HOMOLOGY THEORY OF MULTIPLICATIVE HOM-LIE ALGEBRAS

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In this article, we establish Serre-Hochschild spectral sequences of Hom-Lie algebras. By using these spectral sequences, we describe homology groups of finite dimensional multiplicative Hom-Lie algebras in terms of homology groups of Lie algebras and abelian Hom-Lie algebras.

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1. INTRODUCTION

In recent years, many mathematicians and physicists studied various Hom-type algebras. M. Hassanzadeh, I. Shapiro and S. Sütlü studied the cyclic homology of Hom-associative algebras in [7]. B. Guan, L. Chen and B. Sun introduced Hom-Lie superalgebras in [3]. A. Makhlouf and S. Silvestrov studied Hom-associative, Hom-Leibniz and Hom-Lie admissible algebraic structures in [9] and Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras in [10]. The theory of quantum Hom-algebra was established in [15],[16],[17]. The notion of Hom-Lie algebras originated from the q-deformation of Witt algebras and Virasoro Lie algebras (see [5]). Hom-Lie algebras are studied extensively. Especially, the representation theory of Hom-Lie algebras was well-developed. For example, the irreducible representations of simple Hom-Lie algebras were obtained in [2]. The Ado theorem for a nilpotent Hom-Lie algebra was proved in [11]. D. Yau constructed the universal enveloping algebra of a Hom-Lie algebra in [14]. In addition, the cohomology of Hom-Lie algebras was studied in [1] and [12]. From these articles, we know that the low dimension cohomology groups can be interpreted as the central extensions and derivations of Hom-Lie algebras. Dually, there is a homology theory of Hom-Lie algebras, which was developed by many researchers. Especially, D. Yau defined the Chevalley-Eilenberg type complex of a Hom-Lie algebra in [13], which is the main object we focus on in this paper. Since Serre-Hochschild spectral sequence of Lie algebras plays a very important role in the homology theory of Lie algebras (see

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[6]), we construct a counterpart Serre-Hochschild spectral sequence of Hom-Lie algebras. Using this spectral sequence, we establish the bridge between the homology groups of Hom-Lie algebras and that of Lie algebras.

This paper is arranged as follows: in Section 2, we recall some basic definitions of Hom-Lie algebras and provide some examples of different type of Hom-Lie algebras. We prove that every multiplicative Hom-Lie algebra is a semi-direct product of a regular Hom-Lie algebra and a module over this regular Hom-Lie algebra.

In Section 3, the definition of homology groups of a multiplicative Hom-Lie algebra are given. In addition, we investigate the module structures on the Hom-chains.

In Section 4, we establish the Serre-Hochschild spectral sequence of Hom-Lie algebras. By using this method, we describe the homology groups of finite dimensional multiplicative Hom-Lie algebras in terms of homology groups of Lie algebras and abelian Hom-Lie algebras, see the formula (16).

In this article, $\mathbf{k}$ is an algebraically closed field of characteristic zero. In addition, all the vectors and algebras are over the field $\mathbf{k}$. $\mathbb{Z}$ is the ring of integers and $\mathbb{Z}_+$ is the set of non-negative integers.

2. PRELIMINARY

In this section, we recall some basic concept and prove some elementary results related to Hom-Lie algebras. First, let us recall the definitions of various Hom-Lie algebras.

Definition 2.1. Suppose that $\mathfrak{g}$ is a vector space with an endomorphism α and $[\cdot,\cdot]_\mathfrak{g} : \wedge^2\mathfrak{g} \to \mathfrak{g}$ is a skew-symmetric map. Then the triple $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ is called a Hom-Lie algebra if it satisfies the *Hom-Jacobi Identity*:

$$[\alpha(x), [y,z]_\mathfrak{g}]_\mathfrak{g} + [\alpha(y), [z,x]_\mathfrak{g}]_\mathfrak{g} + [\alpha(z), [x,y]_\mathfrak{g}]_\mathfrak{g} = 0 \tag{1}$$

for $x, y, z \in \mathfrak{g}$.

1. A *Hom-subalgebra* $\mathfrak{h}$ of a Hom-Lie $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ is a α-invariant subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[\mathfrak{h},\mathfrak{h}]_\mathfrak{g} \subset \mathfrak{h}$.

2. A Hom-subalgebra $\mathfrak{h}$ is called a *Hom-ideal* of $\mathfrak{g}$ if $[\mathfrak{h},\mathfrak{g}]_\mathfrak{g} \subset \mathfrak{h}$.

