C^+ \in ((\text{mod-}R)^{op}, \text{Ab})$. According to [19, Theorem 2], there is an exact sequence of abelian groups

$$0 \to C^+ \otimes A \to C^+ \otimes B.$$

Then $f : A \to B$ is a $\mathcal{C}$-adjoint preenvelope of A.

(2) Let $C \in \mathcal{C}$ be any functor. Since $f : A \to B$ is a pure epimorphism, we obtain a spilt monomorphism $f^+ : B^+ \to A^+$ by [19, Theorem 2]. So the sequence of abelian groups

$$0 \to B^+ \otimes C \to A^+ \otimes C$$

is exact. Thus $f : A \to B$ is a $\mathcal{C}$-adjoint precover of B. □

In general, the $\mathcal{C}$-adjoint preenvelopes of functors are not always monomorphisms and the $\mathcal{C}$-adjoint precovers of functors are not always epimorphisms. However, we have the following result.

Proposition 3.4. *Let $\mathcal{C}$ be a class of functors in (mod-R, Ab).*

(1) *If C_0^{++} is a cogenerator in (mod-R, Ab) for some $C_0 \in \mathcal{C}$, then any $\mathcal{C}$-adjoint preenvelope in (mod-R, Ab) is a monomorphism.*

(2) *If C_0^+ is a cogenerator in ((mod-R)^{op}, Ab) for some $C_0 \in \mathcal{C}$, then any $\mathcal{C}$-adjoint precover in (mod-R, Ab) is an epimorphism.*

Proof. (1) Let $f : A \to B$ in (mod-R, Ab) be any $\mathcal{C}$-adjoint preenvelope of A. Then for $C_0 \in \mathcal{C}$, the sequence of abelian groups

$$0 \to C_0^+ \otimes A \xrightarrow{1_{C_0^+} \otimes f} C_0^+ \otimes B$$

is exact. Consider the following commutative diagram:

$$\begin{array}{ccccc} (C_0^+ \otimes B)^+ & \longrightarrow & (C_0^+ \otimes A)^+ & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \\ (B, C_0^{++}) & \longrightarrow & (A, C_0^{++}) & & \end{array}$$

Hence $(B, C_0^{++}) \to (A, C_0^{++})$ is an epimorphism. So $f : A \to B$ is a monomorphism since C_0^{++} is a cogenerator.

(2) Let $f : A \to B$ in (mod-R, Ab) be any $\mathcal{C}$-adjoint precover of B. Then for $C_0 \in \mathcal{C}$, the sequence of abelian groups $0 \to B^+ \otimes C_0 \xrightarrow{f^+ \otimes 1_{C_0}} A^+ \otimes C_0$ is exact. Consider the following commutative diagram:

$$\begin{array}{ccccc} (A^+ \otimes C_0)^+ & \longrightarrow & (B^+ \otimes C_0)^+ & \longrightarrow & 0 \\ \cong \downarrow & & \cong \downarrow & & \\ (A^+, C_0^+) & \longrightarrow & (B^+, C_0^+) & & \end{array}$$

Thus $(A^+, C_0^+) \to (B^+, C_0^+)$ is an epimorphism. Since C_0^+ is a cogenerator, $f^+ : B^+ \to A^+$ is a monomorphism. So $f : A \to B$ is an epimorphism. $\square$

The next result considers the closure properties of adjoint preenvelopes and adjoint precovers of functors in (mod-R, Ab). The case of modules has been proved in [14, Proposition 2.4].

Proposition 3.5. *Let $\mathcal{C}$ be a class of functors in (mod-R, Ab).*

(1) *If $\mathcal{C}$ is closed under direct limits and $A_i \to B_i$ in (mod-R, Ab) is a $\mathcal{C}$-adjoint preenvelope of A_i for $i \in I$, where I is a direct set, then $\varinjlim A_i \to \varinjlim B_i$ in (mod-R, Ab) is a $\mathcal{C}$-adjoint preenvelope of $\varinjlim A_i$.*

(2) *If $\mathcal{C}$ is closed under direct sums and $A_i \to B_i$ in (mod-R, Ab) is a $\mathcal{C}$-adjoint preenvelope of A_i for $i \in I$, then $\oplus_{i\in I} A_i \to \oplus_{i \in I} B_i$ is a $\mathcal{C}$-adjoint preenvelope of $\oplus_{i\in I} A_i$.*

(3) *If $\mathcal{C}$ is closed under finite direct sums and $A_i \to B_i$ in (mod-R, Ab) is a $\mathcal{C}$-adjoint precover of B_i for $i = 1, 2, \cdots, n$, then $\oplus_{i=1}^n A_i \to \oplus_{i=1}^n B_i$ is a $\mathcal{C}$-adjoint precover of $\oplus_{i=1}^n B_i$.*

