ON CLEAN FREE MODULES AND A CHARACTERIZATION OF CLEAN RINGS

ESMAEIL ROSTAMI and SINA HEDAYAT

Communicated by Lucian Beznea

In this paper, the concept of clean ring is generalized to modules. We call a free R-module, R^n, clean, whenever every element of R^n can be written as the sum of a unimodular and an idempotent row. We show that when R is Noetherian, the R-module R^n is clean if and only if R can be expressed as a finite direct product of indecomposable rings R_i, say $R = \bigoplus_{i=1}^{t} R_i$, such that each R_i has at most $2^n - 1$ maximal ideals. We also give a new characterization of clean rings.

AMS 2020 Subject Classification: 13C10, 13E05.

Key words: clean ring, clean free module, indecomposable ring, unimodular row.

1. INTRODUCTION

Throughout this paper, all rings are assumed to be commutative with identity, $\mathrm{Nil}(R)$, $\mathrm{J}(R)$ and $\mathrm{Max}(R)$ are nilradical, Jacobson radical and the set of all maximal ideals of R, respectively. Nicholson in [6] studied lifting idempotents in a noncommutative ring and he considered a class of rings in which idempotents can be lifted by every left ideal. Then he defined a ring R to be clean ring if every element is the sum of an idempotent and a unit, and he showed that in a clean ring, idempotents can be lifted by every left ideal, and the converse is true if idempotents are central. After Nicholson many authors considered clean rings and gave several characterizations for this class of rings. Local rings and zero-dimensional rings are clean. Also every direct product of clean rings and homomorphic image of a clean ring is a clean ring. For more information and details see [1, 5]. In Section 3 of this paper, we give a new characterization of commutative clean rings.

In [3], the authors call a module clean when its endomorphism ring is clean. If, however, we only want to consider free modules, it is possible to give a different approach using unimodular rows. In fact, when we consider $R^{(\mathcal{I})}$ as a free R-module, a unimodular row is a natural extension of unit element of R. Thus in Section 2 of this note, we extend the concept of clean ring to free modules of arbitrary rank by using unimodular rows. Then, under some restrictions on the ring, we give necessary and sufficient conditions under which a free module is clean.

MATH. REPORTS **25(75)** (2023), *1*, 103–112
doi: 10.59277/mrar.2023.25.75.1.103

2. CLEAN FREE MODULES

In this section, for a positive integer n, we consider $R^n = R \oplus R \oplus ... \oplus R$, n times, as an R-module, and we say that $(a_1, ..., a_n) \in R^n$ is a unimodular row if there exist $b_1, b_2, ..., b_n \in R$ such that $\sum_{i=1}^{n} a_i b_i = 1_R$, or equivalently, $\langle a_1, ..., a_n \rangle = R$.

Definition 2.1. We say that $(e_1, ..., e_n) \in R^n$ is an *idempotent row* if its components are idempotent elements of R (i.e., $e_i^2 = e_i$ for $i = 1, 2, ..., n$).

Definition 2.2. Let R be a ring and n a positive integer. The R-module R^n is called *clean* if every row in R^n is the sum of a unimodular row and an idempotent row.

When we consider R as an R-module, the unimodular rows of R-module $R = R^1$ are precisely unit elements of R. Thus R is a clean R-module if and only if R is a clean ring. We can extend the above definitions to free R-modules with infinite basis, as follows: let F be the free R-module $R^{(\mathcal{I})}$ with an infinite basis $\{e_i\}_{i\in\mathcal{I}}$. Then an infinite row $(a_i)_{i\in\mathcal{I}}$ in $R^{(\mathcal{I})}$ is called a unimodular row (respectively, an idempotent row), if $\langle \{a_i\}_{i\in\mathcal{I}} \rangle = R$ (respectively, $a_i^2 = a_i$ for all $i \in \mathcal{I}$). In this case, the R-module $R^{(\mathcal{I})}$ is called clean if every row in $R^{(\mathcal{I})}$ is the sum of a unimodular and an idempotent.

