DEPTH AND STANLEY DEPTH OF POWERS OF THE EDGE IDEALS OF SOME CATERPILLAR AND LOBSTER TREES

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Let S be a ring of polynomials in finitely many variables over a field. In this paper, we give lower bounds for depth and Stanley depth of modules of the type S/I^t for $t \geq 1$, where I is the edge ideal of some caterpillar and lobster trees. These new bounds are much sharper than the existing bounds for the classes of ideals we considered.

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INTRODUCTION

Let K be a field and $S = K[x_1, \ldots, x_m]$ be the polynomial ring in m variables over K. Let N be a finitely generated $\mathbb{Z}^m$-graded S-module. Let $uK[Z]$ be the K-subspace generated by all elements of the form uy where u is a homogeneous element in N, y is a monomial in $K[Z]$ and $Z \subseteq \{x_1, x_2, \ldots, x_m\}$. If $uK[Z]$ is a free $K[Z]$-module then it is called a Stanley space of dimension $|Z|$. A decomposition $\mathcal{D}$ of the K-vector space N as a finite direct sum of Stanley spaces is called a Stanley decomposition of N. Let

$$\mathcal{D} \,:\, N = \bigoplus_{j=1}^{r} u_j K[Z_j].$$

The Stanley depth of $\mathcal{D}$ is $\operatorname{sdepth}(\mathcal{D}) = \min\{|Z_j|\}$. The number

$$\operatorname{sdepth}(N) := \max\{\operatorname{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } N\},$$

is called the Stanley depth of N. If $\mathfrak{m} := (x_1, x_2, \ldots, x_m)$ then the depth of N is defined to be the common length of all maximal N-sequences in $\mathfrak{m}$. In [19], Stanley conjectured that $\operatorname{sdepth}(N) \geq \operatorname{depth}(N)$. This conjecture was later disproved by Duval et al. [7] in 2016. Stanley depth has been studied extensively in the last two decades, see, for example, [12, 13, 15, 16, 18]. Let I be a monomial ideal of S. It is known, in general, that the depth of the

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powers of I, depth(S/I^t), stabilize for large t. Indeed, this follows from the general theorems that apply to any graded ideal of S. In particular, by [3] $\min\{\text{depth}(S/I^t)\} \leq n - l(I)$, where $l(I)$ is the analytic spread of I, and the minimum is taken over all powers t. In [2], Brodmann showed that for sufficiently large t, depth(S/I^t) is a constant, and this constant is bounded above by $n - l(I)$. However, relatively little is known about depth(S/I^t) for specific values of t other than $t = 1$. For some classes of powers of monomial ideals for which values or bounds are known, we refer the readers to [10, 8, 14].

Let G be a finite, undirected and simple graph on m vertices $v_1, v_2, \ldots, v_m$. The *edge ideal* $I(G)$ of the graph G is the ideal of S generated by all monomials of the form x_ix_j such that $\{v_i, v_j\}$ is an edge of G. Let $m \geq 2$. A *path* P_m of length $m-1$ is a graph on m vertices such that the vertex set of P_m can be ordered in a way that whenever two vertices are consecutive in the list, there is an edge between them. A *tree* is a graph in which any two vertices are connected by exactly one path. The *diameter* of a connected graph G is the maximum distance between any two vertices, where the distance between two vertices is given by the minimum length of a path connecting the vertices. For $t \geq 1$, Morey gave a lower bound for depth of $S/I^t(T)$ when T is a tree in [14], in terms of the diameter of T. Later on, in [16] Pournaki et. al. proved that this lower bound also serves as a lower bound for Stanley depth of $S/I^t(T)$. This lower bound being dependent on the diameter of a tree is weak in general.

The main focus of this paper is to give a better lower bound for some classes of trees. These bounds are independent of the diameters of the trees we considered and are much better than the bounds given in [14, 16]. Note that the lower bound for the depth of an edge ideal of a tree also provide a lower bound on the power for which the depth stabilizes. Our work encompasses the computation of lower bounds for depth and Stanley depth of the powers of the edge ideals associated with some classes of caterpillar and lobster trees. The lower bound for the caterpillar trees depends on the power of the edge ideal, the number of leaves and the order of the path, see Theorem 2.7 and Corollary 2.8, while for the lobster trees it depends upon the power of the edge ideal and the number of near leaves, see Theorem 3.5 and Corollary 3.6. These parameters collectively make much sharper bounds than the bounds given in [14, 16]. We gratefully acknowledge the use of the computer algebra system CoCoA ([6]).

1. DEFINITIONS AND NOTATIONS

We start this section with a review of some notations and definitions, for more details, see [9, 20]. Note that by abuse of notation, x_i will at times be used to denote both a vertex of a graph G and the corresponding variable of the polynomial ring S. Let G be a graph with $V(G) := \{x_1, x_2, \dots, x_m\}$ and edge set $E(G)$. For a vertex x_i of G the set $N(x_i) := \{x_j \mid x_i x_j \in E(G)\}$ is called the *neighborhood* of the vertex x_i. A vertex x_i is called a *leaf* (or *pendant vertex*) if $N(x_i)$ has cardinality one and x_i is called *isolated* if $N(x_i) = \emptyset$. The *parity* of an integer is its attribute of being even or odd. A graph with one vertex and no edges is called a *trivial graph.* An *internal vertex* is a vertex in a tree which is not a leaf. A graph with one internal vertex and k leaves is called a *k-star*, denoted by $\mathcal{S}_k$. Note that $\mathcal{S}_0$ is a trivial graph. A *caterpillar tree* is a tree in which the removal of all pendant vertices results in a path. A *lobster tree* is a tree with the property that the removal of pendant vertices leaves a caterpillar.

Definition 1.1. Let $n \geq 1$ and $k \geq 2$ be integers and P_n be a path on n vertices $\{u_1, u_2, \dots, u_n\}$ that is $E(P_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\}$ (for $n = 1$, $E(P_n) = \emptyset$). We define a graph on nk vertices by attaching $k-1$ pendant vertices at each u_i. We denote this graph by $P_{n,k}$.

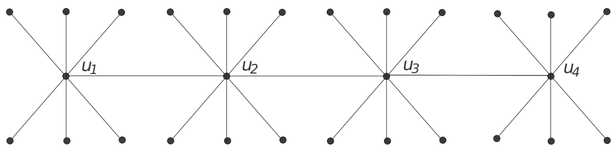

Figure 1 – $P_{4,7}$

For example of $P_{n,k}$, see Fig. 1.

Let $n \geq 2$, $k \geq 2$ and $l \geq 1$ be integers with $l \in [k] := \{1, 2, \dots, k\}$. Let $P_{n,k,l}$ be a graph which is obtained by removing $k-l$ pendant vertices attached to the vertex u_n of the graph $P_{n,k}$. Note that $P_{n,k,k} = P_{n,k}$. For examples of $P_{n,k,l}$, see Fig. 2. It is easy to see that $P_{n,k,l}$ belongs to the family of caterpillar graphs.

Remark 1.2. In Definition 1.1, $P_{1,k}$ represents a $(k-1)$-star.

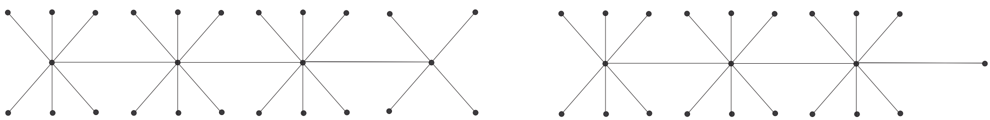

Figure 2 – From left to right, $P_{4,7,5}$ and $P_{4,7,1}$, respectively.

Definition 1.3. Let $r \geq 2$ and $p \geq 1$ be integers. Let $\mathcal{S}_r$ be a star on $r+1$ vertices say $\{v_1, v_2, \dots, v_r, v_{r+1}\}$ with v_{r+1} as a central vertex. We define a graph by adding p pendant vertices to each vertex v_i with $1 \leq i \leq r$. We denote this graph by $\mathcal{S}_{r,p}$.