3. A Hom Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ is said to be *multiplicative* if $\alpha([a,b]_\mathfrak{g}) = [\alpha(a), \alpha(b)]_\mathfrak{g}$.

4. A *regular* Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ is a multiplicative Hom-Lie algebra with an invertible α.

It is obvious that a Hom-subalgebra $\mathfrak{h}$ of a Hom-Lie $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is itself a Hom-Lie algebra with the restriction map $\alpha_{\mathfrak{h}}$ and restriction bracket. In the sequel of this paper, to simplify notations, we usually abbreviate the Hom-Lie triple $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ to $\mathfrak{g}$.

Example 2.2. Let W_1 be the Lie algebra generated by the set $\{e_i\}_{i\geq -1}$ together with the brackets: $[e_i, e_j] = (j-i)e_{i+j}$. For any $k \in \mathbb{Z}_+$, we define $\alpha_k(e_i) = e_{i+k}$. Then $(W_1, [\cdot,\cdot], \alpha_k)$ is a Hom-Lie algebra, but it is not multiplicative for positive k.

Example 2.3. *(Yau twist)* Let $\mathfrak{g}$ be a Lie algebra with bracket $[\cdot,\cdot]$. Suppose α is an endomorphism of $\mathfrak{g}$. Define $[x,y]_{\mathfrak{g}} := \alpha([x,y]) = [\alpha(x), \alpha(y)]$. Then $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a multiplicative Hom-Lie algebra. In particular, any Lie algebra is a Hom-Lie algebra with $\alpha = id$.

Definition 2.4. Suppose that $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a *Yau twist* of a Lie $\mathfrak{g}_L$ by its endomorphism α, then we call it a Hom-Lie algebra of Lie type of the Lie algebra $\mathfrak{g}_L$.

Every regular Hom-Lie algebra is a Hom-Lie algebra of Lie type. In fact, if $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a regular Hom-Lie algebra, then it is a Hom-Lie algebra of Lie type of $\mathfrak{g}_L$, where $\mathfrak{g}_L = \mathfrak{g}$ as vector spaces, whose bracket is given by $[x,y] = \alpha^{-1}([x,y]_{\mathfrak{g}})$ for $x, y \in \mathfrak{g}_L$. Thus the category of Lie algebras is equivalent to the category of regular Hom-Lie algebras.

Example 2.5. Suppose that $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a multiplicative Lie algebra. For arbitrary $s \in \mathbb{Z}^+$, $\alpha^s([x,y]) = [\alpha^s(x), \alpha^s(y)] = 0$ for $x \in \ker \alpha^s$ and $y \in \mathfrak{g}$. Thus $\ker \alpha^s$ is a Hom-ideal of $\mathfrak{g}$.

Next, we recall representations of Hom-Lie algebras. A representation of a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is also called a module over it or a $\mathfrak{g}$-module.

Definition 2.6. Suppose that $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a Hom-Lie algebra and M is a vector space with an endomorphism α_M. Let ρ_M be a k-linear map from $\mathfrak{g}$ to $\mathfrak{gl}(M)$. Then the triple (M, ρ_M, α_M) is called a $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$-module if the following compatible conditions hold:

$$\rho_M([x,y]_{\mathfrak{g}})\alpha_M(m) = \rho_M(\alpha(x))\rho_M(y)m - \rho_M(\alpha(y))\rho_M(x)m, \tag{2}$$

$$\alpha_M(\rho(x)m) = \rho_M(\alpha(x))\alpha_M(m) \tag{3}$$

for any $x,\ y \in \mathfrak{g}$, $m \in \mathfrak{g}$.

In the sequel, to simplify notations, we will use the pair (M, α_M) to substitute the triple (M, ρ_M, α_M) and $\rho_M(x)(m)$ is abbreviated as $x \cdot m$, or xm, if there is no ambiguous of the action of $\mathfrak{g}$ on M. Sometimes, we simply call M a $\mathfrak{g}$-module. We use $(\mathbf{k}, id)$ to denote the *trivial module* of a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$.