PROPOSITION 2.3. *Let $\mathcal{I}$ be an infinite set. Then $R^{(\mathcal{I})}$ is always a clean R-module.*

Proof. Let $\alpha = (a_i)_{i\in\mathcal{I}}$ be an arbitrary row in $R^{(\mathcal{I})}$. For $j \in \mathcal{I}$, we define $\alpha(j)$ to be a row in $R^{(\mathcal{I})}$ whose components are equal to the components of α except in the j^{th} position in which it has -1. Clearly $\alpha(j)$ is a unimodular row. Now let j be an index in $\mathcal{I}$ such that $a_j = 0$. Then $\alpha = (a_i)_{i\in\mathcal{I}} = \alpha(j) + e_j$, where e_j has all components 0 except the j^{th} component, which is 1. So every row in $R^{(\mathcal{I})}$ is the sum of a unimodular and an idempotent, as desired. □

Therefore, in the rest of this article, we focus on free modules of finite rank.

THEOREM 2.4. *Let n be a positive integer and R a ring. Then R has at most $2^n - 1$ maximal ideals if and only if every row of R^n is the sum of a unimodular row and a row consists of 0's and 1's.*

Furthermore, if R is indecomposable, R^n is clean if and only if R has at most $2^n - 1$ maximal ideals.

Proof. ($\Rightarrow$) Let $(r_1, r_2, ..., r_n) \in R^n$. Taking the set $\{0, 1\}^n$ as an index set, we define $I_{(i_1,i_2,...,i_n)}$ to be the ideal of R generated by $r_1 - i_1, r_2 - i_2, ..., r_n - i_n$

for every $(i_1, i_2, ..., i_n) \in \{0,1\}^n$. Let $(i_1, i_2, ..., i_n) \neq (j_1, j_2, ..., j_n)$ be in $\{0,1\}^n$ and $\mathfrak{m}$ be a maximal ideal of R, such that $I_{(i_1,i_2,...,i_n)} + I_{(j_1,j_2,...,j_n)} \subseteq \mathfrak{m}$. Thus $j_k - i_k = (a_k - i_k) - (a_k - j_k) \in \mathfrak{m}$ for $k = 1, 2, ..., n$. Now since $(i_1, i_2, ..., i_n) \neq (j_1, j_2, ..., j_n)$, there exists $k \in \{1, 2, ..., n\}$, such that $j_k - i_k = 1$ or $j_k - i_k = -1$, a contradiction. Thus,

$$\{I_{(i_1,i_2,...,i_n)} : (i_1, i_2, ..., i_n) \in \{0,1\}^n\}$$

is a family with 2^n pairwise comaximal ideals of R. Since R has at most $2^n - 1$ maximal ideals, there exists $(i_1, i_2, ..., i_n) \in \{0,1\}^n$ such that $I_{(i_1,i_2,...,i_n)} = R$. Thus $(a_1, a_2, ..., a_n) = (a_1 - i_1, a_2 - i_2, ..., a_n - i_n) + (i_1, i_2, ..., i_n)$, where $(a_1 - i_1, a_2 - i_2, ..., a_n - i_n)$ is a unimodular row and the row $(i_1, i_2, ..., i_n)$ consists of 0's and 1's.

($\Leftarrow$) Assuming that R has at least 2^n maximal ideals, we can choose a subset $A = \{\mathfrak{m}_{(i_1,i_2,...,i_n)} \mid (i_1, i_2, ..., i_n) \in \{0,1\}^n\}$ of $\mathrm{Max}(R)$ with 2^n distinct elements which are indexed by $\{0,1\}^n$. For every $j \in \{1, 2, ..., n\}$, we define A_j to be the intersection of all elements of A whose index has 0 in j^{th} position and A_j' to be the intersection of all elements of A whose index has 1 in j^{th} position. Clearly, for all $j \in \{1, 2, ..., n\}$, A_j and A_j' are comaximal ideals of R, so there exist $\alpha_j \in A_j$ and $\beta_j \in A_j'$ such that $\alpha_j - \beta_j = 1$(or $\beta_j = \alpha_j - 1$).

Now, we want to show that for every $(i_1, i_2, ..., i_n) \in \{0,1\}^n$, $I_{(i_1,i_2,...,i_n)} \neq R$, where $I_{(i_1,i_2,...,i_n)} = \langle \alpha_1 - i_1, \alpha_2 - i_2, ..., \alpha_n - i_n \rangle$.