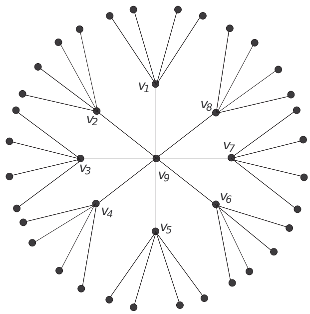

Figure 3 – $\mathcal{S}_{8,4}$

For example of $\mathcal{S}_{r,p}$, see Fig. 3.

Let $q \geq 0$ with $q \leq p$ be an integer, then $\mathcal{S}_{r,p,q}$ is a graph which is obtained by removing $p - q$ leaves from exactly one v_i. Clearly, $\mathcal{S}_{r,p,p} = \mathcal{S}_{r,p}$. For examples of $\mathcal{S}_{r,p,q}$, see Fig. 4.

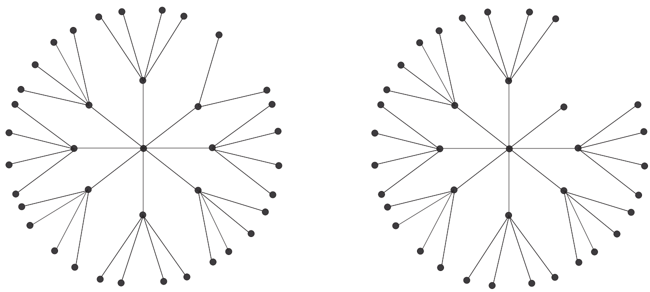

Figure 4 – From left to right, $\mathcal{S}_{8,4,2}$ and $\mathcal{S}_{8,4,0}$, respectively

In order to make the paper self contained, we recall some known results that we use in this paper.

Lemma 1.4 (Depth Lemma, [4, Proposition 1.2.9]). *If*

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0,$$

is a short exact sequence of $\mathbb{Z}^m$-graded S-modules, then

1. $\operatorname{depth} A_2 \geq \min\{\operatorname{depth} A_1, \operatorname{depth} A_3\}$,

2. $\operatorname{depth} A_1 \geq \min\{\operatorname{depth} A_2, \operatorname{depth} A_3 + 1\}$,

3. $\operatorname{depth} A_3 \geq \min\{\operatorname{depth} A_2, \operatorname{depth} A_1 - 1\}$.

A. Rauf proved the following lemma for Stanley depth.

Lemma 1.5 ([18, Lemma 2.2]). *If*

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0,$$

is a short exact sequence of $\mathbb{Z}^m$-graded S-modules, then

$$\operatorname{sdepth} A_2 \geq \min\{\operatorname{sdepth} A_1, \operatorname{sdepth} A_3\}.$$

Lemma 1.6 ([11, Lemma 3.6]). *Let I be a monomial ideal of S. If $S' = S[y]$ is the polynomial ring over S in the variable y, then* $\operatorname{depth}(S'/IS') = \operatorname{depth}(S/I) + 1$ *and* $\operatorname{sdepth}(S'/IS') = \operatorname{sdepth}(S/I) + 1$.

Lemma 1.7 ([15, Lemma 1.1]). *Let $I \subset K[x_1, \dots, x_r] = S_1$ and $J \subset K[x_{r+1}, \dots, x_m] = S_2$ be monomial ideals, where $1 < r < m$. Then*

$$\operatorname{depth}_S(S/(IS + JS)) = \operatorname{depth}_{S_1}(S_1/I) + \operatorname{depth}_{S_2}(S_2/J).$$

Lemma 1.8 ([18, Theorem 3.1]). *Let $I \subset K[x_1, \dots, x_r] = S_1$ and $J \subset K[x_{r+1}, \dots, x_m] = S_2$ be monomial ideals, where $1 < r < m$. Then*

$$\operatorname{sdepth}_S(S/(IS + JS)) \geq \operatorname{sdepth}_{S_1}(S_1/I) + \operatorname{sdepth}_{S_2}(S_2/J).$$

The following two lemmas play a key role in the proofs of our main theorems.

Lemma 1.9 ([14, Lemma 2.10]). *Let G be a graph and $I = I(G)$. Let x_i be a leaf of G and x_j be the unique neighbor of x_i. Then $(I^t : x_i x_j) = I^{t-1}$, for any $t \geq 2$.*

Lemma 1.10 ([14, Lemma 2.5]). *Let I be a square-free monomial ideal in a polynomial ring S and let M be a monomial in S. If y is a variable such that y does not divide M and J is the extension in R of the minor of I formed by setting $y = 0$, then $((I^t : M), y) = ((J^t : M), y)$, for any $t \geq 1$.*

Proposition 1.11 ([1, Theorems 2.6 and 2.9]). *If $I = I(\mathcal{S}_{m-1})$, which is a square-free monomials ideal of S, then* $\operatorname{depth}(S/I) = \operatorname{sdepth}(S/I) = 1$ *and* $\operatorname{depth}(S/I^t), \operatorname{sdepth}(S/I^t) \geq 1$.

Lemma 1.12 ([14, Lemma 2.6]). *Let G be a bipartite graph and $I = I(G)$. Then for all $t \geq 1$,*

$$\operatorname{depth}(S/I^t) \geq 1.$$

Theorem 1.13 ([5, Theorem 1.4]). *For a finitely generated $\mathbb{Z}^m$-graded S-module N, if* $\operatorname{sdepth}(N) = 0$ *then* $\operatorname{depth}(N) = 0$.

A forest is a graph with each connected component a tree. The following theorems give lower bounds for depth and Stanley depth of powers of an edge ideal corresponding to a forest.

THEOREM 1.14 ([14, Theorem 3.4]). *Let G be a forest having s number of connected components $G_1, G_2\ , \dots, G_s$. Let $I = I(G)$ and d_j be the diameter of G_j and suppose $d = \max_j\{d_j\}$. Then for $t \geq 1$*

$$\operatorname{depth}(S/I^t) \geq \max\big\{\lceil\frac{d-t+2}{3}\rceil + s - 1, s\big\}.$$

THEOREM 1.15 ([16, Theorem 2.7]). *Let G be a forest having s number of connected components $G_1, G_2\ , \dots, G_s$. Let $I = I(G)$ and d_j be the diameter of G_j and suppose $d = \max_j\{d_j\}$. Then for $t \geq 1$*

$$\operatorname{sdepth}(S/I^t) \geq \max\big\{\lceil\frac{d-t+2}{3}\rceil + s - 1, s\big\}.$$

Let v be a vertex of G, v is called a *near leaf* of G if v is not a leaf and $N(v)$ contains at most one vertex that is not a leaf. Let a denote the number of near leaves of G. The bounds for depth and Stanley depth are strengthened by the following results in the same papers.

COROLLARY 1.16 ([14, Corollary 3.7]). *Let G be a forest having s number of connected components $G_1, G_2\ , \dots, G_s$. Let $I = I(G)$ and d_j be the diameter of G_j and suppose $d = \max_j\{d_j\}$, and let a be the number of near leaves of a component of diameter d. Then for $t \geq 1$*

$$\operatorname{depth}(S/I^t) \geq \max\big\{\lceil\frac{d-t+a}{3}\rceil + s - 1, s\big\}.$$

COROLLARY 1.17 ([16, Corollary 3.2]). *Let G be a forest having s number of connected components $G_1, G_2\ , \dots, G_s$. Let $I = I(G)$ and d_j be the diameter of G_j and suppose $d = \max_j\{d_j\}$, and let a be the number of near leaves of a component of diameter d. Then for $t \geq 1$*

$$\operatorname{sdepth}(S/I^t) \geq \max\big\{\lceil\frac{d-t+a}{3}\rceil + s - 1, s\big\}.$$

If T is a tree, then the following corollary is an immediate consequence of the Corollary 1.16 and 1.17.

COROLLARY 1.18. *Let T be a tree and d be the diameter of T and let a be the number of near leaves of T. If $I = I(T)$, then for $t \geq 1$*

$$\operatorname{depth}(S/I^t), \operatorname{sdepth}(S/I^t) \geq \max\big\{\lceil\frac{d-t+a}{3}\rceil, 1\big\}.$$

The bound in Corollary 1.18 depends on the diameter of T and the number of near leaves in T. If I is the edge ideal of $P_{n,k}$ or $\mathcal{S}_{r,p}$, we give lower bounds for depth and Stanley depth of S/I^t as Corollary 2.8 and Corollary 3.6. We observe that our bounds are much sharper than the bounds given in Corollary 1.18.

