ARMENDARIZ MODULES AND NILTORSIONLESS MODULES

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In this paper, we introduce the concepts niltorsionless modules and NS-modules (which are generalisations of NI-rings). We prove that the class of niltorsionless modules contains the classes of semiprime and, in particular, the class of regular modules. We prove that over an NI-ring, every module is an NS-module. An example is provided to show that the converse is false. We also prove that over an NI-ring, $\frac{M}{\mathrm{NilRej}(M)}$ is reduced. Further, the concepts weak Armendariz modules and weakly semicommutative modules (analogues of the corresponding ring-theoretic concepts) are briefly studied.

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1. INTRODUCTION

Marks [10] called a ring R an *NI-ring* if the set $\mathrm{Nil}(R)$ of its nilpotent elements is an ideal of R. NI-rings have been a focus of attention recently. Hizem [7, Theorem 1], established the following characterization of NI-rings: a ring R is an NI-ring if and only if it is a nil power serieswise Armendariz ring, i.e., whenever power series $f(X) = \sum a_i X^i$, $g(X) = \sum b_j X^j$ in $R[[X]]$ satisfy $f(X)g(X) \in \mathrm{Nil}(R)[[X]]$, then $a_i b_j \in \mathrm{Nil}(R)$, for all i and j. An Armendariz ring which is not an NI-ring was constructed by Antoine [2, Example 4.8]. Chun et al. [6], studied rings satisfying the condition 'the set of nilpotent elements form a subring (which may not contain the identity of the ring)'. Armendariz rings as well as NI rings satisfy this condition.

We denote the factor ring $R/\mathrm{Nil}(R)$ of an NI-ring R by $\overline{R}$; it is clear that $\overline{R}$ is a reduced ring, i.e., it has no nonzero nilpotent elements. In Section 3, we prove a module-theoretic analogue of this result (Theorem 3.10).

After introduction of the notion of an Armendariz ring by Rege et al. [13], a large number of generalizations of that concept were introduced and studied by many authors. (See, for example, the references in [11].) In paragraph 4.7 of [13], the possibility of extending the concept of an Armendariz ring to modules was mentioned. A study of Armendariz modules and semicommutative

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modules (also called *zero-insertive (ZI)*) was carried out by Buhphang et al. [3]. The concept of a weak Armendariz ring, a ring-theoretic analogue of the concept of an Armendariz ring, was introduced and studied by Liu et al. [8]. Weakly semicommutative rings have also been studied by Liang et al. [9]. In Section 4, we introduce and briefly study the module theoretic extensions of the concepts of weak Armendariz and weakly semicommutative rings.

The dual (right) R-module $M^* := \mathrm{Hom}_R(M, R)$ associated with a left R-module M has played an important role in the study of some known concepts of interest to us - namely, torsionless modules, (Zelmanowitz) regular modules and semiprime modules (see, [14] and [15]). We use M^* to define and study the three concepts mentioned in the abstract.

2. PRELIMINARIES

By a ring we mean an associative ring with an identity element; R always denotes a ring. The set of all nonzero idempotents of R is denoted by $\overline{I}(R)$. Unless otherwise mentioned, by a module we mean a unitary left module. Module homomorphisms are written on the opposite side of the scalars. All our left-sided concepts and results have right-sided counterparts. For unexplained concepts and results we refer to [1] and Section 2A of [12].

Remark 2.1. Let M be a left R-module. By the *standard Morita context* of M we mean the quadruple $(R, M, M^*, E(M))$. Here $E(M)$ denotes the ring of endomorphisms of the left R-module M. There is a natural structure of a left $E(M)-$, right $R-$, bimodule on $M^* := \mathrm{Hom}_R(M, R)$. For $m, n \in M$ and $q \in M^*$, we define the element $[q, n]$ of $E(M)$ by $m[q, n] = (mq)n$. The 'generalized associativity situation' in the Morita context is exploited without explicit mention.

Next, we recall some definitions. If M and Q are left R-modules then the *reject* of M in Q is defined in [1, Section 8], as the R-submodule $\underset{f \in \mathrm{Hom}(M,Q}{\cap} Kerf$ of M; it is denoted by $Rej_M(Q)$. By $Rej(M)$ we mean the R-submodule $Rej_M(R)$ of M.