Let $(i_1, i_2, ..., i_n)$ be an arbitrary row in $\{0,1\}^n$. For every $j \in \{1, 2, ..., n\}$, if $i_j = 0$, then $\alpha_j - i_j = \alpha_j \in \mathfrak{m}_{(i_1,i_2,...,i_n)}$, and if $i_j = 1$, then $\alpha_j - i_j = \beta_j \in \mathfrak{m}_{(i_1,i_2,...,i_n)}$. So we have $I_{(i_1,i_2,...,i_n)} \subseteq \mathfrak{m}_{(i_1,i_2,...,i_n)}$. Therefore, $(\alpha_1 - i_1, \alpha_2 - i_2, ..., \alpha_n - i_n) \in R^n$ is not a unimodular row. Thus $(\alpha_1, \alpha_2, ..., \alpha_n) \in R^n$ is not the sum of a unimodular row and a row consists of $0's$ and $1's$, a contradiction.

The last statement follows because the set of idempotent elements of an indecomposable ring is $\{0,1\}$. □

Lemma 2.5. *Let $R = \bigoplus_{i=1}^m R_i$ be a ring decomposition of R. Then R^n is a clean R-module if and only if each R_i^n is a clean R_i-module.*

Proof. Without loss of generality, we can assume that $m = 2$. First, we can easily see that $((a_1, b_1), (a_2, b_2), ..., (a_n, b_n)) \in (R_1 \bigoplus R_2)^n$ is a unimodular row if and only if there exist $(c_1, d_1), (c_2, d_2), ..., (c_n, d_n) \in R_1 \bigoplus R_2$ such that $\sum_{i=1}^n (a_i, b_i)(c_i, d_i) = (1_{R_1}, 1_{R_2})$ if and only if there exist $c_1, c_2, ..., c_n \in R_1$ and $d_1, d_2, ..., d_n \in R_2$ such that $\sum_{i=1}^n a_i c_i = 1_{R_1}$ and $\sum_{i=1}^n b_i d_i = 1_{R_2}$ if and only if both $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are unimodular rows in R_1^n and R_2^n, respectively. Also, it is easily seen that

$$((e_1, f_1), (e_2, f_2), ..., (e_n, f_n)) \in (R_1 \bigoplus R_2)^n$$

is an idempotent row if and only if both $(e_1, e_2, ..., e_n)$ and $(f_1, f_2, ..., f_n)$ are idempotent rows in R_1^n and R_2^n, respectively. The rest of the proof is easy to check. $\square$

THEOREM 2.6. *Let $R = \bigoplus_{i=1}^m R_i$ be a ring decomposition of R such each R_i is an indecomposable ring and n be a positive integer. Then R^n is clean R-module if and only if each R_i has at most $2^n - 1$ maximal ideals.*

Proof. By Theorem 2.4 and Lemma 2.5, R^n is a clean R-module if and only if each R_i^n is a clean R_i-module if and only if each of R_i has at most $2^n - 1$ maximal ideals. $\square$

We remark that if R is a clean ring, then R^n is a clean R-module for each positive integer n. Now, let m and n be two positive integers such that $m < n$. If R^m is a clean R-module, then R^n is a clean R-module as well. To see this, Let $(r_1, r_2, ..., r_n)$ be an arbitrary row in R^n. Then $(r_1, r_2, ..., r_m)$ is a row in R^m, and so there exists a unimodular row $(a_1, a_2, ..., a_m)$ and an idempotent row $(e_1, e_2, ..., e_m)$ in R^m such that $(r_1, r_2, ..., r_m) = (a_1, a_2, ..., a_m) + (e_1, e_2, ..., e_m)$. Since $(a_1, a_2, ..., a_m)$ is a unimodular row in R^m, we have that $(a_1, a_2, ..., a_m, r_{m+1}, ..., r_n)$ is also a unimodular row in R^n. Thus,

$$(r_1, r_2, ..., r_n) = (a_1, a_2, ..., a_m, r_{m+1}, ..., r_n) + (e_1, e_2, ..., e_m, 0, ..., 0).$$

The converse of the above remark is not true, in general, as the next example shows.

Example 2.7. Let m and n be positive integers such that $m < n$. Let $p_1, p_2, ..., p_r$ be r prime numbers. The ring

$$R = \mathbb{Q}^{p_1, p_2, \ldots, p_r} = \{\frac{a}{b} \in \mathbb{Q} \mid p_1 \nmid b, p_2 \nmid b, ..., p_r \nmid b\}$$

is a semilocal indecomposable subring of $\mathbb{Q}$ with exactly r distinct maximal ideals (note that $\mathbb{Q}^{p_1, p_2, \ldots, p_r}$ is a domain). So by Theorem 2.4, for $r = 2^n - 1$, the R-module R^n is a clean R-module, while R^m is not.