2. POWERS OF EDGE IDEAL OF A SUBCLASS OF CATERPILLAR TREE

Let $n, k \geq 2$ and $l \in [k]$. We define $A_i := \{y_{1i}, y_{2i}, \dots, y_{(k-1)i}\}$, for $1 \leq i \leq n-1$, and $A_n := \{y_{1n}, y_{2n}, \dots, y_{(l-1)n}\}$, where $A_n = \emptyset$ if $l = 1$. Let $\bar{A}_i := \{u_i\} \cup A_i$, $\bar{A}_n := \{u_n\} \cup A_n$ and $A := \bar{A}_1 \cup \bar{A}_2 \cup \cdots \cup \bar{A}_n$. Let S be the polynomial ring over a field K in variables of set A that is $S := K[A]$. Let $I = I(P_{n,k})$, in this section, we give lower bounds for depth and Stanley depth of S/I^t for $t \geq 1$. We denote by $G(I)$, the minimal set of monomial generators of the monomial ideal I. If $l \geq 2$, then

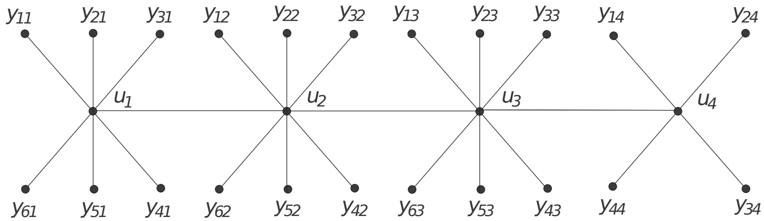

Figure 5 – Graph $P_{4,7,5}$ with labelled vertices.

$$G(I(P_{n,k,l})) = \bigcup_{i=1}^{n-1} \left\{u_i u_{i+1}, u_i y_{1i}, u_i y_{2i}, \dots, u_i y_{(k-1)i}\right\} \cup \left\{u_n y_{1n}, u_n y_{2n}, \dots, u_n y_{(l-1)n}\right\}.$$

If $l = 1$, then

$$G(I(P_{n,k,1})) = \bigcup_{i=1}^{n-1} \left\{u_i u_{i+1}, u_i y_{1i}, u_i y_{2i}, \dots, u_i y_{(k-1)i}\right\}.$$

Note that $P_{n,k,k} = P_{n,k}$. Also for $1 \leq j \leq n-1$, we have

$$I(P_{j,k}) := I(P_{n,k,l}) \cap K[\bar{A}_1 \cup \cdots \cup \bar{A}_{j-1} \cup \bar{A}_j].$$

LEMMA 2.1. *If $I = I(P_{n,k,l})$ then for $t \geq 1$, $(I^t, u_n) = (I^t(P_{n-1,k}), u_n)$.*

Proof. The inclusion $(I^t(P_{n-1,k}), u_n) \subseteq (I^t, u_n)$ is clear. Conversely, if $u \in I^t$ is a monomial which is not divisible by u_n, then, by the definition of $G(I)$, it follows that $u \in I^t(P_{n-1,k})$. □

Remark 2.2. Let $t \geq 1$. From Proposition 1.11 it follows,

$$\operatorname{depth}(S/I(P_{1,k})) = \operatorname{sdepth}(S/I(P_{1,k})) = 1$$

and $\operatorname{depth}(S/I^t(P_{1,k})), \operatorname{sdepth}(S/I^t(P_{1,k})) \geq 1$.

LEMMA 2.3. *Let $k \geq 2$, $l \in [k]$ and $I = I(P_{2,k,l})$. We have that*

$$\operatorname{depth}(S/I)\,,\ \operatorname{sdepth}(S/I) \geq l.$$

Proof. Clearly, u_1 and u_2 have $k-1$ and $l-1$ pendant vertices, respectively. Consider the short exact sequence:

$$0 \longrightarrow S/(I:u_2) \xrightarrow{\cdot u_2} S/I \longrightarrow S/(I,u_2) \longrightarrow 0.$$

Now $(I : u_2) = (x : x \in N(u_2))$ and $S/(I : u_2) \cong K[A_1 \cup \{u_2\}]$, thus $\operatorname{depth}(S/(I : u_2)) = k$. $S/(I,u_2) \cong (K[\bar{A}_1]/I(P_{1,k}))[A_2]$. Thus by Lemma 1.6 and Proposition 1.11 $\operatorname{depth}(S/(I,u_2)) = 1 + l - 1 = l$. By Depth Lemma, $\operatorname{depth}(S/I) \geq \min\{\operatorname{depth}(S/(I : u_2)), \operatorname{depth}(S/(I,u_2))\} \geq l$. Proof for Stanley depth is similar using Lemma 1.5. □

PROPOSITION 2.4. *Let $n, k \geq 2$, $l \in [k]$ and $I = I(P_{n,k,l})$. We have that*

$$\operatorname{depth}(S/I), \operatorname{sdepth}(S/I) \geq \begin{cases} \left(\frac{n-2}{2}\right)k + l\,, & \text{if } n \text{ is even;} \\ \left(\frac{n-1}{2}\right)k + 1\,, & \text{if } l \geq 2 \text{ and } n \text{ is odd;} \\ \left(\frac{n-1}{2}\right)k\,, & \text{if } l = 1 \text{ and } n \text{ is odd.} \end{cases}$$

Proof. For $n = 2$, the conclusion follows from Lemma 2.3. For $n = 3$, we consider the following short exact sequence

$$0 \longrightarrow S/(I:u_3) \xrightarrow{\cdot u_3} S/I \longrightarrow S/(I,u_3) \longrightarrow 0.$$

Now

$$S/(I:u_3) \cong S/((y : y \in N(u_3)) + I(P_{1,k})) \cong \big(K[\bar{A}_1]/I(P_{1,k})\big)[A_2 \cup \{u_3\}].$$

From Lemma 1.6 and Proposition 1.11 it follows that

$$\operatorname{depth} S/(I:u_3) = k + \operatorname{depth}(K[\bar{A}_1]/I(P_{1,k})) = k + 1.$$

Now, since $S/(I,u_3) \cong S/(I(P_{2,k}),u_3) \cong \big(K[\bar{A}_1 \cup \bar{A}_2]/(I(P_{2,k})\big)[A_3]$, by Lemma 1.6 and Lemma 2.3 it follows that

$$\operatorname{depth}(S/(I,u_3)) = l - 1 + \operatorname{depth}(K[\bar{A}_1 \cup \bar{A}_2]/(I(P_{2,k})) \geq l - 1 + k = k + l - 1.$$

So by Depth Lemma

$$\operatorname{depth}(S/I) \geq \begin{cases} k+1\,, & \text{if } l \geq 2; \\ k\,, & \text{if } l = 1. \end{cases}$$

Similarly, from Lemma 1.5 it follows that

$$\operatorname{sdepth}(S/I) \geq \begin{cases} k+1\,, & \text{if } l \geq 2; \\ k\,, & \text{if } l = 1. \end{cases}$$

For $n \geq 4$, we consider the following short exact sequence

$$0 \longrightarrow S/(I : u_n) \xrightarrow{\cdot u_n} S/I \longrightarrow S/(I, u_n) \longrightarrow 0.$$

Notice that

$$\begin{aligned} S/(I : u_n) &= S/((y : y \in N(u_n)) + I(P_{n-2,k})) \\ &\cong \big(K[A\backslash(\bar{A}_{n-1} \cup \bar{A}_n)]/I(P_{n-2,k})\big)[A_{n-1} \cup u_n]. \end{aligned}$$

and

$$S/(I, u_n) \cong \big(K[A\backslash\bar{A}_n]/I(P_{n-1,k})\big)[A_n].$$

Case 1: n is even.

For $k \geq 2$, since $P_{n-2,k} = P_{n-2,k,k}$ and $P_{n-1,k} = P_{n-1,k,k}$, using induction on n and Lemma 1.6, it follows that:

$$\begin{aligned} \operatorname{depth}(S/(I : u_n)) &= \operatorname{depth}\big(K[A\backslash(\bar{A}_{n-1} \cup \bar{A}_n)]/I(P_{n-2,k})\big) + (k-1) + 1 \\ &\geq \big((\frac{n-2-2}{2})k + k\big) + k = (\frac{n}{2})k, \end{aligned}$$

and

$$\begin{aligned} \operatorname{depth}(S/(I, u_n)) &= \operatorname{depth}\big(K[A\backslash\bar{A}_n]/I(P_{n-1,k})\big) + (l-1) \geq \\ &\big((\frac{n-2}{2})k + 1\big) + (l-1) = (\frac{n-2}{2})k + l. \end{aligned}$$

Thus by Depth Lemma,

$$\operatorname{depth}(S/I) \geq (\frac{n-2}{2})k + l.$$

Case 2: n is odd.