If I is a Hom-ideal of a multiplicative Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$, then $(I, \alpha|_I)$ is a $\mathfrak{g}$-module with the action: $x \cdot y = [x, y]_{\mathfrak{g}}$ for $x \in \mathfrak{g}$ and $y \in I$. In particular, $I = \mathfrak{g}$ is a $\mathfrak{g}$-module. This module is called a *adjoint module.* A representation of a multiplicative Hom-Lie algebra can be characterized by another multiplicative Hom-Lie algebra. To describe this, let M be a vector space, $\rho_M \in \mathrm{Hom}_{\mathbf{k}}(\mathfrak{g}, \mathfrak{gl}(M))$, $\alpha_M \in \mathfrak{gl}(M)$. Define a bracket on $\mathfrak{g} \oplus M$ by

$$[(x, m_1), (y, m_2)]_{(\mathfrak{g},M)} = ([x, y]_{\mathfrak{g}}, \rho_M(x)m_2 - \rho_M(y)m_1]). \tag{4}$$

and a $\mathbf{k}$-linear map by

$$\alpha_{\ltimes}(x, m_1) = (\alpha(x), \alpha_M(m_1)),$$

where $x, y \in \mathfrak{g}$ and $m_1, m_2 \in M$. Then we have the following proposition.

Proposition 2.1. *Suppose that $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a multiplicative Hom-Lie algebra. Then $(\mathfrak{g} \oplus M, [\cdot,\cdot]_{(\mathfrak{g},M)}, \alpha_{\ltimes})$ is a multiplicative Hom-Lie algebra if and only if (M, α_M) is a $\mathfrak{g}$-module with the action ρ_M.*

Proof. If (M, α_M) is a $\mathfrak{g}$-module, $(\mathfrak{g} \oplus M, [\cdot,\cdot]_{(\mathfrak{g},M)}, \alpha_{\ltimes})$ is a multiplicative Hom-Lie algebra by [12, Proposition 4.5]. On the other hand, if $(\mathfrak{g} \oplus M, [\cdot,\cdot]_{(\mathfrak{g},M)}, \alpha_{\ltimes})$ is a multiplicative Hom-Lie algebra, then M is a Hom-ideal of $\mathfrak{g} \oplus M$. Thus (M, α_M) is a $\mathfrak{g}$-module with the action ρ_M. □

The multiplicative Hom-Lie algebra $(\mathfrak{g} \oplus M, [\cdot,\cdot]_{(\mathfrak{g},M)}, \alpha_{\ltimes})$ in Proposition 2.1 is called a semi-direct product of $\mathfrak{g}$ and its representation M. Every finite-dimensional multiplicative Hom-Lie algebra is a semi-direct product of a regular Hom-Lie algebra and its representation. To prove this claim, let us fix some terms. Suppose that α is an endomorphism of a vector space V and

$$0 \subset \ker\alpha \subset \ker\alpha^2 \subset \cdots \subset \ker\alpha^s \subset \cdots$$

is an ascending chain of subspaces of V. If there is k such that $\ker(\alpha^k) = \ker(\alpha^l)$ for any $l \geq k$, then there is a minimal integer s such that $\ker\alpha^s = \ker\alpha^t$ for any $t \geq s$. We call this s a null degree of α. The null degree of α is denoted by $n(\alpha)$.

With this notion, we can prove the following key lemma for Section 4.

Lemma 2.1. *Suppose that $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a finite dimensional multiplicative Hom-Lie algebra with null degree $n(\alpha) = s$. Then $\mathfrak{g} = R \ltimes \ker\alpha^s$, where $R \simeq \mathfrak{g}/\ker(\alpha^s)$ is a regular Hom-subalgebra of $\mathfrak{g}$.*

Proof. Let $\mathfrak{g} = R \oplus \ker\alpha^s$ be the Jordan decomposition of α. It is obvious that $\alpha|_R$ is invertible. We claim that R is closed under the bracket of $\mathfrak{g}$. Indeed, for $a, b \in R$, write $[a,b]_{\mathfrak{g}} = c + x$, where $c \in R$ and $x \in \ker \alpha^s$. Since the restriction of α on R is an isomorphism, there exists $a', b', c' \in R$ such that

$$\alpha^s(a') = a,\ \alpha^s(b') = b,\ \alpha^s(c') = c.$$

Thus, $\alpha^s([a,b]_{\mathfrak{g}} - c) = \alpha^{2s}([a',b']_{\mathfrak{g}} - c') = 0$. Therefore, $[a',b']_{\mathfrak{g}} - c' \in \ker \alpha^{2s} = \ker \alpha^s$. This implies that $x = [a,b]_{\mathfrak{g}} - c = \alpha^s([a',b']_{\mathfrak{g}} - c') = 0$. Hence, R is a Hom-subalgebra of $\mathfrak{g}$. Since $\ker \alpha^s$ is a Hom-ideal of $\mathfrak{g}$, $\mathfrak{g} = R \ltimes \ker \alpha^s$. $\square$