Proof. (1) For any $C \in \mathcal{C}$, $i \in I$, the sequence of abelian groups

$$0 \to C^+ \otimes A_i \to C^+ \otimes B_i$$

is exact by hypothesis. Thus the sequence

$$0 \to \varinjlim (C^+ \otimes A_i) \to \varinjlim (C^+ \otimes B_i)$$

is exact. Consider the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \varinjlim (C^+ \otimes A_i) & \longrightarrow & \varinjlim (C^+ \otimes B_i) \\ & & \cong \downarrow & & \cong \downarrow \\ & & C^+ \otimes \varinjlim A_i & \longrightarrow & C^+ \otimes \varinjlim B_i \end{array}$$

Then we have an exact sequence

$$0 \to C^+ \otimes \varinjlim A_i \to C^+ \otimes \varinjlim B_i,$$

and $\varinjlim B_i \in \mathcal{C}$ by hypothesis. Thus $\varinjlim A_i \to \varinjlim B_i$ in (mod-R, Ab) is a $\mathcal{C}$-adjoint preenvelope of $\varinjlim A_i$.

(2) It is obvious by (1) since direct sums are special direct limits.

(3) For any $C \in \mathcal{C}$, $i \in I$, the sequence of abelian groups $0 \to B_i^+ \otimes C \to A_i^+ \otimes C$ is exact by hypothesis. Then the sequence $0 \to \oplus_{i=1}^n (B_i^+ \otimes C) \to \oplus_{i=1}^n (A_i^+ \otimes C)$ is exact. Consider the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \oplus_{i=1}^n (B_i^+ \otimes C) & \longrightarrow & \oplus_{i=1}^n (A_i^+ \otimes C) \\ & & \downarrow \cong & & \downarrow \cong \\ & & (\oplus_{i=1}^n B_i)^+ \otimes C & \longrightarrow & (\oplus_{i=1}^n A_i)^+ \otimes C \end{array}$$

Then we get an exact sequence $0 \to (\oplus_{i=1}^n B_i)^+ \otimes C \to (\oplus_{i=1}^n A_i)^+ \otimes C$, and $\oplus_{i=1}^n A_i \in \mathcal{C}$ by hypothesis. Hence $\oplus_{i=1}^n A_i \to \oplus_{i=1}^n B_i$ is a $\mathcal{C}$-adjoint precover of $\oplus_{i=1}^n B_i$. $\square$

Recall that a functor $G \in$ (mod-R, Ab) is called FP-injective if we have $\mathrm{Ext}^1(F, G) = 0$ for each finitely presented functor $F \in$ (mod-R, Ab). Equivalently, G is isomorphic to a functor of the form $- \otimes M$, where M is a left R-module [11, Lemma 1.3]. A functor $F \in$ (mod-R, Ab) is said to be FP-projective [12, Definition 2.1] if $\mathrm{Ext}^1(F, G) = 0$ for any FP-injective functor $G \in$ (mod-R, Ab). The flat functors of ((mod-R)^{op}, Ab) have been characterized by Crawley-Boevey [5, Theorem 1.4] as those functors isomorphic to $(-, M)$ for some right R-module M.

PROPOSITION 3.6. *The following are true for any ring R.*

(1) *Every functor in (mod-R, Ab) has an FP-injective adjoint preenvelope.*

(2) *Every functor in (mod-R, Ab) has a flat adjoint precover.*

Proof. (1) Let F be any functor in (mod-R, Ab). Then there exists an exact sequence $0 \to F \to E$ in (mod-R, Ab) with E injective. For any FP-injective functor G in (mod-R, Ab), G^+ is flat in ((mod-R)^{op}, Ab) by [12, Proposition 2.9]. According to [19, Theorem 3], the sequence of abelian groups

$$0 \to G^+ \otimes F \to G^+ \otimes E$$

is exact. So $F \to E$ is an FP-injective adjoint preenvelope in (mod-R, Ab).

(2) Let F be any functor in (mod-R, Ab). Then there exists an exact sequence $H \to F \to 0$ in (mod-R, Ab) with H projective. For any flat functor Q in (mod-R, Ab), the sequence of abelian groups

$$0 \to F^+ \otimes Q \to H^+ \otimes Q$$

is exact. So $H \to F$ is a flat adjoint precover in (mod-R, Ab). □

Remark 3.7. (1) The class of FP-injective functors satisfies the condition of Proposition 3.5 (1) by [12, Proposition 2.3] and Proposition 3.6;

(2) The class of flat functors satisfies the condition of Proposition 3.5 (3) by Proposition 3.6;

(3) The class of FP-projective functors satisfies the condition of Proposition 3.5 (2), (3).

Definition 3.8 ([13, Definition 3.1]). (1) A functor $Q \in ((\text{mod-}R)^{op}, \text{Ab})$ is called $\mathfrak{F}$-projective if $\text{Ext}^1(Q, H) = 0$ for any flat functor $H \in ((\text{mod-}R)^{op}, \text{Ab})$;

(2) A functor $F \in ((\text{mod-}R)^{op}, \text{Ab})$ is called $\mathfrak{F}$-flat if $\text{Tor}_1(F, G) = 0$ for any FP-injective functor $G \in (\text{mod-}R, \text{Ab})$;

(3) A functor $T \in (\text{mod-}R, \text{Ab})$ is called $\mathfrak{F}$-injective if $\text{Ext}^1(G, T) = 0$ for any FP-injective functor $G \in (\text{mod-}R, \text{Ab})$.