Again by induction on n and Lemma 1.6,

$$\begin{aligned} \operatorname{depth}(S/(I : u_n)) &= \operatorname{depth}\big(K[A\backslash(\bar{A}_{n-1} \cup \bar{A}_n)]/I(P_{n-2,k})\big) + (k-1) + 1 \\ &\geq \big((\frac{n-3}{2})k + 1\big) + k = (\frac{n-1}{2})k + 1, \end{aligned}$$

and

$$\begin{aligned} \operatorname{depth}(S/(I, u_n)) &= \operatorname{depth}\big(K[A\backslash\bar{A}_n]/I(P_{n-1,k})\big) + (l-1) \geq \\ &\big((\frac{n-1-2}{2})k + k\big) + (l-1) = (\frac{n-1}{2})k + (l-1). \end{aligned}$$

Thus by Depth Lemma,

$$\operatorname{depth}(S/I) \geq \begin{cases} (\frac{n-1}{2})k + 1, & \text{if } l \geq 2; \\ (\frac{n-1}{2})k, & \text{if } l = 1. \end{cases}$$

Proof for the Stanley depth is similar by using Lemma 1.5 instead of Depth Lemma. $\square$

COROLLARY 2.5. *If $n \geq 2$, $k \geq 2$ and $I = I(P_{n,k})$ then*

$$\operatorname{depth}(S/I), \operatorname{sdepth}(S/I) \geq \begin{cases} \left(\frac{n}{2}\right)k, & \text{if } n \text{ is even;} \\ \left(\frac{n-1}{2}\right)k+1, & \text{if } n \text{ is odd.} \end{cases}$$

Example 2.6. By using CoCoA (for sdepth we use SdepthLib.coc [17]) it has been noticed that the equality may hold in some cases. For instance, $\operatorname{depth}(S/I(P_{4,4})) = \operatorname{sdepth}(S/I(P_{4,4})) = 8 = \left(\frac{4}{2}\right)4$, and $\operatorname{depth}(S/I(P_{5,3})) = \operatorname{sdepth}(S/I(P_{5,3})) = 7 = \left(\frac{5-1}{2}\right)3+1$.

For convenience, we label the vertices of $A_n = \{y_{1n}, y_{2n}, \dots, y_{(l-1)n}\}$ by $s_1, s_2, \dots, s_{l-1}$. Set $S_i := K[A]/(s_1, s_2, \dots, s_i)$ and $I_i := I \cap S_i$.

THEOREM 2.7. *Let $n, k \geq 2$, $l \in [k]$, $t \geq 1$, and $I = I(P_{n,k,l})$. We have that*

$$\operatorname{depth}(S/I^t), \operatorname{sdepth}(S/I^t) \geq \begin{cases} \max\left\{1, \left(\frac{n-t-1}{2}\right)k + l - 1\right\}, & \text{if } n \text{ and } t \text{ have opposite parity;} \\ \max\left\{1, \left(\frac{n-t}{2}\right)k\right\}, & \text{if } n \text{ and } t \text{ have the same parity and } 2 \leq l \leq k; \\ \max\left\{1, \left(\frac{n-t}{2}\right)k - 1\right\}, & \text{if } n \text{ and } t \text{ have the same parity and } l = 1. \end{cases}$$

Proof. Since $P_{n,k,l}$ is a bipartite graph, from Lemma 1.12 it follows that $\operatorname{depth}(S/I^t) \geq 1$ for all $t \geq 1$. We use induction on n and t. For $n \geq 2$ and $t = 1$, the result follows from Proposition 2.4. For $n = 2$ and $t \geq 1$, the result follows from Lemma 1.12. Let $n = 3$. For $t \geq 3$, the result again follows from Lemma 1.12. If $t = 2$ then we need to prove the desired inequality. Let $I = I(P_{3,k,l})$. We will prove that

$$\operatorname{depth}(S/I^2) \geq \max\left\{1, \left(\frac{3-2-1}{2}\right)k + l - 1\right\} = \max\{1, l-1\}.$$

If $l = 1$, then $\max\{1, l-1\} = 1$ and from Lemma 1.12 we have that $\operatorname{depth}(S/I^2) \geq 1$. Assume that $l \geq 2$ and consider the following short exact sequence

$$0 \longrightarrow S/(I^2 : u_3) \xrightarrow{\cdot u_3} S/I^2 \longrightarrow S/(I^2, u_3) \longrightarrow 0.$$

By Lemma 2.1, $S/(I^2, u_3) \cong S/(I^2(P_{2,k}), u_3) \cong (K[A \backslash (\bar{A}_3)]/I^2(P_{2,k}))[A_3]$. Therefore,

$$\operatorname{depth}(S/(I^2, u_3)) \geq 1 + (l-1) = l.$$

We consider the following family of short exact sequences:

$$0 \longrightarrow S_0/(I_0^2 : u_3s_1) \xrightarrow{\cdot s_1} S_0/(I_0^2 : u_3) \longrightarrow S_0/((I_0^2 : u_3), s_1) \longrightarrow 0,$$
$$0 \longrightarrow S_1/(I_1^2 : u_3s_2) \xrightarrow{\cdot s_2} S_1/(I_1^2 : u_3) \longrightarrow S_1/((I_1^2 : u_3), s_2) \longrightarrow 0,$$
$$\vdots$$
$$0 \longrightarrow S_{l-2}/(I_{l-2}^2 : u_3s_{l-1}) \xrightarrow{\cdot s_{l-1}} S_{l-2}/(I_{l-2}^2 : u_3) \longrightarrow S_{l-2}/((I_{l-2}^2 : u_3), s_{l-1}) \longrightarrow 0.$$

By Lemma 1.9, $\operatorname{depth}(S_i/(I_i^2 : u_3s_{i+1})) = \operatorname{depth}(S_i/I_i)$ and by Proposition 2.4

$$\operatorname{depth}(S_i/(I_i^2 : u_3s_{i+1})) \geq k+1.$$

Since $S_{l-2}/((I_{l-2}^2 : u_3), s_{l-1}) \cong S_{l-1}/(I_{l-1}^2 : u_3)$, consider the following short exact sequence

$$0 \to S_{l-1}/(I_{l-1}^2 : u_3u_2) \xrightarrow{\cdot u_2} S_{l-1}/(I_{l-1}^2 : u_3) \to S_{l-1}/((I_{l-1}^2 : u_3), u_2) \to 0.$$

By Lemma 1.9, $\operatorname{depth}(S_{l-1}/(I_{l-1}^2 : u_3u_2)) = \operatorname{depth}(S_{l-1}/I_{l-1})$, here $l = 1$ and by Proposition 2.4

$$\operatorname{depth}(S_{l-1}/(I_{l-1}^2 : u_3u_2)) \geq k.$$

Clearly, $S_{l-1}/((I_{l-1}^2 : u_3), u_2) \cong \Big(K[A\backslash(\bar{A}_2 \cup \bar{A}_3)]/I^2(P_{1,k})\Big)[A_2 \cup \{u_3\}]$, therefore by Lemma 1.6 and Proposition 1.11, we have

$$\begin{aligned}\operatorname{depth}(S_{l-1}/((I_{l-1}^2 : u_3), u_2)) &= \operatorname{depth}\big(K[A\backslash(\bar{A}_2 \cup \bar{A}_3)]/I^2(P_{1,k})\big) + (k-1) + 1 \\ &\geq k+1.\end{aligned}$$

Depth Lemma implies

$$\operatorname{depth}(S/I^2) \geq l.$$

Now let $n \geq 4$, $t \geq 2$ and $I = I(P_{n,k,l})$. We consider two cases:

CASE 1: n and t have the same parity.
(a). Let $l = 1$. Consider the following short exact sequence

$$0 \longrightarrow S/(I^t : u_n) \xrightarrow{\cdot u_n} S/I^t \longrightarrow S/(I^t, u_n) \longrightarrow 0.$$

By Lemma 2.1, $S/(I^t, u_n) = S/(u_n, I^t(P_{n-1,k}))$. For $k \geq 2$, since $P_{n-1,k} = P_{n-1,k,k}$, $n-1$ and t have the opposite parity, using induction on n, it follows that:

$$\operatorname{depth}(S/(I^t, u_n)) \geq \Big(\frac{n-1-t-1}{2}\Big)k + (k-1) = \Big(\frac{n-t}{2}\Big)k - 1.$$

We consider another short exact sequence as follows

$$0 \longrightarrow S/(I^t : u_nu_{n-1}) \xrightarrow{\cdot u_{n-1}} S/(I^t : u_n) \longrightarrow S/((I^t : u_n), u_{n-1}) \longrightarrow 0.$$

Since u_{n-1} is the unique neighbor of u_n, from Lemma 1.9 it follows that $(I^t : u_n u_{n-1}) = I^{t-1}$. Now n and $t-1$ have the opposite parity, thus, by induction on t

$$\begin{aligned}\operatorname{depth}(S/(I^t : u_n u_{n-1})) &= \operatorname{depth}(S/I^{t-1}) \\ &\geq \Big(\frac{n-(t-1)-1}{2}\Big)k + 1 - 1 \\ &= \Big(\frac{n-t}{2}\Big)k,\end{aligned}$$

and by Lemma 1.10 we have

$$S/((I^t : u_n), u_{n-1}) \cong \big(K[A\backslash(\bar{A}_{n-1} \cup \bar{A}_n)]/I^t(P_{n-2,k})\big)[A_{n-1} \cup \{u_n\}].$$

By induction on n and Lemma 1.6

$$\operatorname{depth}(S/((I^t : u_n), u_{n-1}) \geq \Big((\frac{n-t-2}{2})k\Big) + k \geq \Big(\frac{n-t}{2}\Big)k.$$

Thus, by Depth Lemma we have,

$$\operatorname{depth}(S/I^t) \geq \Big(\frac{n-t}{2}\Big)k - 1.$$

(b). Let $l \geq 2$. Consider the following short exact sequence

$$0 \longrightarrow S/(I^t : u_n) \xrightarrow{\cdot u_n} S/I^t \longrightarrow S/(I^t, u_n) \longrightarrow 0.$$

By Lemma 2.1, $S/(I^t, u_n) \cong \big(K[A\backslash\bar{A}_n]/(I^t(P_{n-1,k}))\big)[A_n]$. Therefore, by Lemma 1.6 we have

$$\operatorname{depth}(S/(I^t, u_n)) = \operatorname{depth}(K[A\backslash\bar{A}_n]/(I^t(P_{n-1,k})) + |A_n|$$

Here $n-1$ and t have the opposite parity, so by induction on n,

$$\operatorname{depth}(S/(I^t, u_n)) \geq \Big((\frac{n-t-2}{2})k + k - 1\Big) + (l-1) \geq \Big(\frac{n-t}{2}\Big)k + l - 2.$$

Now, we find lower bound for depth of module $S/(I^t : u_n)$.

Let $0 \leq i \leq l-2$. By Lemma 1.10, $S_i/((I_i^t : u_n), s_{i+1}) \cong S_{i+1}/(I_{i+1}^t : u_n)$ where $S_0 = S$ and $I_0 = I$. We consider the following family of short exact sequences:

$$0 \longrightarrow S_0/(I_0^t : u_n s_1) \xrightarrow{\cdot s_1} S_0/(I_0^t : u_n) \longrightarrow S_0/((I_0^t : u_n), s_1) \longrightarrow 0,$$

$$0 \longrightarrow S_1/(I_1^t : u_n s_2) \xrightarrow{\cdot s_2} S_1/(I_1^t : u_n) \longrightarrow S_1/((I_1^t : u_n), s_2) \longrightarrow 0,$$

$$\vdots$$

$$0 \longrightarrow S_{l-2}/(I_{l-2}^t : u_n s_{l-1}) \xrightarrow{\cdot s_{l-1}} S_{l-2}/(I_{l-2}^t : u_n) \longrightarrow S_{l-2}/((I_{l-2}^t : u_n), s_{l-1}) \longrightarrow 0.$$

By Lemma 1.9, $\operatorname{depth}(S_i/(I_i^t : u_n s_{i+1})) = \operatorname{depth}(S_i/I_i^{t-1})$. Here n and $t-1$ have the opposite parity so, by using induction on t,

$$\operatorname{depth}(S_i/(I_i^t : u_n s_{i+1})) \geq \Big\lceil\frac{n-(t-1)-1}{2}\Big\rceil k + (l-1-i) \geq \Big\lceil\frac{n-t}{2}\Big\rceil k.$$

Again by Lemma 1.10, $S_{l-2}/((I_{l-2}^t : u_n), s_{l-1}) \cong S_{l-1}/(I_{l-1}^t : u_n)$. Now consider the following short exact sequence

$$0 \longrightarrow S_{l-1}/(I_{l-1}^t : u_n u_{n-1}) \xrightarrow{\cdot u_{n-1}} S_{l-1}/(I_{l-1}^t : u_n) \longrightarrow S_{l-1}/((I_{l-1}^t : u_n), u_{n-1}) \longrightarrow 0,$$

by Lemma 1.9, $\operatorname{depth}(S_{l-1}/(I_{l-1}^t : u_n u_{n-1})) = \operatorname{depth}(S_{l-1}/I_{l-1}^{t-1})$. By using induction on t,

$$\operatorname{depth}(S_{l-1}/(I_{l-1}^t : u_n u_{n-1})) \geq \Big\lceil\frac{n-(t-1)-1}{2}\Big\rceil k + \big(l-(l-1)\big)-1 = \Big\lceil\frac{n-t}{2}\Big\rceil k.$$

Clearly, $S_{l-1}/((I_{l-1}^t : u_n), u_{n-1}) \cong S'/I_{l-1}^t S'$, where $S' = K[A\backslash(A_n \cup \{u_{n-1}\})]$. Thus u_n and all variables in A_{n-1} are regular on $S'/I_{l-1}^t S'$. Since

$$S'/I_{l-1}^t S' \cong \Big(K[A\backslash(\bar{A}_{n-1} \cup \bar{A}_n)]/I^t(P_{n-2,k})\Big)[A_{n-1} \cup \{u_n\}],$$

therefore, by Lemma 1.6, we get $\operatorname{depth}(S'/I_{l-1}^t S') = \operatorname{depth}\Big(K[A\backslash(\bar{A}_{n-1} \cup \bar{A}_n)]/I^t(P_{n-2,k})\Big) + (k-1) + 1$. Here $n-2$ and t have the same parity, so by induction on n

$$\operatorname{depth}(S'/I_{l-1}^t S') \geq \Big\lceil\frac{n-2-t}{2}\Big\rceil k + (k-1) + 1 = \Big\lceil\frac{n-t}{2}\Big\rceil k.$$

Depth Lemma implies

$$\operatorname{depth}(S/I^t) \geq \Big\lceil\frac{n-t}{2}\Big\rceil k.$$

CASE 2: n and t have the opposite parity.