Therefore, it is possible to define the *rank of cleanness* of a ring R as the smallest positive integer m (if there exists any) so that R^m is a clean R-module. Of course, by Theorem 2.4, for an indecomposable ring R, the rank of cleanness of R is m if and only if $2^{m-1} \leq |\operatorname{Max}(R)| \leq 2^m - 1$. We will not refer to this definition again.

LEMMA 2.8. *Let R be a ring and let n be a positive integer. If R^n is a clean R-module, then for every ideal I of R, $(R/I)^n$ is a clean R/I-module.*

Proof. It is clear that, if $(a_1, ..., a_n) \in R^n$ is a unimodular (resp., an idempotent) row, then $(a_1 + I, ..., a_n + I) \in (R/I)^n$ is a unimodular (resp., an idempotent) row. Now if R^n is a clean R-module and $(r_1 + I, r_2 + I, ..., r_n + I)$ is an arbitrary row in $(R/I)^n$, then $(r_1, r_2, ..., r_n)$ is the sum of a unimodular and an idempotent row in R^n, and so $(r_1 + I, r_2 + I, ..., r_n + I)$ is the sum of a unimodular and an idempotent row in $(R/I)^n$. Thus $(R/I)^n$ is a clean R/I-module. □

Although, in general, the converse of Lemma 2.8 is clearly false, it is true if I is contained in $\mathrm{Nil}(R)$. Before proving this, recall that for any ideal I of R, we say that idempotents can be lifted modulo I (or idempotents in R/I lift to the idempotents of R) if for each $x \in R$ with $x + I = x^2 + I$, there exists some $e^2 = e \in R$ such that $x + I = e + I$.

PROPOSITION 2.9. *Let R be a ring and let n be a positive integer. Let I be an ideal of R that is contained in $\mathrm{J}(R)$ and let $(R/I)^n$ be a clean R/I-module. If idempotents in R/I lift to idempotents in R, then R^n is a clean R-module.*

Proof. Let $(a_1 + I, a_2 + I, ..., a_n + I)$ be a unimodular row in $(R/I)^n$, so there exist $b_1 + I, b_2 + I, ..., b_n + I \in R/I$, such that $\sum_{i=1}^{n}(a_i + I)(b_i + I) = 1 + I$. This gives $1 - (\sum_{i=1}^{n} a_i b_i) \in I$. Now since $I \subseteq \mathrm{J}(R)$, $\sum_{i=1}^{n} a_i b_i = 1 - (1 - (\sum_{i=1}^{n} a_i b_i))$ is a unit element of R. Thus $(a_1, a_2, ..., a_n)$ is a unimodular row in R^n. Now let $(r_1, r_2, ..., r_n)$ be an arbitrary row in R^n. Thus there exist a unimodular row $(a_1 + I, a_2 + I, ..., a_n + I)$ and an idempotent row $(e_1 + I, e_2 + I, ..., e_n + I)$ in $(R/I)^n$ such that $(r_1 + I, r_2 + I, ..., r_n + I) = (a_1 + I, a_2 + I, ..., a_n + I) + (e_1 + I, e_2 + I, ..., e_n + I)$. Since idempotents in R/I lift to idempotents in R, we may assume $e_i^2 = e_i$ for $i = 1, 2, ..., n$. By above statement, since $(r_1 - e_1 + I, r_2 - e_2 + I, ..., r_n - e_n + I) = (a_1 + I, a_2 + I, ..., a_n + I)$ is a unimodular row in $(R/I)^n$, hence $(r_1 - e_1, r_2 - e_2, ..., r_n - e_n)$ is a unimodular row in R^n. Therefore, $(r_1, r_2, ..., r_n) = (r_1 - e_1, r_2 - e_2, ..., r_n - e_n) + (e_1, e_2, ..., e_n)$ which is a sum of a unimodular and an idempotent row. Therefore R^n is a clean R-module. □

COROLLARY 2.10. *Let R be a ring and let n be a positive integer. If I is an ideal of R that is contained in $\mathrm{Nil}(R)$ and $(R/I)^n$ is a clean R/I-module, then R^n is a clean R-module.*

Proof. Since $I \subseteq \mathrm{Nil}(R)$, idempotents can be lifted modulo I [2, Proposition 27.1], and the result follows from Proposition 2.9. □

By assuming that idempotents in R/I lift to idempotents in R, we may replace $\mathrm{Nil}(R)$ by $\mathrm{J}(R)$ in Corollary 2.10.