The next lemma is frequently used in the sequel.

Lemma 3.9 ([13, Lemma 2.1]). *Let R be a ring and $n \geqslant 0$.*

(1) $(\text{Tor}_n(F, G), C) \cong \text{Ext}^n(F, (G(\cdot), C)) \cong \text{Ext}^n(G, (F(\cdot), C))$ *for any* $F \in ((\text{mod-}R)^{op}, \text{Ab})$, $G \in (\text{mod-}R, \text{Ab})$, *and any injective abelian group* C*;*

(2) $\text{Tor}_n((G(\cdot), C), F) \cong (\text{Ext}^n(F, G), C)$ *for any finitely presented functor* $F \in (\text{mod-}R, \text{Ab})$, $G \in (\text{mod-}R, \text{Ab})$, *and any injective abelian group* C*;*

(3) *If R is a right coherent ring, then* $\text{Tor}_n(F, (G(\cdot), C)) \cong (\text{Ext}^n(F, G), C)$ *for any finitely presented functor* $F \in ((\text{mod-}R)^{op}, \text{Ab})$, $G \in ((\text{mod-}R)^{op}, \text{Ab})$, *and any injective abelian group* C*.*

Let $(\text{mod-}R, \text{Ab})^{\text{fp}}$ denote the full subcategory of (mod-R, Ab) consisting of finitely presented functors. Then, by [16, Proposition 2.27], an injective object of $(\text{mod-}R, \text{Ab})^{\text{fp}}$ is a finitely presented FP-injective object of (mod-R, Ab).

In the following, we will see that those functors in Definition 3.8 are closely related to cokernels of a flat adjoint preenvelope in $((\text{mod-}R)^{op}, \text{Ab})$ or kernels of an FP-injective adjoint precover in (mod-R, Ab).

Theorem 3.10. *Let R be a ring. The following assertions hold.*

(1) *Let R be a right coherent ring and Q a finitely presented functor in $((\text{mod-}R)^{op}, \text{Ab})$. Then Q is $\mathfrak{F}$-projective if and only if Q is a cokernel of a flat adjoint preenvelope in $((\text{mod-}R)^{op}, \text{Ab})$.*

(2) *Let F be any functor in (mod-R, Ab). Then F^+ is $\mathfrak{F}$-flat in $((\text{mod-}R)^{op}, \text{Ab})$ if and only if F is a kernel of an FP-injective adjoint precover in (mod-R, Ab).*

(3) *Let F be a finitely presented functor in (mod-R, Ab). Then F is $\mathfrak{F}$-injective if and only if F is a kernel of an injective adjoint precover in (mod-R, $Ab)^{fp}$.*

Proof. (1) "$\Rightarrow$" For a $\mathfrak{F}$-projective functor $Q \in ((\text{mod-}R)^{op}, \text{Ab})$, there exists an exact sequence

$$0 \to A \to B \to Q \to 0$$

in $((\text{mod-}R)^{op}, \text{Ab})$ with B flat by [18, Corollary 2.11]. For any flat functor F, the induced sequence

$$\text{Tor}_1(Q, F^+) \to A \otimes F^+ \to B \otimes F^+$$

is exact. According to Lemma 3.9, there is an isomorphism

$$\text{Tor}_1(Q, F^+) \cong \text{Ext}^1(Q, F)^+ = 0.$$

Then $A \to B$ is a flat adjoint preenvelope of A in $((\text{mod-}R)^{op}, \text{Ab})$.

"$\Leftarrow$" Let $f : A \to B$ be any flat adjoint preenvelope in $((\text{mod-}R)^{op}, \text{Ab})$. By hypothesis, we have an exact sequence $A \xrightarrow{f} B \to Q \to 0$ in $((\text{mod-}R)^{op}$, Ab) with $Q = \text{Coker} f$ as well as a monomorphism $\iota : \text{Im} f \to B$ and an epimorphism $\pi : A \to \text{Im} f$, such that $f = \iota\pi$. For any flat functor $F \in ((\text{mod-}R)^{op}, \text{Ab})$, $F^+ \in (\text{mod-}R, \text{Ab})$. Applying the functor $- \otimes F^+$ to the above exact sequence yields the following commutative diagram.

$$\begin{array}{ccccc} 0 & \longrightarrow & A \otimes F^+ & \xrightarrow{f \otimes 1_{F^+}} & B \otimes F^+ \\ & & {\scriptstyle \pi \otimes 1_{F^+}} \downarrow & \nearrow {\scriptstyle \iota \otimes 1_{F^+}} & \\ & & \text{Im} f \otimes F^+ & & \end{array}$$