Definitions 2.2. A module M is *regular* [14] (resp., *semiprime*) if given a non-zero element $m \in M$, there exists $q \in M^*$ such that $(mq)m = m$ (resp., $(mq)m \neq 0$). M is *torsionless* if given $m \in M$, there exists $q \in M^*$ such that $mq \neq 0$.

Remark 2.3. Denoting the right ideal mM^* of R by J_m, it is easy to check that M is torsionless $\Leftrightarrow$ $Rej(M) = 0 \Leftrightarrow J_m \neq 0$, for all $m \neq 0$.

In the next definition, we introduce the concept of antisemiprime module.

Definition 2.4. A left R-module M is *antisemiprime* if for a given non-zero element $m \in M$ there exists a non-zero $q \in M^*$ such that $q(mq) \neq 0$.

Antiregular modules were defined and studied by Choudhuri et al. in a series of papers beginning with [4]. Their endomorphism rings were studied in [5].

Definition 2.5. A module M is *antiregular* if for each non-zero element m of M, there is a non-zero element $q \in M^*$ such that $q(mq) = q$.

In the next proposition, we record some implications. We recall the proof of (a) for the sake of completeness.

PROPOSITION 2.6. *The following implications hold for a module.*

(a) *Regular* $\Rightarrow$ *antiregular.*

(b) *Antiregular modules are semiprime as well as antisemiprime.*

(c) *If a module is semiprime or antisemiprime then it is torsionless.*

Proof. (a) Suppose for a non-zero element m of a left R-module M we have $(mq)m = m$ for some $q \in M^*$. Then $0 \neq mq = (mq)(mq)$ implies that $mq \in \bar{I}(R)$, the set of all non-zero idempotents of R. Write $\tilde{q} = q(mq) \in M^*$. Then $m\tilde{q} = (mq)^2 = mq \neq 0 \Rightarrow \tilde{q} \neq 0$. We also have $\tilde{q}(m\tilde{q}) = q(mq)^3 = q(mq) = \tilde{q} \neq 0$ showing that M is antiregular.

(b) Let m be a non-zero element of an antiregular module M. Now there exists $q \in M^*$ satisfying $q(mq) = q \neq 0$, proving M is antisemiprime. Further, $[(mq)m]q = (mq)(mq) = mq \neq 0$ yields $(mq)m \neq 0$. Hence, M is semiprime.

(c) Let $m \in M, m \neq 0$. If M is semiprime (resp., antisemiprime), there exists $q \in M^*$ satisfying $(mq)m \neq 0$ (resp., $q(mq) \neq 0$). In either case $mq \neq 0$, showing that M is torsionless. $\square$

Remark 2.7. It is clear that for every ring R the modules ${}_RR$ and R_R are antisemiprime, so the term 'antisemiprime ring' is redundant.

Next, we define four new concepts.

Definition 2.8. The *NilReject* of a left R-module M is the subset $\{m \in M \mid mq \in \mathrm{Nil}(R), \forall q \in M^*\}$ of M.

We denote the NilReject of a left R-module M by $\mathrm{NilRej}_M(R)$ (and by $\mathrm{NilRej}(M)$ if there is no possibility of confusion). The conditions 'NilReject is a submodule' and 'NilReject vanishes' are of interest. In order to study these conditions, we introduce Definitions 2.9 to 2.11.

Definition 2.9. A left R-module M is an *NS-module* if $\mathrm{NilRej}(M)$ is an R-submodule of M.

Definition 2.10. A left R-module M is *niltorsionless* if $\mathrm{NilRej}(M) = 0$.

Definition 2.11. A ring R is *left niltorsionless* if it is niltorsionless as a left R-module.

3. NS-MODULES AND NILTORSIONLESS MODULES

In this section, we record a number of results involving NS-modules and niltorsionless modules and rings.

Remarks 3.1. 1. Niltorsionless modules are, trivially, NS-modules.

2. Let M be a nonzero module for which $M^* = 0$. (The additive group of rationals and finite nontrivial abelian groups, regarded as modules over $\mathbb{Z}$, have this property.) Then $\mathrm{NilRej}(M) = M \neq 0$, showing that M is an NS-module which is not niltorsionless.

3. An analogue of Remark 2.1: M is niltorsionless $\Leftrightarrow$ $J_m = mM^* \nsubseteq \mathrm{Nil}(R), \forall m \neq 0$.

