GAPS IN A SUMSET OF A POLYNOMIAL WITH ITSELF

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In this note, we give some lower and upper bounds for the gaps between consecutive elements of an integer sequence whose elements are expressible in the form $P(m) + P(n)$, where P is a degree d polynomial with integer coefficients, and m, n are nonnegative integers.

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1. INTRODUCTION

Throughout, let $P \in \mathbb{Z}[x]$ be a polynomial of degree d with leading coefficient $a \in \mathbb{N}$. Consider the set of nonnegative integers

$$S = \{P(j) \geq 0 : j \in \{0, 1, 2, \dots\}\}, \tag{1}$$

and the sumset of S with itself

$$\mathcal{S}_P = S + S = \{s_1 < s_2 < s_3 < \dots\}, \tag{2}$$

so that each $s_j \in \mathcal{S}_P$ is expressible in the form $P(m) + P(n)$ with some $m, n \in \mathbb{N} \cup \{0\}$. In this note, we show that

THEOREM 1.1. *For each $\varepsilon > 0$ there are infinitely many $j \in \mathbb{N}$ such that*

$$s_{j+1} - s_j > \frac{1}{a}\left(\frac{195}{898} - \varepsilon\right) \log s_j \tag{3}$$

if $d = 2$ and

$$s_{j+1} - s_j > (4 - \varepsilon) \frac{d\Gamma(2/d) a^{2/d}}{\Gamma(1/d)^2} s_j^{1-2/d} \tag{4}$$

if $d \geq 3$. On the other hand, there is a positive constant $\gamma = \gamma(P)$ such that

$$s_{j+1} - s_j < \gamma s_j^{(1-1/d)^2} \tag{5}$$

for each $j \in \mathbb{N}$.

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The most known case of this problem is that with $P(x) = x^2$, when $\{s_j\}_{j=1}^{\infty}$ is the sequence of integers expressible by the sum of two perfect squares. Those are the integers whose prime decomposition contains primes of the form $4k+3$ in even powers only. For $P(x) = x^2$ the lower bound of the form $s_{j+1} - s_j \gg \log s_j$ is due to Richards [11]. Then, the constant implicit in $\gg$ has been improved by Dietmann and Elsholtz [5], and by Kalmynin and Konyagin [9] (both preprints were included in a subsequent paper [6]). The upper bound $s_{j+1} - s_j \ll s_j^{1/4}$ is due to Bambah and Chowla [1], with subsequent improvements by Uchiyama [14], Shiu [12], [13], and Jameson [8]. See also the related results of Kaplan and Williams [10] (on numbers expressible by a quadratic form), and Diananda [4], where the upper bound in the case $P(x) = x^d$ has been obtained.

Despite all the work, the gap between (3) and (5) for $d = 2$ is a huge one. However, for $d \geq 3$ it is not so large, since both bounds are exponential: the exponents of s_j in (4) and (5) are $1 - 2/d$ and $1 - 2/d + 1/d^2$, respectively. It would be of interest to find the correct growth of the gaps for $d \geq 3$ when the problems seem easier, even though the arithmetical structure of the set $\mathcal{S}_P$ is much more complicated. See, for instance, the paper of Broughan [2] for the characterization of integers expressible by the sum of two cubes.

Of course, there are infinitely many bounded gaps between s_j and s_{j+1}. To see this, simply select any two fixed integers $m_1, m_2 \geq 0$ such that $0 \leq P(m_1) < P(m_2)$. Taking $g = P(m_2) - P(m_1) \in \mathbb{N}$ and infinitely many $n \in \mathbb{N}$ satisfying $P(n) \geq 0$, we see that the elements $P(n) + P(m_1)$ and $P(n) + P(m_2)$ both belong to $\mathcal{S}_P$. Hence, for infinitely many $j \in \mathbb{N}$, we have $1 \leq s_{j+1} - s_j \leq g$.