Then we get an exact sequence of abelian groups $0 \to \text{Im} f \otimes F^+ \xrightarrow{\iota \otimes 1_{F^+}} B \otimes F^+$ since $\text{A} \otimes F^+ \xrightarrow{f \otimes 1_{F^+}} B \otimes F^+$ is a monomorphism and $\text{A} \otimes F^+ \xrightarrow{\pi \otimes 1_{F^+}} \text{Im} f \otimes F^+$ is an isomorphism. Again applying the functor $- \otimes F^+$ to exact sequence $0 \to \text{Im} f \to B \to Q \to 0$, which yields the following exact sequence

$$0 = \text{Tor}_1(B, F^+) \to \text{Tor}_1(Q, F^+) \to \text{Im} f \otimes F^+ \xrightarrow{\iota \otimes 1_{F^+}} B \otimes F^+.$$

Thus $\text{Tor}_1(Q, F^+) = 0$. By Lemma 3.9, we have

$$\text{Ext}^1(Q, F)^+ \cong \text{Tor}_1(Q, F^+) = 0.$$

So $\text{Ext}^1(Q, F) = 0$ for any flat functor $F \in ((\text{mod-}R)^{op}, \text{Ab})$. Thus Q is $\mathfrak{F}$-projective in $((\text{mod-}R)^{op}, \text{Ab})$.

(2) and (3) are similar to the proof of (1) and we omit them. □

Remark 3.11. Mao proved that Q is $\mathfrak{F}$-projective in ((mod-$R)^{op}$, Ab) if and only if Q is a cokernel of a flat preenvelope in ((mod-$R)^{op}$, Ab), see [13, Statement 3.6]; a cokernel F of a flat preenvelope in ((mod-$R)^{op}$, Ab) is $\mathfrak{F}$-flat, the converse holds if F is finitely presented, see [13, Theorem 3.8]; T is $\mathfrak{F}$-injective in (mod-R, Ab) if and only if T is a kernel of an FP-injective precover in (mod-R, Ab), see [13, Theorem 3.3].

Corollary 3.12. *If F is $\mathfrak{F}$-flat in* ((mod-$R)^{op}$, Ab)*, then F is a cokernel of a flat adjoint preenvelope in* ((mod-$R)^{op}$, Ab)*. The converse holds if R is a right coherent ring and F is a finitely presented functor in* ((mod-$R)^{op}$, Ab)*.*

Proof. We firstly assume that F is $\mathfrak{F}$-flat in ((mod-$R)^{op}$, Ab). Then there exists an exact sequence

$$0 \to A \to B \to F \to 0$$

in ((mod-$R)^{op}$, Ab) with B flat by [18, Corollary 2.11]. For any flat functor $G \in$ ((mod-$R)^{op}$, Ab), $G^+ \in$ (mod-R, Ab) is FP-injective by [12, Proposition 2.9]. Applying the functor $- \otimes G^+$ to the above exact sequence, we obtain the following exact sequence

$$0 = \mathrm{Tor}_1(F, G^+) \to A \otimes G^+ \to B \otimes G^+.$$

Thus $A \to B$ is a flat adjoint preenvelope in ((mod-$R)^{op}$, Ab).

The converse is obvious by Theorem 3.10 (1) since any $\mathfrak{F}$-projective functor in ((mod-$R)^{op}$, Ab) is $\mathfrak{F}$-flat by [13, Corollary 3.9]. □

4. RELATIONSHIPS BETWEEN ADJOINT PREENVELOPES, ADJOINT PRECOVERS AND PREENVELOPES AND PRECOVERS

Let $\mathcal{C}$ be a class of functors in (mod-R, Ab) and $\mathcal{D}$ a class of functors in ((mod-$R)^{op}$, Ab). We write

$$^{\top}\mathcal{C} = \{F \in ((\text{mod-}R)^{op}, \text{Ab}) \mid \mathrm{Tor}_1(F, C) = 0 \text{ for any functor } C \in \mathcal{C}\},$$
$$\mathcal{D}^{\top} = \{F \in (\text{mod-}R, \text{Ab}) \mid \mathrm{Tor}_1(D, F) = 0 \text{ for any functor } D \in \mathcal{D}\}.$$

Note that if $\mathcal{C}$ is a class of functors in (mod-R, Ab), then $\mathcal{C}^+ \subseteq$ ((mod-$R)^{op}$, Ab), where $\mathcal{C}^+ = \{C^+ \in ((\text{mod-}R)^{op}, \text{Ab}) \mid C \text{ is any functor in } \mathcal{C}\}$.

Lemma 4.1. *Let $\mathcal{C}$ be a class of functors in (mod-R, Ab) and $0 \to F \to G \to H \to 0$ an exact sequence in (mod-R, Ab) with $G \in \mathcal{C}$.*

(1) *If $H \in (\mathcal{C}^+)^{\top}$, then $F \to G$ is a $\mathcal{C}$-adjoint preenvelope of F.*

(2) *If $F^+ \in {}^{\top}\mathcal{C}$, then $G \to H$ is a $\mathcal{C}$-adjoint precover of H.*

Proof. (1) For any $C \in \mathcal{C}$, $C^+ \in \mathcal{C}^+$ in ((mod-R)^{op}, Ab). Applying the functor $C^+ \otimes -$ to the exact sequence $0 \to F \to G \to H \to 0$ yields the exact sequence of abelian groups

$$0 = \mathrm{Tor}_1(C^+, H) \to C^+ \otimes F \to C^+ \otimes G.$$

Thus $F \to G$ is a $\mathcal{C}$-adjoint preenvelope of F.