4. It is easy to see that

$$A := \{a \in R | aR \subset \mathrm{Nil}(R)\} = \{a \in R | Ra \subset rmNil(R)\}.$$

(So we can write $\mathrm{NilRej}(R)$ unambiguously in place of the subset A of R.) It follows that R is left niltorsionless $\Leftrightarrow$ $A = 0$ $\Leftrightarrow$ the ring R is right niltorsionless. In view of this, we talk simply of niltorsionless rings.

5. Clearly, if R is reduced, then $\mathrm{NilRej}(R) = 0$, and so R is niltorsionless.

6. We have, trivially $\mathrm{NilRej}(R) \subseteq \mathrm{Nil}(R)$. We also have: if R is a NI-ring, $\mathrm{NilRej}(R) = \mathrm{Nil}(R)$.

7. Subrings of niltorsionless rings need not be niltorsionless. For example, consider $R = M_2(K)$ the ring of 2×2 matrices over a field K which is niltorsionless since it is von Neumann regular. However, the subring $UT_2(K)$ of 2×2 upper triangular matrices over K is not niltorsionless since there does not exist any upper triangular matrix A with entries in K such that AE_{12} is non nilpotent.

We note in part (a) of the next proposition that the class of niltorsionless modules contains (over a given ring) the class of antiregular modules; we note in part (c) that modules belonging to this larger class are also semiprime as well as antisemiprime.

Proposition 3.2. *Let M be a left R-module.*

(a) *If M is antiregular, then M is niltorsionless.*

(b) *If M is regular, then M is niltorsionless.*

(c) *Niltorsionless modules are semiprime as well as antisemiprime.*

Proof. (a) Let $m \in M, m \neq 0$. Since M is antiregular there exists $q \in M^*$ such that $q(mq) = q \neq 0$. Hence, $(mq) \in \overline{I}(R)$ yielding $mq \notin \mathrm{Nil}(R)$.

(b) follows from (a).

(c) Let m be a non-zero element of a niltorsionless module M. Now there exists $q \in M^*$ such that $mq \notin \mathrm{Nil}(R)$. Hence, we have $(mq)(mq) \neq 0$, yielding $(mq)m \neq 0$ as well as $q(mq) \neq 0$. Therefore, M is semiprime as well as antisemiprime. □

Remark 3.3. It follows from the above proposition that antiregular rings (in particular, (von Neumann) regular rings) are niltorsionless and niltorsionless rings are semiprime.

In Proposition 3.4, we record a result concerning modules over NI-rings.

Proposition 3.4. *Let M be a left module over an NI-ring R. Then*

1. *M is an NS-module.*
2. $\mathrm{Nil}(R)M \leq \mathrm{NilRej}(M)$.

Proof. Since $\mathrm{NilRej}(M) = \bigcap_{q \in M^*} q^{-1}(\mathrm{Nil}(R))$, it is an R-submodule of M. Next let $t \in \mathrm{Nil}(R), m \in M$ and $q \in M^*$. Since R is an NI-ring, $(tm)q = t(mq) \in \mathrm{Nil}(R)$ for each $q \in M^*$ implying that $tm \in \mathrm{NilRej}(M)$. Hence, $\mathrm{Nil}(R)M \leq \mathrm{NilRej}(M)$. □

Examples 3.5. A ring over which all (left) modules are NS-modules need not be an NI-ring. (We use semisimple in the sense of Bourbaki.) If M is a left module over a semisimple ring R, then M is semisimple and projective, and therefore regular. Hence, by Proposition 3.2(b), M is niltorsionless and hence is an NS-module. However, semisimple rings are NI-rings exactly when they are (finite) products of division rings, by the Wedderburn structure theorem.

We also have:

Proposition 3.6. *Let M and W be left R-modules, and let $\beta : M \to W$ be an R-homomorphism. Then* $\mathrm{NilRej}(M)\beta \leq \mathrm{NilRej}(W)$.

Proof. Let $m \in \mathrm{NilRej}(M)$. Note that if $q \in W^*$, then $\beta \circ q \in M^*$. Hence $(m\beta)q = (m)(\beta \circ q) \in \mathrm{Nil}(R)$ yielding $m \in \mathrm{NilRej}(W)$. □

Corollary 3.7. NilRej(M) *is invariant under* $E(M)$.

