AN UNIFIED DIFFERENCE METHOD FOR THE NUMERICAL SOLUTION OF THE TROESCH'S BOUNDARY VALUE PROBLEM

PRAMOD KUMAR PANDEY

Communicated by Lucian Beznea

In this article, the unified difference and quasi qudratization method are developed and discussed to obtain an approximate numerical solution for the Troesch's problem, a nonlinear differential equation and corresponding boundary value problems. The degree of accuracy in numerical solution by the proposed method is good and comparable to other existing methods in literature for the range of values of parameter in Troesch's problem.

AMS 2020 Subject Classification: 65L10, 65L12.

Key words: Boundary Value Problem, Dirichlet boundary condition, quasi qudratization, Troesch's problem, unified difference method.

1. INTRODUCTION

A second order differential equation arises in the studies models of physical sciences. The Troesch's problem is defined by a Troesch's parameter, second order nonlinear differential equation and boundary conditions. This problem arises in the investigation of confinement of a plasma column by a radiation pressure [5], theory of gas porous electrodes [3] and defined as,

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} = \lambda \sinh(\lambda u(x)), \quad 0 < x < 1 \tag{1}$$

and boundary conditions are

$$u(0) = 0 \quad \text{and} \quad u(1) = 1.$$

The analytical solution of Troesch's problem (1) is developed and studied in [13]. In practice, it is often required to consider well-suited schemes for the numerical solution of different types of nonlinear problems. The difficulty in computing the solution to Troesch's problem is studied in [13, 1] and showed sensitivity towards numerical methods. So Troesch's problem became a test case for methods of solving unstable two-point boundary value problems because of difficulties and sensitivity in the computation of its numerical solution. In order to overcome such sensitivity and difficulties, there are several numerical techniques for the solution of Troesch's problem reported in the

MATH. REPORTS **25(75)** (2023), *1*, 123–132
doi: 10.59277/mrar.2023.25.75.1.123

literature such as: finite difference method [10], transformation method [8], shooting method [4, 11], homotopy perturbation method [12, 7], non-standard finite difference method [14], asymptotic approximation method [6] and logarithmic finite difference method [9] and references therein. Recently, efficient and accurate work on numerical solution method for large values of Troesch's parameter were reported in [7, 9].

In this article, we have assumed the existence and uniqueness of the solution of the problem (1). Inspired by the logarithmic finite difference method, in this article we consider the unified difference method for the numerical solution of the considered boundary value problems (1). We consider moderate value of Troesch's parameter in numerical computation and discuss the findings in the approximate solution.

We have presented our work in this article as follows. In Section 2, we derive a unified difference method. We have discussed derivation and convergence of the proposed method under appropriate condition in Section 3 and Section 5, respectively. In Section 5, we introduce quasi quadratization, an iterative method for the solution of system of equations. The application of the proposed method on the model problems and numerical results were produced to show the efficiency in Section 6. Discussion and conclusion on the performance of the proposed method are presented in Section 7.

2. THE DIFFERENCE METHOD

Let us consider following two points boundary value problem,

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} = f(x, u(x), u'(x)), \quad 0 < x < 1 \tag{2}$$

subject to boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 1.$$

In fact, problem (1), is a particular case of problem (2). To solve problem (2) numerically by the present procedure, first let us consider N a positive integer, define $h = \frac{1}{N+1}$ a uniform step length and grid points $x_i = i \cdot h, i = 0, 1, \cdots, N+1$ in the domain $[0, 1]$ of the considered problem. Our aim is to determine the numerical solution of the problem at these grid points. In our discussion and derivation, u_i and f_i denote, respectively, the numerical approximation of solution of problem $u(x)$ and forcing function $f(x, u(x), u'(x))$ at grid point $x = x_i$ for different values of $i = 0, 1, \cdots, N+1$. We may write problem (2) at these grid points $x = x_i$, $i = 0, 1, \cdots, N+1$. That is

$$u_i'' = f_i, \quad a \le x_i \le b \tag{3}$$

and boundary conditions are

$$u_0 = 0 \quad \text{and} \quad u_{N+1} = 1.$$

Define the following approximations,

$$\overline{u}' = \frac{u_{i+1} - u_{i-1}}{2h} \tag{4}$$

and

$$\overline{f}_i = f(x_i, u_i, \overline{u}'_i) \tag{5}$$

Thus, by the application of approximations (4-5) and following the ideas in [15], we propose our unified finite difference method for the numerical solution of the considered problem (3),

$$u_{i+1} - (1 + \exp(\lambda h))u_i + \exp(\lambda h)u_{i-1} = h^2 \exp(\frac{\lambda h}{2})\overline{f}_i \tag{6}$$

Hence, we have obtained (6), a system of equations in the variables u_i, $i = 1, 2, \cdots, N$. The solution of a system of equations (6) is the approximate numerical solution of the considered problem (3). To solve the system of equations (6), we employ an appropriate iterative method.