(2) The exact sequence $0 \to F \to G \to H \to 0$ in (mod-R, Ab) induces an exact sequence $0 \to H^+ \to G^+ \to F^+ \to 0$ in ((mod-R)^{op}, Ab). For any $C \in \mathcal{C}$, we obtain an exact sequence of abelian groups

$$0 = \mathrm{Tor}_1(F^+, C) \to H^+ \otimes C \to G^+ \otimes C.$$

Thus $G \to H$ is a $\mathcal{C}$-adjoint precover of H. □

Let $\mathcal{C}$ be a class of functors in (mod-R, Ab). We write

$$^{\perp}\mathcal{C} = \{T \in (\text{mod-}R, \text{Ab}) \mid \mathrm{Ext}^1(T, C) = 0 \text{ for any functor } C \in \mathcal{C}\},$$
$$\mathcal{C}^{\perp} = \{T \in (\text{mod-}R, \text{Ab}) \mid \mathrm{Ext}^1(C, T) = 0 \text{ for any functor } C \in \mathcal{C}\}.$$

Recall from [6] that a functor $F \in$ (mod-R, Ab) is said to have a special $\mathcal{C}$-preenvelope provided that there exists an exact sequence $0 \to F \to G \to H \to 0$ in (mod-R, Ab) with $G \in \mathcal{C}$ and $H \in {}^{\perp}\mathcal{C}$. A functor $F \in$ (mod-R, Ab) is said to have a special $\mathcal{C}$-precover provided that there exists an exact sequence $0 \to Q \to G \to F \to 0$ in (mod-R, Ab) with $G \in \mathcal{C}$ and $Q \in \mathcal{C}^{\perp}$. Compared with above, we introduce the corresponding Tor edition, and the definition as follows.

Definition 4.2. Let $\mathcal{C}$ be a class of functors in (mod-R, Ab). A functor $F \in$ (mod-R, Ab) is said to have a special $\mathcal{C}$-adjoint preenvelope provided that there exists an exact sequence $0 \to F \to G \to H \to 0$ in (mod-R, Ab) with $G \in \mathcal{C}$ and $H \in (\mathcal{C}^+)^{\top}$. A functor $F \in$ (mod-R, Ab) is said to have a special $\mathcal{C}$-adjoint precover provided that there exists an exact sequence $0 \to Q \to G \to F \to 0$ in (mod-R, Ab) with $G \in \mathcal{C}$ and $Q^+ \in {}^{\top}\mathcal{C}$.

Furthermore, we will investigate the relationships between (resp. special) adjoint preenvelopes (adjoint precovers) and (resp. special) preenvelopes (precovers).

THEOREM 4.3. *Let $\mathcal{C}$ be a class of functors in (mod-R, Ab).*

(1) *If $\mathcal{C}^{++} \subseteq \mathcal{C}$, and $\varphi : F \to G$ in (mod-R, Ab) is a (resp. special) $\mathcal{C}$-preenvelope of F, then $\varphi : F \to G$ is a (resp. special) $\mathcal{C}$-adjoint preenvelope of F.*

(2) *If $\mathcal{C}^{++} \subseteq \mathcal{C}$, and $\varphi : F \to G$ in (mod-R, Ab) with $F \in \mathcal{C}$ is a (resp. special) $\mathcal{C}$-adjoint precover of G if and only if $\varphi^{++} : F^{++} \to G^{++}$ is a (resp. special) $\mathcal{C}$-precover of G^{++}.*

Proof. (1) Firstly, we assume that $\varphi : F \to G$ is a $\mathcal{C}$-preenvelope of F. For any functor $C \in \mathcal{C}$, there is an exact sequence $(G, C) \to (F, C) \to 0$. By hypothesis, $\mathcal{C}^{++} \subseteq \mathcal{C}$. Then we have an exact sequence $(G, C^{++}) \to (F, C^{++}) \to 0$. Consider the following commutative diagram:

$$\begin{array}{ccccc} (G, C^{++}) & \longrightarrow & (F, C^{++}) & \longrightarrow & 0 \\ \cong \downarrow & & \cong \downarrow & & \\ (C^{+} \otimes G)^{+} & \longrightarrow & (C^{+} \otimes F)^{+} & & \end{array}$$

So the sequence $(C^{+} \otimes G)^{+} \to (C^{+} \otimes F)^{+} \to 0$ is exact, which implies the exactness of the sequence $0 \to C^{+} \otimes F \to C^{+} \otimes G$. Thus $\varphi : F \to G$ is a $\mathcal{C}$-adjoint preenvelope of F.