Clean rings are pm-rings; i.e. each prime ideal is contained in a unique maximal ideal (see for example [1]). As an extension of this fact, we have the following.

LEMMA 2.11. *Let R be a ring and let n be a positive integer. If R^n is a clean R-module, then every prime ideal of R is contained in at most $2^n - 1$ distinct maximal ideals of R.*

Proof. Let P be a prime ideal of R, since R^n is a clean R-module, $(R/P)^n$ is a clean R/P-module, by Lemma 2.8. Now as a domain, R/P is an indecomposable ring, and therefore by Theorem 2.4, R/P has at most $2^n - 1$ distinct maximal ideal. Thus P is contained in at most $2^n - 1$ distinct maximal ideals of R. $\square$

Before proceeding, we need some notation and terminology. We let $R = \bigoplus_{i=1}^{t} R_i$ be a ring decomposition and suppose that P is a prime ideal of $R = \bigoplus_{i=1}^{t} R_i$. It is clear that there exists $j \in \{1, 2, ..., t\}$ and a prime ideal P_j of R_j such that

$$P = R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_{j-1} \bigoplus P_j \bigoplus R_{j+1} \bigoplus ... \bigoplus R_t.$$

In this case, we say P has *index* j.

Let $\mathcal{A}$ be a collection of ideals of a ring R. We define

$$V(\mathcal{A}) = \{\mathfrak{m} \in \mathrm{Max}(R) \mid \exists \mathfrak{a} \in \mathcal{A}, \mathfrak{a} \subseteq \mathfrak{m}\}.$$

In the next result, which may be considered as a generalization of [1, Theorem 5], some characterizations of a Noetherian ring R satisfying the condition that R^n is a clean R-module are given.

THEOREM 2.12. *Let R be a ring and let n be a positive integer. If R can be expressed as a finite direct product of indecomposable rings R_i, say $R = \bigoplus_{i=1}^{t} R_i$ (e.g., R is Noetherian), then the following are equivalent:*

1. *Each R_i has at most $2^n - 1$ maximal ideals.*
2. *R^n is a clean R-module.*
3. *There is a partition $\{\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_t\}$ for the set of all minimal prime ideals of R such that $\{V(\mathcal{P}_1), V(\mathcal{P}_2), ..., V(\mathcal{P}_t)\}$ is a partition for* Max(R) *and $\mid V(\mathcal{P}_i) \mid \leq 2^n - 1$, for $i = 1, 2, ..., t$.*
4. *Every prime ideal of R is contained in at most $2^n - 1$ maximal ideals and the union of any 2^n distinct maximal ideals contains an idempotent that is not contained in their intersection.*

Proof. (1)⇔(2) It is clear by Theorem 2.6.

(1)⇒(3) Let $\{\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_t\}$ be the partition for the set of all minimal prime ideals of R such that for every $i = 1, 2, ..., t$, $\mathcal{P}_i$ contains all minimal prime ideals of R that have the same index i. Since every maximal ideal contains at least one minimal prime ideal and each maximal ideal with index i only contains minimal prime ideals with index i, we have $| V(\mathcal{P}_i) | \leq | \operatorname{Max}(R_i) | \leq 2^n - 1$, for $i = 1, 2, ..., t$, and $\{V(\mathcal{P}_1), V(\mathcal{P}_2), ..., V(\mathcal{P}_t)\}$ is a partition for $\operatorname{Max}(R)$.