(a). Let $l = 1$. For this, consider the following short exact sequence:

$$0 \longrightarrow S/(I^t : u_n) \xrightarrow{\cdot u_n} S/I^t \longrightarrow S/(I^t, u_n) \longrightarrow 0.$$

By Lemma 2.1, $S/(I^t, u_n) = S/(u_n, I^t(P_{n-1,k}))$. Here $n-1$ and t have the same parity, so by induction on n,

$$\operatorname{depth}(S/(I^t, u_n)) \geq \Big\lceil\frac{n-t-1}{2}\Big\rceil k.$$

For the depth of module $S/(I^t : u_n)$, we consider another short exact sequence as follows:

$$0 \longrightarrow S/(I^t : u_n u_{n-1}) \xrightarrow{\cdot u_{n-1}} S/(I^t : u_n) \longrightarrow S/((I^t : u_n), u_{n-1}) \longrightarrow 0,$$

since u_{n-1} is the unique neighbor of u_n thus by Lemma 1.9 we have $(I^t : u_n u_{n-1}) = I^{t-1}$. Now n and $t-1$ have the same parity, thus, by induction on t

$$\begin{aligned}\operatorname{depth}(S/(I^t : u_n u_{n-1})) &= \operatorname{depth}(S/I^{t-1})\\ &\geq \Big(\frac{n-(t-1)}{2}\Big)k - 1\\ &= \Big(\frac{n-t-1}{2}\Big)k + k - 1\\ &> \Big(\frac{n-t-1}{2}\Big)k,\end{aligned}$$

and by Lemma 1.10 we have

$$S/((I^t : u_n), u_{n-1}) \cong \big(K[A\backslash(\bar{A}_{n-1} \cup \bar{A}_n)]/I^t(P_{n-2,k})\big)[A_{n-1} \cup u_n],$$

by induction on n and Lemma 1.6

$$\begin{aligned}\operatorname{depth}(S/((I^t : u_n), u_{n-1})) &\geq \Big((\frac{n-t-3}{2})k + k - 1\Big) + k\\ &= (\frac{n-t-1}{2})k - 1 + k\\ &> \Big(\frac{n-t-1}{2}\Big)k.\end{aligned}$$

Thus by Depth Lemma we have,

$$\operatorname{depth} S/I^t \geq \Big(\frac{n-t-1}{2}\Big)k.$$

(b). Let $l \geq 2$. Consider the short exact sequence

$$0 \longrightarrow S/(I^t : u_n) \xrightarrow{\cdot u_n} S/I^t \longrightarrow S/(I^t, u_n) \longrightarrow 0.$$

By Lemma 2.1, $S/(I^t, u_n) \cong \big(K[A\backslash\bar{A}_n]/(I^t(P_{n-1,k}))\big)[A_n]$.

$$\operatorname{depth}(S/(I^t, u_n) = \operatorname{depth}\Big(K[A\backslash\bar{A}_n]/(I^t(P_{n-1,k}))\Big) + |A_n|.$$

Here $n-1$ and t have the same parity, so by induction on n,

$$\operatorname{depth}(S/(I^t, u_n)) \geq (\frac{n-t-1}{2})k + l - 1.$$

Consider again the following family of short exact sequences:

$$0 \longrightarrow S_0/(I_0^t : u_n s_1) \xrightarrow{\cdot s_1} S_0/(I_0^t : u_n) \longrightarrow S_0/((I_0^t : u_n), s_1) \longrightarrow 0,$$
$$0 \longrightarrow S_1/(I_1^t : u_n s_2) \xrightarrow{\cdot s_2} S_1/(I_1^t : u_n) \longrightarrow S_1/((I_1^t : u_n), s_2) \longrightarrow 0,$$
$$\vdots$$
$$0 \longrightarrow S_{l-2}/(I_{l-2}^t : u_n s_{l-1}) \xrightarrow{\cdot s_{l-1}}$$

$$S_{l-2}/(I^t_{l-2} : u_n) \longrightarrow S_{l-2}/((I^t_{l-2} : u_n), s_{l-1}) \longrightarrow 0.$$

By Lemma 1.9, $\operatorname{depth}(S_i/(I^t_i : u_n s_{i+1})) = \operatorname{depth}(S_i/I^{t-1}_i)$. Here n and $t-1$ have the same parity so, by using induction on t,

$$\begin{aligned}\operatorname{depth}(S_i/(I^t_i : u_n s_{i+1})) &\geq \Big(\frac{n-(t-1)}{2}\Big)k\\ &= \Big(\frac{n-t+1}{2}\Big)k\\ &> \Big(\frac{n-t-1}{2}\Big)k + l - 1.\end{aligned}$$

Since $S_{l-2}/((I^t_{l-2} : u_n), s_{l-1}) \cong S_{l-1}/(I^t_{l-1} : u_n)$, consider the following short exact sequence

$$\begin{aligned}0 \longrightarrow S_{l-1}/(I^t_{l-1} : u_n u_{n-1}) &\xrightarrow{\cdot u_{n-1}}\\ &S_{l-1}/(I^t_{l-1} : u_n) \longrightarrow S_{l-1}/((I^t_{l-1} : u_n), u_{n-1}) \longrightarrow 0.\end{aligned}$$

By Lemma 1.9, $\operatorname{depth}(S_{l-1}/(I^t_{l-1} : u_n u_{n-1})) = \operatorname{depth}(S_{l-1}/I^{t-1}_{l-1})$, here $l = 1$ and n and $t-1$ have the same parity, thus by induction on t,

$$\operatorname{depth}(S_{l-1}/(I^t_{l-1} : u_n u_{n-1})) \geq \Big(\frac{n-(t-1)}{2}\Big)k - 1 \geq \Big(\frac{n-t-1}{2}\Big)k + l - 1.$$

Clearly $S_{l-1}/((I^t_{l-1} : u_n), u_{n-1}) \cong S'/I^t_{l-1}S'$, where $S' = K[A\backslash(A_n \cup \{u_{n-1}\})]$. Thus u_n and all variables in A_{n-1} are regular on $S'/I^t_{l-1}S'$. Since

$$S'/I^t_{l-1}S' \cong \Big(K[A\backslash(\bar{A}_{n-1} \cup \bar{A}_n)]/I^t(P_{n-2,k})\Big)[A_{n-1} \cup \{u_n\}],$$

therefore $\operatorname{depth}(S'/I^t_{l-1}S') = \operatorname{depth}\Big(K[A\backslash(\bar{A}_{n-1} \cup \bar{A}_n)]/I^t(P_{n-2,k})\Big) + (k-1) + 1$. Since $n-2$ and t have the opposite parity, so by induction on n and Lemma 1.6, we get

$$\operatorname{depth}(S'/I^t_{l-1}S') \geq \Big(\frac{n-2-t-1}{2}\Big)k + (k-1) + k \geq \Big(\frac{n-t-1}{2}\Big)k + l - 1.$$

Depth Lemma implies

$$\operatorname{depth}(S/I^t) \geq \Big(\frac{n-t-1}{2}\Big)k + l - 1.$$

This completes the proof for depth. Note that from Lemma 1.12 and Theorem 1.13 we have that $\operatorname{sdepth}(S/I^t) \geq 1$, for all $t \geq 1$. Proof for the Stanley depth is similar by using Lemma 1.5 instead of Depth Lemma. $\square$

Corollary 2.8. *If $n \geq 2$, $t \geq 1$ and $I = I(P_{n,k})$ then*

$$\operatorname{depth}(S/I^t), \operatorname{sdepth}(S/I^t) \geq \begin{cases} \max\{1, \left(\frac{n-t+1}{2}\right)k - 1\}, & \text{if } n \text{ and } t \text{ have opposite parity;} \\ \max\{1, \left(\frac{n-t}{2}\right)k\}, & \text{if } n \text{ and } t \text{ have same parity.} \end{cases}$$

A comparison of the actual values of depth with lower bound in Corollary 2.8 is shown in the following example.

Example 2.9. By using CoCoA we have, $\operatorname{depth}(S/I^2(P_{4,4})) = 5$ and $\operatorname{depth}(S/I^2(P_{5,3})) = 6$, while by our Corollary 2.8, $\operatorname{depth}(S/I^2(P_{4,4})) \geq 4$ and $\operatorname{depth}(S/I^2(P_{5,3})) \geq 5$.

Also, this new bound is much sharper than the one given in Corollary 1.18, as shown in the following example.

Example 2.10. Let $I = I(P_{n,k})$ with $n = 50$ and $k = 10$. Clearly $P_{n,k}$ has two near leaves and its diameter is 51. Let $t = 15$. By Corollary 1.18

$$\operatorname{depth}(S/I^{15}), \operatorname{sdepth}(S/I^{15}) \geq \lceil \frac{51 - 15 + 2}{3} \rceil = 13.$$

Whereas our Corollary 2.8 shows that

$$\operatorname{depth}(S/I^{15}), \operatorname{sdepth}(S/I^{15}) \geq \left(\frac{50 - 15 + 1}{2}\right)10 - 1 = 179.$$

Comparison shows a noteworthy difference between both the lower bounds.