Secondly, we assume that $\varphi : F \to G$ is a special $\mathcal{C}$-preenvelope of F. Then there exists an exact sequence $0 \to F \to G \to H \to 0$ in (mod-R, Ab) with $H \in {}^{\perp}\mathcal{C}$. For any $C \in \mathcal{C}$, there is an isomorphism by Lemma 3.9,

$$\mathrm{Tor}_1(C^{+}, H)^{+} \cong \mathrm{Ext}^1(H, C^{++}) = 0.$$

So $\mathrm{Tor}_1(C^{+}, H) = 0$. Thus $\varphi : F \to G$ is a special $\mathcal{C}$-adjoint preenvelope of F.

(2) Firstly, the morphism $\varphi : F \to G$ in (mod-R, Ab) with $F \in \mathcal{C}$ is a $\mathcal{C}$-adjoint precover of G if and only if $0 \to G^{+} \otimes C \to F^{+} \otimes C$ is exact if and only if $(F^{+} \otimes C)^{+} \to (G^{+} \otimes C)^{+} \to 0$ is exact for any $C \in \mathcal{C}$. Consider the following commutative diagram:

$$\begin{array}{ccccc} (F^{+} \otimes C)^{+} & \longrightarrow & (G^{+} \otimes C)^{+} & \longrightarrow & 0 \\ \cong \downarrow & & \cong \downarrow & & \\ (C, F^{++}) & \longrightarrow & (C, G^{++}) & & \end{array}$$

So the sequence $(C, F^{++}) \to (C, G^{++})$ is exact if and only if $\varphi^{++} : F^{++} \to G^{++}$ is a $\mathcal{C}$-precover of G^{++} as $F^{++} \in \mathcal{C}$.

Secondly, the morphism $\varphi : F \to G$ in (mod-R, Ab) is a special $\mathcal{C}$-adjoint precover of G if and only if there exists an exact sequence $0 \to Q \to F \to G \to 0$ in (mod-R, Ab) with $Q^{+} \in {}^{\top}\mathcal{C}$. For any $C \in \mathcal{C}$, again by Lemma 3.9, we have

$$\mathrm{Ext}^1(C, Q^{++}) \cong \mathrm{Tor}_1(Q^{+}, C)^{+} = 0.$$

The above are equivalent to the condition that there exists an exact sequence $0 \to Q^{++} \to F^{++} \to G^{++} \to 0$ in (mod-R, Ab) with $Q^{++} \in \mathcal{C}^{\perp}$, which is equivalent to $\varphi^{++} : F^{++} \to G^{++}$ is a special $\mathcal{C}$-precover of G^{++}. □

Remark 4.4. (1) If $\mathcal{C}$ is a class of FP-injective functors, then Theorem 4.3 holds by [12, Proposition 2.9].

(2) Let R be a right coherent ring. If $\mathcal{C}$ is a class of flat functors, then Theorem 4.3 holds by [13, Corollary 2.2].

(3) It is known that every functor in (mod-R, Ab) has an (resp. special) FP-injective preenvelope by [18, Remark 3.2] and [12, Theorem 2.12]. Thus every functor F in (mod-R, Ab) has an (resp. special) FP-injective adjoint preenvelope by Theorem 4.3 (1).

Next, we discuss the connections between adjoint preenvelopes and precovers as well as adjoint precovers and preenvelopes.

THEOREM 4.5. *Let $\mathcal{C}$ be a class of functors in (mod-R, Ab) and $\varphi : F \to G$ a morphism in (mod-R, Ab).*

(1) $\varphi : F \to G$ in (mod-R, Ab) is a (resp. special) $\mathcal{C}$-adjoint preenvelope of F if and only if $\varphi^+ : G^+ \to F^+$ in ((mod-R)^{op}, Ab) is a (resp. special) $\mathcal{C}^+$-precover of F^+.

(2) $\varphi : F \to G$ in (mod-R, Ab) is a (resp. special) $\mathcal{C}$-adjoint precover of G if and only if $\varphi^+ : G^+ \to F^+$ in ((mod-R)^{op}, Ab) is a (resp. special) $\mathcal{C}^+$-preenvelope of G^+.

Proof. (1) Firstly, the morphism $\varphi : F \to G$ in (mod-R, Ab) is a $\mathcal{C}$-adjoint preenvelope of F if and only if the sequence of abelian groups $0 \to C^+ \otimes F \to C^+ \otimes G$ is exact if and only if $(C^+ \otimes G)^+ \to (C^+ \otimes F)^+ \to 0$ is exact for any $C \in \mathcal{C}$. Consider the following commutative diagram:

$$\begin{array}{ccccc} (C^+ \otimes G)^+ & \longrightarrow & (C^+ \otimes F)^+ & \longrightarrow & 0 \\ \cong \downarrow & & \cong \downarrow & & \\ (C^+, G^+) & \longrightarrow & (C^+, F^+) & & \end{array}$$

So the sequence $(C^+, G^+) \to (C^+, F^+) \to 0$ is exact if and only if $\varphi^+ : G^+ \to F^+$ is a $\mathcal{C}^+$-precover of F^+.