(3)⇒(2) Let $\{\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_t\}$ be a partition for the set of all minimal prime ideals of R such that $\{V(\mathcal{P}_1), V(\mathcal{P}_2), ..., V(\mathcal{P}_t)\}$ is a partition for $\operatorname{Max}(R)$ and $| V(\mathcal{P}_i) | \leq 2^n - 1$, for every $i = 1, 2, ..., t$. We define $I_i = \bigcap_{P \in \mathcal{P}_i} P$, for each $i = 1, 2, ..., t$. Clearly, for two distinct parts $\mathcal{P}_i$ and $\mathcal{P}_j$, the ideals I_i and I_j are comaximal and the intersection $\bigcap_i I_i$ is $\operatorname{Nil}(R)$. Thus, by the Chinese Remainder Theorem, $R/\operatorname{Nil}(R) \cong \prod_{i=1}^{t} R/I_i$. On the other hand, by the definition of I_i, we have $| \operatorname{Max}(R/I_i) | \leq 2^n - 1$. Now Theorem 2.4 implies that each $(R/I_i)^n$ is a clean R/I_i-module. Therefore $(R/\operatorname{Nil}(R))^n$ is a clean $R/\operatorname{Nil}(R)$-module and so by Corollary 2.10, R^n is a clean R-module.

(2) ⇒ (4) Since (1) and (2) are equivalent, every prime ideal of R is contained in at most $2^n - 1$ distinct maximal ideals of R, by Lemma 2.11. Let a family of 2^n distinct maximal ideals be given. Let $i \in \{1, 2, ..., t\}$ be the index of one of the maximal ideals, and e the idempotent of R consisting of a 0 at i^{th} position and 1's elsewhere. Then the index of a maximal ideal $\mathfrak{m}$ of R is i if and only if $e \in \mathfrak{m}$. Not all of the 2^n maximal ideals can contain e because R_i has at most $2^n - 1$ maximal ideals by Theorem 2.6 So (4) holds.

(4)⇒(1) If there exists some j such that $| \operatorname{Max}(R_j) | \geq 2^n$, then we can choose 2^n distinct maximal ideals $\mathfrak{m}_1^{'}, \mathfrak{m}_2^{'}, ..., \mathfrak{m}_{2^n}^{'}$ in $\operatorname{Max}(R_j)$. For $k = 1, 2, 3, ..., 2^n$, we define

$$\mathfrak{m}_k = R_1 \oplus R_2 \oplus ... \oplus R_{j-1} \oplus \mathfrak{m}_k^{'} \oplus R_{j+1} \oplus ... \oplus R_t.$$

Clearly $\{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_{2^n}\}$ is a subset of $\operatorname{Max}(R)$ with 2^n distinct elements. But since each R_i is indecomposable, so every idempotent element in R is a row whose components are 0 or 1. Now if e is an idempotent row whose j^{th} position is 0, then e is in all $\mathfrak{m}_k$, $k = 1, 2, ..., 2^n$, and if e is an idempotent row whose j^{th} position is 1, then for $k = 1, 2, ..., 2^n$, e is not in any $\mathfrak{m}_k$, which contradicts the assumption. □

In the following, we shall consider free modules over polynomial or power series rings and determine when such modules are clean.

THEOREM 2.13. *Let R be a ring, and n a positive integer.*
(1) $(R[x])^n$ is never a clean $R[x]$-module.
(2) $(R[[x]])^n$ is a clean $R[[x]]$-module if and only if R^n is a clean R-module.

Proof. (1) Let $\mathfrak{m}$ be a maximal ideal of R. We have $R[x]/\mathfrak{m}[x] \cong (R/\mathfrak{m})[x]$. Now $(R/\mathfrak{m})[x]$ has infinitely many maximal ideals, and since $(R/\mathfrak{m})[x]$ is indecomposable, by Theorem 2.4, $(R[x]/\mathfrak{m}[x])^n$ is never a clean $R[x]/\mathfrak{m}[x]$-module. So by Lemma 2.8, $(R[x])^n$ is never a clean $R[x]$-module.

(2) Since $R \cong R[[x]]/\langle x \rangle$, if $(R[[x]])^n$ is a clean $R[[x]]$-module, then R^n is a clean R-module, by Lemma 2.8.