3. DEVELOPMENT OF THE FINITE DIFFERENCE METHOD

In this section, we will discuss the development of the proposed unified finite difference method (6). There are diverse methods to form a discrete problem from a continuous problem i.e. difference equations from differential equations. To begin, consider the problem defined by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \beta \frac{\mathrm{d}y}{\mathrm{d}x} = 0, \quad 0 \le x \le 1 \tag{7}$$

and boundary conditions are

$$y(0) = 0 \quad \text{and} \quad y(1) = 1 \tag{8}$$

where β is a constant. The exact solution of the problem (7-8) is

$$y(x) = \frac{\exp(\beta x) - 1}{\exp(\beta) - 1}. \tag{9}$$

If we assume approximate solution of the type (9) for the problem (7-8) then we can derive approximate difference equation [15] for the numerical solution of the problem (7-8) in the following form,

$$y_{i+1} - (1 + \exp(\beta h))y_i + \exp(\beta h)y_{i-1} = 0 \tag{10}$$

Let us rewrite (2) in the following second order differential equation,

$$u''(x) - \lambda u'(x) = f(x, u, u').$$

We considerthe following linear combination of $u(x)$ the solution of the problem and forcing function $f(x, u(x), u'(x))$ at grid node x_i,

$$u_{i+1} - (1 + \exp(\lambda h))u_i + \exp(\lambda h)u_{i-1} - h^2 b_0 f_i = 0 \quad \text{and} \quad b_0 \neq 0, \tag{11}$$

where b_0 is a constant to be determined under appropriate conditions. Let us expand each term of (11) in a Taylor series about grid node x_i up to fourth terms and simplify. We obtain

$$h(1-\exp(\lambda h))u_i' + \frac{h^2}{2}(1+\exp(\lambda h))u_i'' + \frac{h^3}{6}(1-\exp(\lambda h))u_i''' + O(h^4) - h^2 b_0 f_i = 0. \tag{12}$$

Substituting $1 + h\lambda + \frac{h^2\lambda^2}{2} + ..$ for $\exp(\lambda h)$ and f_i for $u_i'' - \lambda u_i'$ in (12), we obtain

$$h^2(1 + \frac{h\lambda}{2} - b_0)f_i + O(h^4) = 0. \tag{13}$$

If we choose $b_0 = 1 + \frac{h\lambda}{2}$, we find the remainder term of $O(h^4)$ in above expression. If we approximate $1 + \frac{h\lambda}{2}$ by $\exp(\frac{\lambda h}{2})$ then we have

$$u_{i+1} - (1 + \exp(\lambda h))u_i + \exp(\lambda h)u_{i-1} = h^2 \exp(\frac{\lambda h}{2})f_i \tag{14}$$

and $O(h^4)$, the remainder term in the above equation (14), is

$$R_i = \frac{h^4}{24}(\lambda^2 f - 2\lambda\frac{\partial f}{\partial x} + 6\frac{\partial^2 f}{\partial x^2} - 4u^{(4)}(x))_i\,.$$

But from (4) and (5), we have

$$\overline{f}_i = f_i + O(h^2). \tag{15}$$

Hence from (14) and (15), we obtain our unified finite difference method

$$u_{i+1} - (1 + \exp(\lambda h))u_i + \exp(\lambda h)u_{i-1} = h^2 \exp(\frac{\lambda h}{2})\overline{f}_i\,. \tag{16}$$

4. QUASI QUADRATIZATION

It is an iterative method which will produce approximate zero of continuous function $f(x)$. Let $x = x_0$ be the approximate zero of $f(x)$. Let quasi quadratize $f(x)$, i.e.

$$f(x_0) \equiv f(x) + (x_0 - x)f'(x) + \frac{(x_0 - x)^2}{2}f''(x) = 0. \tag{17}$$

Solving the quadratic equation (16), we have

$$(x_0 - x) = \frac{-f'(x) \pm \sqrt{(f'(x))^2 - 2f(x)f''(x)}}{f''(x)}. \tag{18}$$

In order to get the next approximation to the correct root, we write (19) as

$$x_{k+1} - x_k = \frac{-f'(x_k) \pm \sqrt{(f'(x_k))^2 - 2f(x_k)f''(x_k)}}{f''(x_k)}. \tag{19}$$

So, we have two approximate zeros of the continuous function f(x).