Secondly, the morphism $\varphi : F \to G$ is a special $\mathcal{C}$-adjoint preenvelope of F if and only if there exists an exact sequence $0 \to F \to G \to H \to 0$ with $H \in (\mathcal{C}^+)^\top$. For any $C \in \mathcal{C}$, there is an isomorphism by Lemma 3.9,

$$\mathrm{Ext}^1(C^+, H^+) \cong \mathrm{Tor}_1(C^+, H)^+ = 0.$$

The above are equivalent to the condition that there exists an exact sequence $0 \to H^+ \to G^+ \to F^+ \to 0$ in ((mod-R)^{op}, Ab) with $H^+ \in (\mathcal{C}^+)^\perp$, which is equivalent to $\varphi^+ : G^+ \to F^+$ is a special $\mathcal{C}^+$-precover of F^+.

(2) Firstly, the morphism $\varphi : F \to G$ in (mod-R, Ab) is a $\mathcal{C}$-adjoint precover of F if and only if the sequence of abelian groups $0 \to G^+ \otimes C \to$

$F^+ \otimes C$ is exact if and only if $(F^+ \otimes C)^+ \to (G^+ \otimes C)^+ \to 0$ is exact for any $C \in \mathcal{C}$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
(F^+ \otimes C)^+ & \longrightarrow & (G^+ \otimes C)^+ & \longrightarrow & 0 \\
\cong\big\downarrow & & \cong\big\downarrow & & \\
(F^+, C^+) & \longrightarrow & (G^+, C^+) & &
\end{array}$$

So the sequence $(F^+, C^+) \to (G^+, C^+) \to 0$ is exact if and only if $\varphi^+ : G^+ \to F^+$ is a $\mathcal{C}^+$-preenvelope of G^+.

Secondly, the morphism $\varphi : F \to G$ is a special $\mathcal{C}$-adjoint precover of G if and only if there exists an exact sequence $0 \to Q \to F \to G \to 0$ with $Q^+ \in {}^{\top}\mathcal{C}$. For any $C \in \mathcal{C}$, again by Lemma 3.9, we have

$$\operatorname{Ext}^1(Q^+, C^+) \cong \operatorname{Tor}_1(Q^+, C)^+ = 0.$$

The above are equivalent to the condition that there exists an exact sequence $0 \to G^+ \to F^+ \to Q^+ \to 0$ in $((\text{mod-}R)^{op}, \text{Ab})$ with $Q^+ \in {}^{\perp}(\mathcal{C}^+)$, which is equivalent to $\varphi^+ : G^+ \to F^+$ is a special $\mathcal{C}^+$-preenvelope of G^+. $\square$

Proposition 4.6. *Let $\mathcal{C}$ be a class of functors in (mod-R, Ab). If every functor F in (mod-R, Ab) has a special $\mathcal{C}$-adjoint preenvelope, then F has a special $(\mathcal{C}^+)^{\top}$-adjoint precover.*

Proof. Let F be any functor in (mod-R, Ab). Then there is an exact sequence $0 \to K \to G \to F \to 0$ in (mod-R, Ab) with G projective. By hypothesis, the functor K has a special $\mathcal{C}$-adjoint preenvelope. Then there exists an exact sequence $0 \to K \to H \to L \to 0$ with $H \in \mathcal{C}$, and $L \in (\mathcal{C}^+)^{\top}$. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & H & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & G & \longrightarrow & T & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & F & = & F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

For any $C \in \mathcal{C}$, applying the functor $C^+ \otimes -$ to the exact sequence $0 \to G \to T \to L \to 0$ gives rise to an exact sequence

$$0 = \mathrm{Tor}_1(C^+, G) \to \mathrm{Tor}_1(C^+, T) \to \mathrm{Tor}_1(C^+, L) = 0.$$

So $T \in (\mathcal{C}^+)^\top$. And $H^+ \in {}^\top((\mathcal{C}^+)^\top)$ since $H^+ \in \mathcal{C}^+$ and $\mathcal{C}^+ \subseteq {}^\top((\mathcal{C}^+)^\top)$. Thus $T \to F$ is a special $(\mathcal{C}^+)^\top$-adjoint precover. □

We study the relationships between preenvelopes (precovers) of modules and adjoint preenvelopes (adjoint precovers) of functors in (mod-R, Ab) (resp. ((mod-R)^{op}, Ab)).

Proposition 4.7. *Let $f : M \to N$ be a left R-module homomorphism with M, N finitely presented. Then the following are equivalent:*

(1) $M \xrightarrow{f} N$ is a pure-injective preenvelope of M;

(2) $- \otimes M \xrightarrow{-\otimes f} - \otimes N$ is an injective adjoint preenvelope of $- \otimes M$ in (mod-R, Ab).