3. POWERS OF EDGE IDEAL OF A SUBCLASS OF LOBSTER TREE

Let $r \geq 2$ and $p, t \geq 1$ be some integers. In this section, we give an upper bound for depth and Stanley depth of $S/I^t(\mathcal{S}_{r,p})$. Our bounds depend only on r and t. We significantly improve the bound for the depth and Stanley depth of $S/I^t(\mathcal{S}_{r,p})$ given in Corollary 1.18. It is easy to see that the diameter of $\mathcal{S}_{r,p}$ is fixed for any r and p. The bound given in Corollary 1.18 depends on t and diameter of $\mathcal{S}_{r,p}$ so this bound becomes weak for bigger values of t. Where as our bound given in Corollary 3.6 being independent of the diameter of $\mathcal{S}_{r,p}$ is better. Before proving the results of this section, we introduce some notations. Let $p \geq 1$ and $q \geq 0$ be integers such that $q \leq p$. Let $1 \leq i \leq r-1$, $B_i := \{x_{1i}, x_{2i}, \dots, x_{pi}\}$, $B_r := \{x_{1r}, x_{2r}, \dots, x_{qr}\}$ ($B_r = \emptyset$, if $q = 0$), $\bar{B}_i := B_i \cup \{v_i\}$ and $\bar{B}_r := B_r \cup \{v_r\}$. Let $B := \{v_{r+1}\} \cup \bar{B}_1 \cup \bar{B}_2 \cup \dots \cup \bar{B}_r$ and define $S := K[B]$.

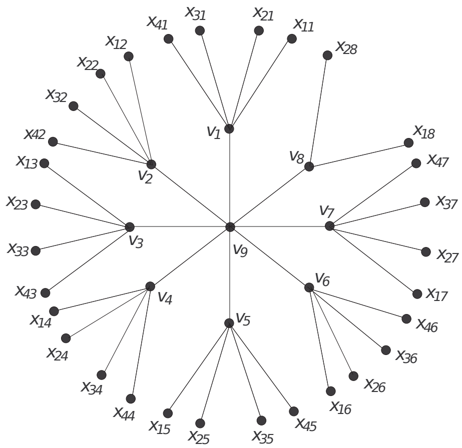

Figure 6 – Graph $\mathcal{S}_{8,4,2}$ with labelled vertices.

If $q \geq 1$, then

$$G(I(\mathcal{S}_{r,p,q})) = \{v_{r+1}v_r, v_r x_{1r}, v_r x_{2r}, \dots, v_r x_{qr}\} \cup \bigcup_{i=1}^{r-1} \{v_{r+1}v_i, v_i x_{1i}, v_i x_{2i}, \dots, v_i x_{pi}\}.$$

If $q = 0$, then

$$G(I(\mathcal{S}_{r,p,q})) = \{v_{r+1}v_r\} \cup \bigcup_{i=1}^{r-1} \{v_{r+1}v_i, v_i x_{1i}, v_i x_{2i}, \dots, v_i x_{pi}\}.$$

Note that for $1 \leq j \leq r-1$, we have that

$$I(\mathcal{S}_{j,k}) := I(\mathcal{S}_{r,p,q}) \cap K[\{v_{r+1}\} \cup \bar{B}_1 \cup \cdots \cup \bar{B}_{j-1} \cup \bar{B}_j].$$

Before proving the main result of this section we prove the result when $t = 1$ in the following lemma.

Lemma 3.1. *Let $r \geq 2$ and $I = I(\mathcal{S}_{r,p,q})$. We have that*

$$\operatorname{depth}(S/I), \operatorname{sdepth}(S/I) \geq \begin{cases} r-1, & \text{if } q = 0; \\ r, & \text{otherwise.} \end{cases}$$

Proof. Consider the short exact sequence

$$0 \longrightarrow S/(I : v_{r+1}) \xrightarrow{\cdot v_{r+1}} S/I \longrightarrow S/(I, v_{r+1}) \longrightarrow 0,$$

we have $S/(I : v_{r+1}) \cong K[B_1 \cup B_2 \cup \cdots \cup B_r \cup \{v_{r+1}\}]$, therefore

$$\operatorname{depth}(S/(I : v_{r+1})) = (r-1)p + q + 1 \geq r.$$

By definition of I, $(I, v_{r+1}) = (I(H), v_{r+1})$, where H is a forest with r connected components, say $H_1, H_2, \ldots, H_{r-1}, H_r$. It can easily be seen that among these r connected components, $r-1$ components are p-star graphs while one component is a q-star. Without loss of generality, we may assume that for $1 \leq i \leq r-1$, $H_i \cong \mathcal{S}_p$ and $H_r \cong \mathcal{S}_q$. If $q = 0$ then $H_r \cong \mathcal{S}_0$ is a trivial graph on one vertex, say v. From Lemma 1.7 and Proposition 1.11 it follows that

$$\begin{aligned}\operatorname{depth}(S/(I, v_{r+1})) &= \operatorname{depth}(K[B\backslash\{v_{r+1}\}]/(I(H))) \\ &= \operatorname{depth}(K[V(H_1)]/I(H_1)) + \cdots + \operatorname{depth}(K[V(H_{r-1})]/I(H_{r-1})) + \\ &\quad \operatorname{depth}(K[v]/(v)) = \underbrace{1 + 1 + \ldots + 1}_{(r-1)-\text{ times}} + 0 = r-1.\end{aligned}$$

If $q \neq 0$ then $H_r \cong \mathcal{S}_q$. From Lemma 1.7 and Proposition 1.11 it follows that

$$\begin{aligned}\operatorname{depth}(S/(I, v_{r+1})) &= \operatorname{depth}(K[B\backslash\{v_{r+1}\}]/(I(H))) = \\ &\operatorname{depth}(K[V(H_1)]/I(H_1)) + \cdots + \operatorname{depth}(K[V(H_{r-1})]/I(H_{r-1})) + \\ &\quad \operatorname{depth}(K[V(H_r)]/I(H_r)) = \underbrace{1 + 1 + \ldots + 1}_{r-\text{ times}} = r.\end{aligned}$$

Hence, by applying Depth Lemma the required result follows. The result for Stanley depth can be proved in the same lines by using Lemma 1.5 instead of Depth Lemma and Lemma 1.8 instead of Lemma 1.7. □

Corollary 3.2. *Let $r \geq 2$ and $I = I(\mathcal{S}_{r,p})$. We have that*

$$\operatorname{depth}(S/I), \operatorname{sdepth}(S/I) \geq r.$$

Example 3.3. We use CoCoA and show that the equality may hold in Corollary 3.2. For instance, we have $\operatorname{depth}(S/I(\mathcal{S}_{4,2})) = \operatorname{sdepth}(S/I(\mathcal{S}_{4,2})) = 4$ and $\operatorname{depth}(S/I(\mathcal{S}_{5,2}))= \operatorname{sdepth}(S/I(\mathcal{S}_{5,2})) = 5$.

Lemma 3.4. *If $I = I(\mathcal{S}_{r,p,q})$ then for $t \geq 1$, $(I^t, v_r) = (I^t(\mathcal{S}_{r-1,p}), v_r)$.*

Proof. The inclusion $(I^t(\mathcal{S}_{r-1,p}), v_r) \subseteq (I^t, v_r)$ is clear. Conversely, if $w \in I^t$ is a monomial which is not divisible by v_r, then, by definition of $G(I)$, it follows that $w \in I^t(\mathcal{S}_{r-1,p})$. □

Now moving towards the main result of this section.