Results similar to those in Theorem 1.1 hold for *integer-valued polynomials*, namely, those $P \in \mathbb{Q}[x]$ for which $P(j) \in \mathbb{Z}$ for each $j \in \mathbb{Z}$. Clearly, each integer-valued polynomial of degree d multiplied by $d!$ belongs to $\mathbb{Z}[x]$ (see, e. g., [3]). So, inequalities (3), (4) and (5) of Theorem 1.1 for the corresponding sumsets of integer values polynomials also hold (although with different constants but with the same exponents $1 - 2/d$ and $(1 - 1/d)^2$ in (4) and (5)).

In the next section, we will give two auxiliary results. With these in hand, the proof of Theorem 1.1 becomes elementary.

2. **AUXILIARY RESULTS**

We begin with a recent result of Kalmynin and Konyagin [9].

LEMMA 2.1. *Let $\mathcal{S}$ be the sequence of all positive integers that can be represented by the sum of two squares of integers. For each $X \geq 2$ let $g(X)$ be the largest gap between two consecutive elements of $\mathcal{S}$ that do not exceed X.*

Then, for each $\varepsilon > 0$ *there exists* $X(\varepsilon)$ *such that*

$$g(X) > \Big(\frac{390}{449} - \varepsilon\Big) \log X$$

for $X \geq X(\varepsilon)$.

In principle, the next result can be verified by a direct calculation. However, we will give a much shorter proof applying a recent result on the cardinality of sumsets from [7]. (Throughout, $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the gamma function.)

LEMMA 2.2. *For each* $\varepsilon > 0$ *there exists* $X_1(\varepsilon)$ *such that the number of distinct elements of the sequence* (2) *not exceeding* X *is less than*

$$(1+\varepsilon)\frac{\Gamma(1/d)^2}{4da^{2/d}\Gamma(2/d)} X^{2/d}$$

for $X \geq X_1(\varepsilon)$.

Proof. Since $P(x) = ax^d + bx^{d-1} + \dots + c$, the number of (not necessarily distinct) elements of the set S defined in (1) is asymptotic to $(X/a)^{1/d}$ as $X \to \infty$. Hence, by [7, Corollary 2] (with $\alpha = 2/d$ there) and (2), it follows that

$$\limsup_{X\to\infty} \frac{\#\{(S+S) \cap [1,X]\}}{X^{2/d}} \leq a^{-2/d} \frac{\Gamma(1/d)^2}{4d\Gamma(2/d)}.$$

This implies the assertion of Lemma 2.2. □

3. PROOF OF THEOREM 1.1

Suppose first that $d = 2$, that is, $P(x) = ax^2 + bx + c \in \mathbb{Z}[x]$. For $s_j \in \mathcal{S}_P$ we have

$$\begin{aligned} s_j = P(m) + P(n) &= am^2 + bm + an^2 + bn + 2c \\ &= a\Big(m + \frac{b}{2a}\Big)^2 + a\Big(n + \frac{b}{2a}\Big)^2 - \frac{b^2}{2a} + 2c. \end{aligned}$$

Hence,

$$4as_j + 2b^2 - 8ac = (2am+b)^2 + (2an+b)^2. \tag{6}$$

Combining (6) with Lemma 2.1, for infinitely many $j \in \mathbb{N}$, we obtain

$$\begin{aligned} 4a(s_{j+1} - s_j) &= (4as_{j+1} + 2b^2 - 8ac) - (4as_j + 2b^2 - 8ac) \\ &> (c_0 - \varepsilon)\log(4as_{j+1} + 2b^2 - 8ac) > (c_0 - \varepsilon)\log s_j, \end{aligned}$$

where $c_0 = 390/449$. This implies (3). Observe that in the special case, when $2a$ divides b, by (6) and the same argument as above, we have

$$s_{j+1} - s_j > a\Big(\frac{390}{449} - \varepsilon\Big) \log s_j,$$

for infinitely many $j \in \mathbb{N}$.