Corollary 3.8. *If* M *is an* NS*-module then* NilRej(M) *is an* $R-E(M)$*-bisubmodule of the bimodule* ${}_RM_{E(M)}$.

Remarks 3.9. 1. Given an NS-module M, we shall use the notation $\overline{M}$ for its factor module $\dfrac{M}{\text{NilRej}(M)}$. If R is an NI-ring, as noted in Proposition 3.4(2), for every left R-module M we have $\text{Nil}(R)M \leq \text{NilRej}(M)$. It follows that $\text{Nil}(R) \leq ann({}_R\overline{M})$ yielding a canonical $\overline{R}$-structure on $\overline{M}$.

2. Let R be an NI-ring, and let M be a left R-module. Let $q \in M^*$. By the definition of NilRej(M), we have $(\text{NilRej}(M))q \leq \text{Nil}(R)$, and hence q induces an R-linear map $q_0 : \overline{M} \to \overline{R}$ defined by $\overline{m}q_0 = \overline{mq}$ which is easily seen to be $\overline{R}$-linear.

The notation introduced in Remarks 3.9 is used in the proof of the following analogue of the result that if R is an NI-ring, the ring $R/Nil(R)$ is reduced.

Theorem 3.10. *If* M *is a left module over an NI-ring* R*, then*

$$\overline{M} = \frac{M}{\text{NilRej}(M)}$$

is reduced as an R*-module.*

Proof. Let $\overline{m}$ be a nonzero element of $\overline{M}$. Then $m \notin \text{NilRej}(M)$. So for some $q \in M^*$ the element mq is non-nilpotent, yielding $\overline{m}q_0 = \overline{mq} \neq \overline{0}$ in $\overline{R}$. Hence $\overline{M}$ is torsionless as a left $\overline{R}$-module. As $\overline{R}$ is a reduced ring, $\overline{M}$ is reduced as an $\overline{R}$-module, by Proposition 2.4 of [12] and – by applying the 'change of rings' result noted in Proposition 3.2(2) of [12] – is reduced as an R-module as well. □

Remark 3.11. Zimmermann proved Proposition 3.12. We reproduce its short proof from [16] (Bemerkung 3.6, p. 33) since [16] is not easily accessible. We use the notation of Remark 2.1.

Proposition 3.12. *If* ${}_RM$ *is regular then* $M_{E(M)}$ *is also regular.*

Proof. Let $m \in M$. Now there exists $q \in M^*$ such that $(mq)m = m$. Clearly, $m[q, m] = (mq)m = m$. So $M_{E(M)}$ is regular. □

Since regular $\Rightarrow$ antiregular $\Rightarrow$ niltorsionless, we can ask whether similar results are valid for the other two classes of modules. The answer in the antiregular case is in the affirmative by Proposition 2.2 of [5]. We consider the niltorsionless case below.

Proposition 3.13. *If ${}_RM$ is niltorsionless then $M_{E(M)}$ is niltorsionless.*

Proof. Let $m \in M, m \neq 0$. Then there exists $q \in M^*$ such that $mq \notin \mathrm{Nil}(R)$. We claim that $[q,m] \notin \mathrm{Nil}(E(M))$. If $[q,m] \in \mathrm{Nil}(E(M))$, then there exists $k \in \mathbb{N}$ such that $[q,m]^k = 0$. This implies $0 = m[q,m]^k = (mq)^k m$ and so $0 = (mq)^{k+1}$. Thus $mq \in \mathrm{Nil}(R)$ which is a contradiction. This proves the claim. Notice that the map $\theta M \to E(M)$ defined by $\theta(n) = [q,n]$ is (by Morita context associativity) right $E(M)-$ linear. Now $\theta(m) = [q,m] \notin \mathrm{Nil}(E(M))$ shows that the right $E(M)-$module M is niltorsionless. □

We note that on putting $M = R$ in the above proposition we recover Remark 3.1(4).

Remarks 3.14. 1. The converse of Proposition 3.13 does not hold. Consider the group of rationals over the ring of integers. Thus, $R = \mathbb{Z}$ and $M = \mathbb{Q}$. Then $M_{E(M)}$ is niltorsionless, since $E(M) = \mathbb{Q}$ However, ${}_RM$ is not niltorsionless, since $M^* = 0$.