5. CONVERGENCE ANALYSIS

In this section, we discuss the convergence of the proposed unified finite difference method (6). Consider the following test problem,

$$u''(x) - \lambda u'(x) = f(x, u, u'), \qquad a < x < b \tag{20}$$

subject to the boundary conditions $u_0 = 0$ and $u_{N+1} = 1$. We assume that

(i) $f(x, u, u')$ is continuous

(ii) $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial u'}$ exist and are continuous

(iii) $\frac{\partial f}{\partial u} > 0$ and $\exists w > 0$ such that $\left|\frac{\partial f}{\partial u'}\right| \le w$.

Then solution of problem (20) exists and it is unique. Let $U(x)$ and $u(x)$ be, respectively, the exact and the approximate solutions of the above considered problem. Applying method (6) to solving problem (20), we obtain the following matrix equation,

$$\mathbf{JU} = \mathbf{RF} + \mathbf{T} \tag{21}$$

where matrix $\mathbf{T}$ is the truncation error term in the method and

$$\mathbf{Ju} = \mathbf{Rf}. \tag{22}$$

Subtract (19) from (18), we have

$$\mathbf{J}(\mathbf{U} - \mathbf{u}) = \mathbf{RF} - \mathbf{Rf} + \mathbf{T} \tag{23}$$

where $\mathbf{U} = [U_1, \cdots, U_N]$ and $\mathbf{u} = [u_1, \cdots, u_N]$. Let define the forcing function at node x_i,

$$F_i = f(x_i, U(x_i), U'(x_i)) \quad \text{and} \quad \overline{F}_i = f(x_i, U(x_i), \overline{U}'(x_i)).$$

We use (14) and linearize the forcing function $f(x, u, u')$. Hence, we have,

$$F_i - f_i = (U_i - u_i)(\frac{\partial f}{\partial u})_i + (U_i' - u_i')(\frac{\partial f}{\partial u'})_i. \tag{24}$$

Further, let us define

$$\epsilon_i = U_i - u_i, \quad i = 1, \cdots, N. \tag{25}$$

From (6) and $i = 1$, we obtain

$$\epsilon_{i+1} - (1 + \exp(\lambda h)\epsilon_i = h^2 \exp(\frac{\lambda h}{2})(\frac{1}{2h}\epsilon_{i+1}(\frac{\partial f}{\partial u'})_i + \epsilon_i(\frac{\partial f}{\partial u})_i \,.$$

Similarly, we can write other equations for $i = 2, 3, \cdots, N$. We use (4), (24), (25) in (23) and simplify. We write the simplified expression in the following matrix equation,

$$(\mathbf{J} - \mathbf{D})\boldsymbol{\epsilon} = \mathbf{T} \tag{26}$$

$$\mathbf{J} = \begin{pmatrix} -(1+\exp(\lambda h)) & 1 & & & 0 \\ \exp(\lambda h) & -(1+\exp(\lambda h)) & 1 & & \\ & & \ddots & & \\ 0 & & & \exp(\lambda h) & -(1+\exp(\lambda h)) \end{pmatrix}_{N\times N},$$

$$\mathbf{D} = h^2 \exp(\frac{\lambda h}{2}) \begin{pmatrix} G_1 & \frac{1}{2h}H_1 & & & 0 \\ -\frac{1}{2h}H_2 & G_2 & \frac{1}{2h}H_2 & & \\ & & \ddots & & \\ 0 & & & -\frac{1}{2h}H_N & G_N \end{pmatrix}_{N\times N}$$

where $G_i = (\frac{\partial f}{\partial u})_i$, $H_i = (\frac{\partial f}{\partial u'})_i$, $\boldsymbol{\epsilon} = [\epsilon_1, \cdots, \epsilon_N]$, and $\mathbf{T} = (T_i)_{N\times 1}$ with

$$T_i = R_i + \frac{h^4}{6}(u^{(3)}\frac{\partial f}{\partial u'})_i, \quad i = 1, \cdots, N.$$

We have determined $\mathbf{J}^{-1} = (j_{l,m})$ explicitly as

$$j_{l,m} = \begin{cases} \frac{(1-\exp(lh\lambda))(\exp(-Nh\lambda)-\exp((1-m)h\lambda))}{(\exp(h\lambda)-1)(\exp(-Nh\lambda)-\exp(h\lambda))}, & l \le m \\ \frac{(1-\exp(mh\lambda))(\exp(-(N-l)h\lambda)-\exp(h\lambda))}{(\exp(h\lambda)-1)(\exp(-Nh\lambda)-\exp(h\lambda))}, & l \ge m. \end{cases}$$