Suppose now that $d \geq 3$. Assume that the sequence $\mathcal{S}_P$ has exactly $l \geq 2$ distinct elements not exceeding X, say $s_1 < s_2 < \cdots < s_l$. Then, at least one gap $s_{j+1} - s_j$, where $j = 1, \dots, l-1$ is at least $(s_l - s_1)/(l-1)$. By the inequality (5), which will be proved later, for each fixed ϵ and each sufficiently large $X \geq X(\epsilon)$, we obtain

$$s_l - s_1 \geq s_l - \frac{\epsilon X}{2} = s_{l+1} - (s_{l+1} - s_l) - \frac{\epsilon X}{2} > s_{l+1} - \gamma s_l^{(1-1/d)^2} - \frac{\epsilon X}{2}$$
$$> X - \gamma X^{(1-1/d)^2} - \frac{\epsilon X}{2} > X - \frac{\epsilon X}{2} - \frac{\epsilon X}{2} = (1-\epsilon)X.$$

So, applying Lemma 2.2 with $\varepsilon = \epsilon/2$ for this largest gap, we deduce that

$$s_{j+1} - s_j \geq \frac{s_l - s_1}{l-1} \geq \frac{(1-\epsilon)X}{l-1} > \frac{(1-\epsilon)X}{(1+\epsilon/2)\frac{\Gamma(1/d)^2}{4da^{2/d}\Gamma(2/d)} X^{2/d}}$$
$$> \frac{(1-2\epsilon)4da^{2/d}\Gamma(2/d)X^{1-2/d}}{\Gamma(1/d)^2},$$

which yields (4) with an appropriate choice of ϵ by $X \geq s_l > s_j$.

In order to prove (5), we will show first that for each sufficiently large $X \in \mathbb{R}$ the open interval $(X, X + \gamma_1 X^{(1-1/d)^2})$, $\gamma_1 > 0$, contains an element of $\mathcal{S}_P$.

Given

$$P(x) = ax^d + bx^{d-1} + \cdots + c \in \mathbb{Z}[x],$$

we first take a real number t satisfying

$$\{t\} = \{(X/a)^{1/d}\} \quad \text{and} \quad \frac{b}{ad} < t \leq 1 + \frac{b}{ad}. \tag{7}$$

(Here and in (9), $\{y\}$ stands for the fractional part of $y \in \mathbb{R}$.) With this choice, as $X \to \infty$, the number $u = (X/a)^{1/d} - t$ is a positive integer and

$$P(u) = X - \Big(\frac{X}{a}\Big)^{(d-1)/d}(adt - b) + O(X^{(d-2)/d}). \tag{8}$$

Hence, $0 < P(u) < X$ and $P(u) \in S$, for each sufficiently large X.

Next, we select $q \in \mathbb{R}$ satisfying

$$\{q\} = \{X^{(d-1)/d^2}(adt-b)^{1/d}a^{(1-2d)/d^2}\} \quad \text{and} \quad \frac{b}{ad} - 1 \leq q < \frac{b}{ad}. \tag{9}$$

Then, $v = X^{(d-1)/d^2}(adt-b)^{1/d}a^{(1-2d)/d^2} - q$ is also a positive integer and

$$P(v) = \Big(\frac{X}{a}\Big)^{(d-1)/d}(adt-b) + (b-adq)\frac{X^{(d-1)^2/d^2}(adt-b)^{(d-1)/d}}{a^{(2d-1)(d-1)/d^2}} + O(X^{(d-1)(d-2)/d^2})$$

as $X \to \infty$.

Now, to evaluate the sum $P(u)+P(v)$, we add the last equality with (8). Then, the term $(X/a)^{(d-1)/d}(adt-b)$ cancels out. The coefficient for $X^{(d-1)^2/d^2}$ is positive by the choice of t and q in (7), (9). Therefore, as the exponents $(d-2)/d$ in (8) and $(d-1)(d-2)/d^2$ are smaller than $(d-1)^2/d^2 = (1-1/d)^2$, the inequalities

$$X < P(u) + P(v) < X + \gamma_1 X^{(1-1/d)^2}$$

hold with some positive number γ_1.

Summarising, we have proved that for each sufficiently large $X \in \mathbb{R}$ the interval $(X, X+\gamma_1 X^{(1-1/d)^2})$ contains an element of the set $\mathcal{S}_P$.

By increasing γ_1 to γ, if necessary, we conclude that $(X, X+\gamma X^{(1-1/d)^2})$ contains an element of $\mathcal{S}_P$ for every $X \geq 1$. Therefore, for every $j \in \mathbb{N}$ choosing $X = s_j$, we deduce (5).

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