2. Let $B = \mathrm{End}(M_{E(M)}) = \mathrm{Biend}({}_RM)$, the biendomorphism ring of ${}_RM$. Using the left B-, right $E(M)$-bimodule structure of M, we deduce from Proposition 3.13 that the condition $M_{E(M)}$ is niltorsionless is sufficient for the niltorsionlessness of ${}_BM$. It is also a necessary condition for the niltorsionlessness of ${}_BM$, since the natural development that leads from R to $E(M)$ to $B = \mathrm{Biend}({}_RM)$ stabilizes, i.e., $E(M) = \mathrm{End}({}_BM) = \mathrm{Biend}(M_{E(M)})$.

We have for a left module M over a reduced ring R, $Rej(M)$=NilRej(M).

Proposition 3.15. *For a module M over a reduced ring we have: M is torsionless $\Leftrightarrow$ M is niltorsionless $\Leftrightarrow$ M is semiprime $\Leftrightarrow$ M is antisemiprime.*

Proof. This is a consequence of Proposition 3.2(c) and Proposition 2.6(c). □

The following result establishes the equivalence of some ring theoretic concepts when R is niltorsionless.

Proposition 3.16. *If R is a niltorsionless ring then the following conditions are equivalent.*

1. *R is reduced.*
2. *R is reversible.*
3. *R is symmetric.*
4. *R is semicommutative.*

5. *R is an NI-ring.*
6. *For all $b \in R, \mathrm{Nil}(R)b \subseteq \mathrm{Nil}(R)$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) hold without the niltorsioness assumption. They are well-known and easy to prove. (See the literature, e.g. [12, Section 2A] for the relevant definitions and proofs.)

(6) $\Rightarrow$ (1). Let $a \in \mathrm{Nil}(R)$. We have $aR \subset \mathrm{Nil}(R)$ yielding $a \in \mathrm{NilRej}(R) = 0$, since R is niltorsionless. $\square$

In the following result, we use the notation $\mathbf{y} = (y_i)$ for an I-tuple belonging to the module ΠM_i.

PROPOSITION 3.17. *Let I be an indexing set and let, for each $i \in I$, the module M_i be niltorsionless over R_i. Then $M = \Pi M_i$ and $W = \oplus M_i$ are niltorsionless over $R = \Pi R_i$.*

Proof. Let $0 \neq \mathbf{m} = (m_i) \in M$. Then $m_j \neq 0$ for some $j \in I$. Since M_j is niltorsionless over R_j, there exists an R_j-homomorphism q_j from M_j to R_j such that $m_j q_j \notin \mathrm{Nil}(R_j)$. Let $q : M \to R$ be defined as $\mathbf{y}q = y_j q_j$, where $y_j q_j$ is regarded as an element of R via the inclusion $R_j \to R$. Then $\mathbf{m}q = m_j q_j \notin Nil(R)$. Therefore, ${}_RM$ is niltorsionless. The proof of the niltorsionlessness of ${}_RW$ is similar. $\square$

PROPOSITION 3.18. *Let I be an indexing set and let, for each $i \in I$, M_i be a niltorsionless module over R, then $M = \Pi M_i$ is niltorsionless over R.*

Proof. Let $0 \neq \mathbf{m} = (m_i) \in M$. Then $m_j \neq 0$, for some $j \in I$. Since M_j is niltorsionless over R, there exists $q_j \in M_j^* = \mathrm{Hom}_R(M_j, R)$ such that $m_j q_j \notin \mathrm{Nil}(R)$. Let $q : M \to R$ be the R-homomorphism defined as $\mathbf{y}q = y_j q_j$. Then $\mathbf{m}q = m_j q_j \notin \mathrm{Nil}(R)$. Therefore ${}_RM$ is niltorsionless. $\square$

4. SOME 'WEAK' CONDITIONS ON MODULES

The concept of a weak Armendariz ring, an analogue of the concept of an Armendariz ring, was introduced and studied by Liu et al. [8]. (This reference may be consulted for the definition and basic properties of weak Armendariz rings.) In the next definition, we propose an extension to modules of this concept.

Definition 4.1. A module ${}_RM$ is *weak Armendariz* if given $f(X) = \sum a_i X^i$ and $m(X) = \sum m_j X^j$ with coefficients in R and M respectively, the condition $f(X)m(X) = 0$ implies $a_i m_j \in \mathrm{NilRej}(M)$ for every i and j.