Proof. Let E be any pure-injective module. Then $- \otimes E$ is an injective object in (mod-R, Ab) by [9, Proposition 1.2] since the functors $- \otimes M$ and $- \otimes N$ are finitely presented by [1, Proposition 6.1]. Consider the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & (M,E)^+ & \longrightarrow & (N,E)^+ \\ & & \downarrow \cong & & \downarrow \cong \\ & & (-\otimes M, -\otimes E)^+ & \longrightarrow & (-\otimes N, -\otimes E)^+ \\ & & \downarrow \cong & & \downarrow \cong \\ & & (-\otimes E)^+ \otimes (-\otimes M) & \longrightarrow & (-\otimes E)^+ \otimes (-\otimes N) \end{array}$$

The left R-module homomorphism $M \xrightarrow{f} N$ is a pure-injective preenvelope of M if and only if the sequence $(N, E) \to (M, E) \to 0$ is exact if and only if $0 \to (M, E)^+ \to (N, E)^+$ is exact. So the sequence of abelian groups

$$0 \to (-\otimes E)^+ \otimes (-\otimes M) \to (-\otimes E)^+ \otimes (-\otimes N)$$

is exact if and only if $- \otimes M \xrightarrow{-\otimes f} - \otimes N$ is an injective adjoint preenvelope of $- \otimes M$ in (mod-R, Ab). □

Proposition 4.8. *Let R be a left coherent ring and $f : M \to N$ a left R-module homomorphism with M, N finitely presented. Then the following are equivalent:*

(1) $M \xrightarrow{f} N$ is a finitely presented pure-projective precover of N;

(2) $(-, M) \xrightarrow{(-,f)} (-, N)$ is a finitely presented projective adjoint precover of $(-, N)$ in ((R-mod)^{op}, Ab).

Proof. For any finitely presented pure-projective left R-module P, the functor $(-, P)$ is a finitely presented projective object in ((mod-R)^{op}, Ab). Consider the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & (P, N)^{+} & \longrightarrow & (P, M)^{+} \\ & & \cong\downarrow & & \cong\downarrow \\ & & ((-, P), (-, N))^{+} & \longrightarrow & ((-, P), (-, M))^{+} \\ & & \cong\downarrow & & \cong\downarrow \\ & & (-, P) \otimes (-, N)^{+} & \longrightarrow & (-, P) \otimes (-, M)^{+} \end{array}$$

The left R-module homomorphism $M \xrightarrow{f} N$ is a finitely presented pure-projective precover of N if and only if the sequence $(P, M) \to (P, N) \to 0$ is exact if and only if $0 \to (P, N)^{+} \to (P, M)^{+}$ is exact. So the sequence of abelian groups $0 \to (-, P) \otimes (-, N)^{+} \to (-, P) \otimes (-, M)^{+}$ is exact if and only if $(-, M) \xrightarrow{(-,f)} (-, N)$ is a finitely presented projective adjoint precover of $(-, N)$ in ((R-mod)^{op}, Ab). $\square$

Finally, we discuss the connections between adjoint preenvelopes and adjoint precovers for some special classes of functors.

Let F be any functor of (mod-R, Ab). There exists a standard morphism $\delta_F : F \to F^{++}$ defined by $(\delta_F)_M(x)(\lambda)$ for any $M \in$ mod-R, $x \in F(M)$, $\lambda \in F^{+}(M)$. If F is an FP-injective functor, then δ_F is a pure monomorphism.

PROPOSITION 4.9. *Let $f : A \to B$ be a morphism in (mod-R, Ab) with A FP-injective. Then $f : A \to B$ is an FP-injective adjoint precover of B if and only if $f^{+} : B^{+} \to A^{+}$ in ((mod-R)^{op}, Ab) is a flat adjoint preenvelope of B^{+}.*

Proof. "⇒" Let F be any flat functor in ((mod-R)^{op}, Ab).

Then $F^{+} \in$ (mod-R, Ab) is FP-injective by [12, Proposition 2.9]. By hypothesis, the sequence of abelian groups

$$0 \to B^{+} \otimes F^{+} \to A^{+} \otimes F^{+}$$

is exact. Again by [12, Proposition 2.9], $A^{+} \in$ (mod-R)^{op}, Ab) is flat. Thus $f^{+} : B^{+} \to A^{+}$ is a flat adjoint preenvelope of B^{+}.

"⇐" Let G be any FP-injective functor in (mod-R, Ab).

Then $G^{+} \in$ ((mod-R)^{op}, Ab) is flat. By hypothesis, the sequence of abelian groups

$$0 \to B^{+} \otimes G^{++} \to A^{+} \otimes G^{++}$$

is exact. Consider the following commutative diagram:

$$
\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & B^{+} \otimes G & \longrightarrow & A^{+} \otimes G \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & B^{+} \otimes G^{++} & \longrightarrow & B^{+} \otimes G^{++}
\end{array}
$$

So the sequence $0 \to B^{+} \otimes G \to A^{+} \otimes G$ is exact. Thus $f : A \to B$ is an FP-injective adjoint precover of B. $\square$

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