Let us assume that

$$F^0 = \max_{x\in[a,b]} \left|\frac{\partial f}{\partial u}\right| \,,\; F^1 = \max_{x\in[a,b]} \left|\frac{\partial f}{\partial u'}\right| \text{ and } M = \max_{x\in[a,b]} \left|(2\lambda^2 f - \lambda\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2})\right|,$$

and $||\mathbf{J}^{-1}\mathbf{D}|| < 1$. Then, by [2],

$$||(\mathbf{J} - \mathbf{D})^{-1}|| \le \frac{||\mathbf{J}^{-1}||}{1 - ||\mathbf{J}^{-1}\mathbf{D}||} \,. \tag{27}$$

Hence, from (26) we have,

$$\|\boldsymbol{\epsilon}\| \le \frac{1}{1 - \|\mathbf{J}^{-1}\|\|\mathbf{D}\|}\|\mathbf{J}^{-1}\|\|\mathbf{T}\|. \tag{28}$$

It is easy to calculate the value of the terms in (28). Thus, the error in proposed unified finite difference method is bounded and $\|\boldsymbol{\epsilon}\| \to 0$ as $h \to 0$. Moreover, we can prove that the rate of convergence of proposed unified finite difference method is at least $O(h^2)$.

6. NUMERICAL RESULTS

To validate the theoretical development, we tested the proposed method on Troesch's problem (1). With different parameters N and λ, we tabulated MAE, the maximum absolute error. We have used the following formula in the computation of MAE,

$$MAE = \max_{1 \le i \le N} |U(x_i) - u_i|$$

where $U(x_i)$ and u_i are, respectively, the exact and the computed value of the the solution of the considered problem.

For the solution of the system of equations (6), we have used Newton-Raphson and quasi quadratization method. All computations were performed on a Windows 2007 Home Basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iterations reached 10^4.

Problem 1. The non linear model problem in [5, 1] with different boundary conditions is given by

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = \lambda \sinh(\lambda u(x)) + f(x), \quad 0 < x < 1 \tag{29}$$

and boundary conditions are

$$u(0) = 0 \quad \text{and} \quad u(1) = 1,$$

where $f(x)$ is calculated so that the analytical solution of the problem is

$$u(x) = \frac{1}{\sinh(\lambda)} \sinh(\lambda x).$$

The MAE computed by method (6) for different values of N and λ are presented in the Tables below. We also presented RMAE, the rate of change maximum absolute error, with change in the parameter λ in the table.

Table 1 – Approximate estimation of the Order (Problem 1)

λ	h	UFD			
		NR		QQ	
		MAE	Order	MAE	Order
	.001	.1248582743(-5)	-	.1247588222(-5)	-
	.0005	.3121135637(-6)	2.0001	.3058244051(-6)	2.0284
16	.00025	.7367088032(-7)	2.0829	.5345156375(-7)	2.5164
	.000125	.1269858213(-7)	2.5364	.8194248614(-10)	9.3494
	.0000625	.1551980037(-8)	3.0325	.1219520361(-12)	9.3922

Table 2 – The maximum absolute error (Problem 1)

h	λ	MAE	
		NR	QQ
.0005	0.5	.6564844154(-15)	.5540273101(-16)
	1.0	.9705723638(-14)	.5545694112(-16)
	5.0	.2669049882(-7)	.2509654880(-7)
	10.0	.1158846439(-6)	.1158322463(-6)

Table 3 – Comparison in the maximum absolute error (Problem 1)

h	λ	MAE		
		UFD		LOG
		NR	QQ	
.0005	1.0	.9705723638(-14)	.5540273101(-16)	.27(-8)
	0.5	.6564844154(-15)	.5545694112(-16)	.2(-9)

Table 4 – Observation in change in the maximum absolute error and λ (Problem 1)

h	λ	UFD			
		NR		QQ	
		MAE	RMAE	MAE	RMAE
.0005	4.0	.7338053829(-7)	-	.4284847531(-8)	-
	8.0	.1559114923(-6)	2.1247	.1132621429(-6)	26.4332
	16.0	.3121135637(-6)	2.0019	.3058244051(-6)	2.7001
	32.0	.6242848153(-6)	2.0002	.6235041106(-6)	2.0388
	64.0	.1248419326(-5)	1.9998	.1248419326(-5)	2.0023