Theorem 3.5. *Let $r \geq 2$, $t \geq 1$, $p \geq 1$ and $0 \leq q \leq p$. If $I = I(\mathcal{S}_{r,p,q})$ then*

$$\operatorname{depth}(S/I^t), \operatorname{sdepth}(S/I^t) \geq \begin{cases} \max\{1, r-t\}, & \text{if } q = 0; \\ \max\{1, r-t+1\}, & \text{otherwise.} \end{cases}$$

Proof. We use induction on r and t. If $r = 2$ and $t \geq 1$, the result follows from Lemma 1.12. If $t = 1$ and $r \geq 2$, the result follows from Lemma 3.1. Assume $r \geq 3$ and $t \geq 2$. Consider the short exact sequence

$$0 \longrightarrow S/(I^t : v_r) \xrightarrow{\cdot v_r} S/I^t \longrightarrow S/(I^t, v_r) \longrightarrow 0 \tag{3.1}$$

by Depth Lemma

$$\operatorname{depth}(S/I^t) \geq \min\{\operatorname{depth}(S/(I^t : v_r)), \operatorname{depth}(S/(I^t, v_r))\}. \tag{3.2}$$

Case 1: $q = 0$. From Lemma 3.4 it follows that $(I^t, v_r) = (I^t(\mathcal{S}_{r-1,p}), v_r)$. Since $\mathcal{S}_{r-1,p} = \mathcal{S}_{r-1,p,p}$ and $p \geq 1$, using induction on r, it follows that

$$\operatorname{depth}(S/(I^t, v_r)) = \operatorname{depth}(K[B\backslash\{v_r\}]/I^t(\mathcal{S}_{r-1,p})) \geq (r-1) - t + 1 = r - t.$$

We consider the short exact sequence

$$0 \longrightarrow S/(I^t : v_r v_{r+1}) \xrightarrow{\cdot v_{r+1}} S/(I^t : v_r) \longrightarrow S/((I^t : v_r), v_{r+1}) \longrightarrow 0, \tag{3.3}$$

since by Lemma 1.9, $(I^t : v_r v_{r+1}) = I^{t-1}$ so by induction on t

$$\operatorname{depth}(S/(I^t : v_r v_{r+1})) = \operatorname{depth}(S/I^{t-1}) \geq r - (t-1) = r - t + 1.$$

Let $R' = K[B\backslash\{v_{r+1}\}]$ and $I' = IR'$. By Lemma 1.10, $S/((I^t : v_r), v_{r+1}) \cong R'/(I^t : v_r) \cong R'/(I')^t$. Clearly, v_r is a regular variable on $R'/(I')^t$ and I' corresponds to the edge ideal of a forest consisting of $r-1$ connected components and each component is a p-star. Therefore, by Lemma 1.6 and Theorem 1.14

$$\begin{aligned}
\operatorname{depth}(S/((I^t : v_r), v_{r+1})) &= \operatorname{depth}(R'/(I')^t) \\
&\geq \max\Big\{\Big\lceil\frac{2-t+2}{3}\Big\rceil + (r-1) - 1, r-1\Big\} + 1 \\
&= (r-1) + 1 \\
&= r \\
&> r - t.
\end{aligned}$$

By applying Depth Lemma on sequence (3.3) we get $\operatorname{depth}(S/(I^t : v_r)) \geq r - t$. From Eq. (3.2) the result follows.

Case 2: $q \geq 1$. Let us label the vertices of $B_r \neq \emptyset$ with $\{y_1, y_2, \dots, y_q\}$. By Lemma 3.4, $(I^t, v_r) = (I^t(\mathcal{S}_{r-1,p}), v_r)$, therefore

$$S/(I^t, v_r) \cong (K[B\backslash\bar{B}_r]/I^t(\mathcal{S}_{r-1,p}))[B_r].$$

Thus by Lemma 1.6 and induction on r

$$\begin{aligned}
\operatorname{depth}(S/(I^t, v_r)) &= \operatorname{depth}(S/I^t(\mathcal{S}_{r-1,p})) + |B_r| \\
&\geq ((r-1) - t + 1) + q = q + r - t,
\end{aligned}$$

Let $R_i = S/(y_1, \dots, y_i)$ and $I_i = IR_i$, where $R_0 = S$ and $I_0 = I$. We consider a family of short exact sequences:

$$0 \longrightarrow R_0/(I_0^t : v_r y_1) \xrightarrow{\cdot y_1} R_0/(I^t : v_r) \longrightarrow R_0/((I^t : v_r), y_1) \longrightarrow 0$$

$$0 \longrightarrow R_1/(I_1^t : v_r y_2) \xrightarrow{\cdot y_2} R_1/(I_1^t : v_r) \longrightarrow R_1/((I_1^t : v_r), y_2) \longrightarrow 0$$

$$0 \longrightarrow R_2/(I_2^t : v_r y_3) \xrightarrow{\cdot y_3} R_2/(I_2^t : v_r) \longrightarrow R_2/((I_2^t : v_r), y_3) \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow R_{q-1}/(I_{q-1}^t : v_r y_q) \xrightarrow{\cdot y_q}$$
$$R_{q-1}/(I_{q-1}^t : v_r) \longrightarrow R_{q-1}/((I_{q-1}^t : v_r), y_q) \longrightarrow 0.$$

For $0 \leq i \leq q-1$, by Lemma 1.9 we have $R_i/(I_i^t : v_r y_{i+1}) \cong R_i/I_i^{t-1}$. Thus by induction on t

$$\text{(3.4)}\quad \operatorname{depth}(R_i/(I_i^t : v_r y_{i+1})) = \operatorname{depth}(R_i/I_i^{t-1}) \geq r - (t-1) + 1 = r - t + 2.$$

By Lemma 1.10, $R_{q-1}/((I_{q-1}^t : v_r), y_q) \cong R_q/(I_q^t : v_r)$, now we have the short exact sequence

$$0 \longrightarrow R_q/(I_q^t : v_r v_{r+1}) \xrightarrow{\cdot v_{r+1}} R_q/(I_q^t : v_r) \longrightarrow R_q/((I_q^t : v_r), v_{r+1}) \longrightarrow 0,$$

by Lemma 1.9 we have $\operatorname{depth}(R_q/(I_q^t : v_r v_{r+1})) = \operatorname{depth}(R_q/I_q^{t-1})$. Thus it is easy to see that $R_q/I_q \cong K[B \backslash B_r]/I(\mathcal{S}_{r,p,0})$. Thus by induction on t and case (1), $\operatorname{depth}(R_q/I_q^{t-1}) \geq r - (t-1) = r - t + 1$. Clearly $R_q/((I_q^t : v_r), v_{r+1}) \cong R''/L^t$, where $R'' = [B \backslash (B_r \cup \{v_{r+1}\})]$ and $L = IR''$ is the edge ideal of a forest consisting of $r-1$ connected components and each component is a p-star. Clearly v_r is a regular variable on R''/L^t. Therefore by Lemma 1.6 and Theorem 1.14

$$\begin{aligned}
\operatorname{depth}(R_q/((I_q^t : v_r), v_{r+1})) &= \operatorname{depth}(R''/L^t) \\
&\geq \max\Big\{\Big\lceil \frac{2 - t + 2}{3} \Big\rceil + (r-1) - 1, r - 1\Big\} + 1 \\
&= (r-1) + 1 \\
&= r \\
&\geq r - t + 1.
\end{aligned}$$

Thus, by Depth Lemma $\operatorname{depth}(S/(I^t : v_r)) \geq r - t + 1$, and hence by Eq. (3.2) $\operatorname{depth}(S/I^t) \geq r - t + 1$. On the same lines by using Lemma 1.5 instead of Depth Lemma one can prove the result for Stanley depth. □

Corollary 3.6. *Let $t \geq 1$, $p \geq 1$ and $I = I(\mathcal{S}_{r,p})$. We have that*

$$\operatorname{depth}(S/I^t), \operatorname{sdepth}(S/I^t) \geq \max\{1, r - t + 1\}.$$

A comparison of the actual values of depth with lower bound in Corollary 3.6 is shown in the following example.

Example 3.7. By using CoCoA we have, $\operatorname{depth}(S/I^2(\mathcal{S}_{4,2})) = 4$ and $\operatorname{depth}(S/I^2(\mathcal{S}_{5,2})) = 5$, while by our Corollary 3.6, $\operatorname{depth}(S/I^2(\mathcal{S}_{4,2})) \geq 3$ and $\operatorname{depth}(S/I^2(\mathcal{S}_{5,2})) \geq 4$.

Also this new bound is much sharper than the one given in Corollary 1.18, as shown in the following example. Note that $\mathcal{S}_{r,p}$ has r near leaves.

Example 3.8. Let $I = I(\mathcal{S}_{r,p})$ with $r = 55$ and $t = 10$. Clearly $d = 4$, thus by Corollary 1.18 we have

$$\operatorname{depth}(S/I^{10}), \operatorname{sdepth}(S/I^{10}) \geq \lceil \frac{4 - 10 + 55}{3} \rceil = 17.$$

While by our Corollary 3.6,

$$\operatorname{depth}(S/I^{10}), \operatorname{sdepth}(S/I^{10}) \geq 55 - 10 + 1 = 46.$$

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