Conversely, note that $(r_1 + xf_1, r_2 + xf_2, ..., r_n + xf_n) \in (R[[x]])^n$, where $r_i \in R$ and $f_i \in R[[x]]$, is a unimodular row in $(R[[x]])^n$ if and only if $(r_1, r_2, ..., r_n)$ is a unimodular row in R^n. Assume that R^n is a clean R-module and $(a_1 + xf_1, a_2 + xf_2, ..., a_n + xf_n)$ is an arbitrary row in $(R[[x]])^n$. Since R^n is a clean R-module, $(a_1, a_2, ..., a_n) = (r_1, r_2, ..., r_n) + (e_1, e_2, ..., e_n)$, where $(r_1, r_2, ..., r_n)$ is a unimodular row and $(e_1, e_2, ..., e_n)$ is an idempotent row in R^n. Thus $(a_1 + xf_1, a_2 + xf_2, ..., a_n + xf_n) = (r_1 + xf_1, r_2 + xf_2, ..., r_n + xf_n) + (e_1, e_2, ..., e_n)$, as desired. So $(R[[x]])^n$ is a clean $R[[x]]$-module. $\square$

3. A NEW CHARACTERIZATION OF CLEAN RINGS

A ring R is said to be *clean* if each element of R is the sum of an idempotent and a unit, and R is said to be *Gelfand* (or *pm*-ring) if each prime ideal is contained in only one maximal. In [1], it was shown that clean rings are Gelfand, but the converse is not true, in general. In this section, we show that a ring R is clean if and only if every indecomposable homomorphic image of R is local. The following result will be used in the sequel.

LEMMA 3.1. *Let I be a proper ideal of R. Then R/I is indecomposable if and only if $R/\sqrt{I}$ is indecomposable.*

Proof. ($\Rightarrow$) Since $(R/I)/\operatorname{Nil}(R/I) = (R/I)/(\sqrt{I}/I) \cong R/\sqrt{I}$, there is no loss of generality in assuming $I = 0$. Now suppose that R is an indecomposable ring, we wish to show that $R/\operatorname{Nil}(R)$ is an indecomposable ring. If, on the contrary, $R/\operatorname{Nil}(R)$ is not indecomposable, then there exists $x \in R$ such that $x(x-1) = x^2 - 1 \in \operatorname{Nil}(R)$, $x \notin \operatorname{Nil}(R)$, and $x - 1 \notin \operatorname{Nil}(R)$. By [2, Proposition 27.1], there exists an idempotent element $e^2 = e \in R$ such that $x + \operatorname{Nil}(R) = e + \operatorname{Nil}(R)$. Now, since R is indecomposable, $e = 0$ or $e = 1$, and so $x \in \operatorname{Nil}(R)$ or $x - 1 \in \operatorname{Nil}(R)$, a contradiction.

($\Leftarrow$) Let R/I be indecomposable, and let $I = A \cap B$ and $A + B = R$ in which A and B are some ideals of R. Thus $\sqrt{I} = \sqrt{A} \cap \sqrt{B}$ and $\sqrt{A} + \sqrt{B} = R$, and so $\sqrt{A} = R$ or $\sqrt{B} = R$. Therefore, we have $A = R$ or $B = R$. $\square$

As an immediate consequence of Lemma 3.1, we see that if I is a primary ideal, or a power of a prime ideal, then R/I is indecomposable.

For a proper ideal I in a ring R we set

$$\Omega(I) = \{\mathfrak{m} \in \mathrm{Max}(R) \mid I \subseteq \mathfrak{m}\}.$$

Recall that the Zariski topology on $\mathrm{Max}(R)$ is the topology obtained by taking the collection of all sets of the form $\Omega(I)$ for every ideal I of R as the closed sets. Also, recall that a topological space is said to be totally disconnected if each of its connected components contains only one point.

THEOREM 3.2 ([4, Theorem I.1]). *Let R be a ring. Then the following conditions are equivalent.*

1. *R is a Gelfand ring and $\mathrm{Max}(R)$ is totally disconnected.*

2. *R is a clean ring.*

Now we want to give a new characterization of clean rings. For this purpose, we first need the following.

PROPOSITION 3.3. *Let $\{P_i\}_{i\in\Lambda}$ be a family of prime ideals of R. Then $R/(\cap_{i\in\Lambda}P_i)$ is decomposable if and only if there is a partition $\Lambda = \Lambda_1 \cup \Lambda_2$ such that $\cap_{i\in\Lambda_1}P_i + \cap_{i\in\Lambda_2}P_i = R$.*

Proof. If $e \in R$ represents a nontrivial idempotent in the quotient, then $e(1-e) \in \cap_{i\in\Lambda}P_i$ and we set

$$\Lambda_1 = \{i \in \Delta : e \in P_i\} \quad \text{and} \quad \Lambda_2 = \{i \in \Delta : 1-e \in P_i\}.$$