Notation 4.2. Let $q : M_1 \to M_2$ be an R-module homomorphism. We denote by $q[X]$ the $R[X]$-linear map from $M_1[X]$ to $M_2[X]$ defined by $\left(\Sigma m_i X^i\right)(q[X]) = \Sigma(m_i q)X^i$. If $q \in M^*$ then $q[X] \in \mathrm{Hom}_{R[X]}(M[X], R[X])$.

Proposition 4.3. *The following conditions are equivalent for a ring R.*

(i) *The ring R is weak Armendariz.*

(ii) *Every left R-module is weak Armendariz.*

(iii) *The left R-module R is weak Armendariz.*

Proof. $(i) \Rightarrow (ii)$. Let M be a left R-module. Assume that for polynomials $f(X) = \sum a_i X^i$ and $m(X) = \sum m_j X^j$ with coefficients in R and M respectively, the condition $f(X)m(X) = 0$ holds. For $q \in M^*$ consider the $R[X]$-linear map $q[X] : M[X] \to R[X]$ defined in 4.2. Notice that $g(X) := m(X)q[X] = \sum (m_j q)X^j \in R[X]$ and we have $f(X)g(X) = f(X)m(X)q[X] = 0$, yielding, since R is a weak Armendariz ring, $(a_i m_j)q = a_i(m_j q) \in \mathrm{Nil}(R)$ for all i, j (and for all $q \in M^*$). This implies that $a_i m_j \in \mathrm{NilRej}(M)$ for all i, j proving that M is a weak Armendariz module.

The implication $(ii) \Rightarrow (iii)$ is trivial. Next assume condition (iii). Suppose that for polynomials $f(X) = \sum a_i X^i$ and $g(X) = \sum b_j X^j$ with coefficients in R we have $f(X)g(X) = 0$. By hypothesis, $a_i.b_j \in \mathrm{NilRej}(R)$ for all i, j. Hence by Remark 3.1(6), $a_i b_j \in \mathrm{Nil}(R)$ for all i, j. $\square$

Armendariz modules are, of course, weak Armendariz. Semicommutative rings are weak Armendariz [8, Corollary 3.4], but need not be Armendariz [13, Example 3.2]. Hence, by Proposition 4.3, weak Armendariz modules need not be Armendariz.

Following [9], we call a ring R *weakly semicommutative* if for elements $a, b \in R$ the condition $ab = 0$ implies $aRb \subset \mathrm{Nil}(R)$. The class of semicommutative rings is properly contained in the class of weakly semicommutative rings. We extend this concept to modules as follows.

Definition 4.4. A left R-module M is *weakly semicommutative* if whenever $a \in R$ and $m \in M$ satisfy the condition $am = 0$, then for each $t \in R$ and each $q \in M^*$ we have $at(mq) \in \mathrm{Nil}(R)$.

Semicommutative modules are weakly semicommutative. In the following proposition, we characterize weakly semicommutative rings through modules over them.

Proposition 4.5. *The following conditions are equivalent for a ring R.*

(i) *The ring R is weakly semicommutative.*

(ii) *Every left R-module is weakly semicommutative.*

(iii) *The left R-module R is weakly semicommutative.*

Proof. $(i) \Rightarrow (ii)$ Let M be a left R-module. Let $am = 0$ and $q \in M^*$. We have $a(mq) = (am)q = 0$. Hence, for each $t \in R$ we must have $at(mq) \in \mathrm{Nil}(R)$, proving that M is weakly semicommutative.

The implication $(ii) \Rightarrow (iii)$ is trivial. Next assume condition (iii). Suppose that for $a, b \in R$ we have $ab = 0$. Since the module $_RR$ is weakly semicommutative, for each $t \in R$ we have $atb \in \mathrm{NilRej}(R)$. Hence, by Remark 3.1(6), $atb \in \mathrm{Nil}(R)$, proving that the ring R is weakly semicommutative. □

We end this paper by stating a result which is an immediate consequence of the definitions of the concepts involved.

Proposition 4.6. *If a niltorsionless module is weak Armendariz (resp., weakly semicommutative), then it is Armendariz (resp., semicommutative).*

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