Table 5 – Observation in change in the maximum absolute error, h and λ (Problem 1)

h	λ	UFD			
		NR		QQ	
		MAE	RMAE	MAE	RMAE
.0005	16.0	.3121135637(-6)	-	.3058244051(-6)	-
.00025	32.0	.1560386305(-6)	2.0002	.1519345608(-6)	2.0129
.000125	64.0	.7800120280(-7)	2.0005	.7371304936(-7)	2.0612
.0000625	128.0	.3898103297(-7)	2.0010	.3415570805(-7)	2.1581
.00003125	256.0	.1947015110(-7)	2.0021	.1463625491(-7)	2.3336

Table 6 – Comparison of the maximum absolute error with Numerov Method (Problem 1)

λ	N	MAE	
		UFD	Numerov Method
368.730026244	500	.1154819262(-3)	.1358319871(-28)
	1000	.2868705037(-4)	.4070522271(-27)
	2000	.7189970767(-5)	.8652980620(-9)
	4000	.1798068770(-5)	.2656571687(-8)
	8000	.4494932455(-6)	Diverge
	16000	.1119155431(-6)	Diverge
	32000	.2570056184(-7)	Diverge

The computational efficiency of the proposed unified finite difference method (6) was tested on Troesch's problem (1). Numerical results are in good agreement for the theoretical development of the proposed method. In numerical experiment for a particular value of λ in the test problem our method converged, but the fourth order Numerov method failed. Thus, it is evident from the tabulated results that method (6) is converging for the moderate value of λ.

7. CONCLUSION

A Troesch's problem (1), a second order nonlinear boundary value problem, was considered for the numerical solution. A unified finite difference method was developed for the numerical solution of the considered problem. The numerical experiment was carried out with the proposed unified finite difference method on the test problem. In making an evaluation of the performance of method, there is a balance between the level of accuracy achieved and computational efficiency of the method. Thus, the numerical results we obtained by the application of the proposed method (6) approves the theoretical development of the proposed method and balance.

REFERENCES

[1] B.A. Troesch, *Intrinsic difficulties in the numerical solution of a boundary value problem.* Internal Report NN-142, TRW, Inc. Redondo Beach, California, 1960.

[2] C.E. Froberg, *Introduction to Numerical Analysis.* Second Ed., Addison-Wesley, New York, 1969.

[3] D. Gidaspow and B.S. Baker, *A model for discharge of storage batteries.* J. Electrochemical Society **120** (1973), *8*, 1005-1010.

[4] D.J. Jones, *Solution of Troesch's and other two point boundary value problems by shooting techniques.* J. Comput. Phys. **12** (1973), 429-434.

[5] E.S. Weibel, *On the confinement of a plasma by magnetostatic fields.* Physics of Fluids **2** (1959), *1*, 52-56.

[6] H. Temimi and H. Kurkcu, *An accurate asymptotic approximation and precise numerical solution of highly sensitive Troesch's Problem.* Appl. Math. Comput. **235** (2014), 253-260.

[7] H. Vazquez-Leal, Y. Khan, G. Fernández-Anaya, A. Herrera-May, A. Sarmiento-Reyes, U. Filobello-Nino, V.-M Jimenez-Fernández, and D. Pereyra-Díaz, *A general solution for Troesch's problem.* Math. Probl. in Engineering **2012**, Article ID 208375.

[8] J.P. Chiou and T.Y. Na, *On the solution of Troesch's nonlinear two-point boundary value problem using an initial value method.* J. Comput. Phys. **19** (1975), 311-316.

[9] M.S. Ismail and K.S. Al-Basyoni, *A logarithmic finite difference Method for Troesch's problem.* Appl. Math. **9** (2018), 550-559.

[10] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations.* Wiley, New York, 1962.

[11] S.H. Chang, *Numerical solution of Troesch's problem by simple shooting method.* Appl. Math. Comput. **216** (2010), *11*, 3303-3306.

[12] S.H. Mirmoradi, I. Hosseinpour, S. Ghanbarpour, and A. Barari, *Application of an approximate analytical method to nonlinear Troesch's problem.* Appl. Math. Sci. **3** (2009), *29-32*, 1579–1585.

[13] S.M. Roberts and J.S. Shipman, *On the closed form solution of Troesch's problem.* J. Comput. Phys. **21** (1976), *3*, 291-304.

[14] U. Erdogan and T. Ozis, *A smart nonstandard finite difference scheme for second order nonlinear boundary value problems.* J. Comput. Phys. **230** (2011), *17*, 6464–6474.

[15] W. Lick and T. Gaskins, *A consistent and accurate procedure for obtaining difference equations from differential equation.* Internat. J. Numer. Methods Engrg. **20** (1984), 1433–1441.

Received May 19, 2019

University of Delhi
Dyal Singh College
Department of Mathematics
Lodhi Road, New Delhi-110003, India
pramod_10p@hotmail.com