So we have $\cap_{i\in\Lambda_1}P_i + \cap_{i\in\Lambda_2}P_i = R$. The converse is obvious. □

THEOREM 3.4. *Let R be a ring. Then the following are equivalent.*

1. *R is a clean ring.*

2. *Every indecomposable homomorphic image of R is local.*

3. *For every nonempty collection $\{P_i\}_{i\in\Lambda}$ of prime ideals of R, if there does not exist a partition $\Lambda = \Lambda_1 \cup \Lambda_2$ such that $\cap_{i\in\Lambda_1}P_i + \cap_{i\in\Lambda_2}P_i = R$, then $R/(\cap_{i\in\Lambda}P_i)$ is a local ring.*

Proof. (1) ⇒ (2) Let R/I be an indecomposable clean ring. Then R/I must be local, by [1, Theorem 3].

(2) ⇒ (1) First, observe that if P is prime, then R/P is indecomposable hence local. So R is a Gelfand ring (which is exactly a ring such that R/P is local for every prime ideal P). By Theorem 3.2, it is sufficient to show that $\mathrm{Max}(R)$ is totally disconnected. Let $\{\mathfrak{m}_\alpha\}_{\alpha\in\Lambda}$ be a subset of $\mathrm{Max}(R)$ with $|\Lambda| \geq 2$. By hypothesis, $R/\cap_{\alpha\in\Lambda}\mathfrak{m}_\alpha$ is decomposable because it is not

local. Thus, there exist ideals A and B of R such that $\cap_{\alpha\in\Lambda}\mathfrak{m}_\alpha = A\cap B$ and $A+B=R$. Clearly, $\{\Omega(A),\Omega(B)\}$ is a separation of $\{\mathfrak{m}_\alpha\}_{\alpha\in\Lambda}$ and so $\{\mathfrak{m}_\alpha\}_{\alpha\in\Lambda}$ is disconnected. Thus $\mathrm{Max}(R)$ is totally disconnected.

(2) $\Rightarrow$ (3) Let $\{P_i\}_{i\in\Lambda}$ be a nonempty collection of prime ideals of R such that there does not exist a partition $\Lambda = \Lambda_1\cup\Lambda_2$ such that $\cap_{i\in\Lambda_1}P_i + \cap_{i\in\Lambda_2}P_i = R$. Then $R/(\cap_{i\in\Lambda}P_i)$ is indecomposable, by Proposition 3.3. Therefore, $R/(\cap_{i\in\Lambda}P_i)$ is a local ring.

(3) $\Rightarrow$ (2) Let I be an ideal of R such that R/I is indecomposable. Thus $R/\sqrt{I}$ is also indecomposable, by Lemma 3.1. Now let $\{P_i\}_{i\in\Lambda}$ be a collection of prime ideals with $\sqrt{I} = \cap_{i\in\Lambda}P_i$. Thus $R/(\cap_{i\in\Lambda}P_i)$ is indecomposable, and so, by Proposition 3.3, there does not exist a partition $\Lambda=\Lambda_1\cup\Lambda_2$ such that $\cap_{i\in\Lambda_1}P_i + \cap_{i\in\Lambda_2}P_i = R$. Therefore, $R/\sqrt{I} = R/(\cap_{i\in\Lambda}P_i)$ is local, and so R/I is local. $\square$

REFERENCES

[1] D.D. Anderson and V.P. Camillo, *Commutative rings whose elements are a sum of a unit and an idempotent*. Comm. Algebra **30** (2002), 3327–3336.

[2] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules*. Second edition, Graduate Texts in Math. **13**, Springer-Verlag, 1992.

[3] V.P. Camillo, D. Khurana, T.Y. Lam, W.K. Nicholson, and Y. Zhou, *Continuous modules are clean*. J. Algebra **304** (2006), 94–111.

[4] F. Couchot, *Indecomposable modules and Gelfand rings*. Comm. Algebra **35** (2007), 231–241.

[5] W. Wm. McGovern, *Neat rings*. J. Pure Appl. Algebra **205** (2006), 243–265.

[6] W.K. Nicholson, *Lifting idempotents and exchange rings*. Trans. Amer. Math. Soc. **299** (1977), 269–278.

Received April 23, 2019

Department of Pure Mathematics
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
e_rostami@uk.ac.ir
s_hedayat@uk.ac.ir