Finally, we recall the tensor product of $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$-modules. Suppose $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a regular Hom-Lie algebra and $(M_1, \alpha_{M_1}), \cdots, (M_n, \alpha_{M_n})$ are $\mathfrak{g}$-modules. Let $M = M_1 \otimes M_2 \otimes \cdots \otimes M_n$ be the tensor product of $M_1, \cdots, M_n$ over $\mathbf{k}$. Define a linear map $\alpha_M : M \to M$ and an action of $\mathfrak{g}$ on M via

$$\alpha_M(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \alpha_{M_1}(x_1) \otimes \alpha_{M_2}(x_2) \otimes \cdots \otimes \alpha_{M_n}(x_n), \tag{5}$$

and

$$h \cdot (x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n} \alpha_{M_1}(x_1) \otimes \cdots \otimes h \cdot x_i \otimes \cdots \otimes \alpha_{M_n}(x_n), \tag{6}$$

for $h \in \mathfrak{g}$ and $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in M$, respectively. Then (M, α_M) is a $\mathfrak{g}$-module by [11, proposition 1.1].

Remark 2.7. Suppose $\mathfrak{g}$ is a Hom-Lie algebra. Then the category of all $\mathfrak{g}$-modules is a symmetric monoidal category with the action given by (6).

3. HOMOLOGY OF MULTIPLICATIVE HOM-LIE ALGEBRAS

In this section, all Hom-Lie algebras are always multiplicative unless otherwise specified. First, let us recall the Chevalley-Elienberg type homology of multiplicative Hom-Lie algebras from [13].

Suppose that (M, α_M) be a module over a multiplicative Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$. Recall that, for $n \in \mathbb{Z}_+$, a Hom-n-chain of the Hom-Lie algebra $\mathfrak{g}$ with coefficients in M is an element in the vector space $C_n(\mathfrak{g}, (M, \alpha_M)) = \wedge^n \mathfrak{g} \otimes M$, where $\wedge^n \mathfrak{g}$ is the nth exterior power of $\mathfrak{g}$. If $n = 0$, then $\wedge^0 \mathfrak{g} = \mathbf{k}$ is a $\mathfrak{g}$-module with a trivial action. The differential d from $C_n(\mathfrak{g}, (M, \alpha_M))$ to $C_{n-1}(\mathfrak{g}, (M, \alpha_M))$ is a $\mathbf{k}$-linear map given by

(7)

$$d(x_1 \wedge x_2 \wedge \cdots \wedge x_n \otimes m) = \sum_{i=1}^{n} (-1)^i \alpha(x_1) \wedge \alpha(x_2) \wedge \cdots \widehat{x_i} \cdots \wedge \alpha(x_n) \otimes x_i.m$$
$$+ \sum_{1 \le i < j \le n} (-1)^{i+j} [x_i, x_j]_{\mathfrak{g}} \wedge \alpha(x_1) \wedge \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \wedge \alpha(x_n) \otimes \alpha_M(m),$$

for $x_1, x_2, \cdots, x_n \in \mathfrak{g}$ and $m \in M$. Since $d^2 = 0$ by [13, Theorem 3.4], $(C_\bullet(\mathfrak{g}, M), d)$ forms a complex. This complex is called *Chevalley-Eilenberg complex of the Hom-Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ with coefficient in M. We use $H_n(\mathfrak{g}, (M, \alpha_M))$ to denote the nth homology group of this Chevalley-Eilenberg complex of the Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ with coefficient in M.

Suppose that a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ is regular. Then it is a Hom-Lie algebra of Lie type of a Lie algebra $\mathfrak{g}_L$. Furthermore, if α_M of the representation (M, α_M) is also invertible, then $H_n(\mathfrak{g}, (M, \alpha_M))$ is the same as the n-th Chevalley-Eilenberg homological group $H_n^{Lie}(\mathfrak{g}_L, M)$ of the Lie algebra with the coefficient in M, where M is a $\mathfrak{g}_L$ module via

$$x \cdot m = \alpha_M^{-1}(\rho_M(x).m), \tag{8}$$

for $x \in \mathfrak{g}_L, m \in M$. In fact, we can construct a morphism from the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}_L$ with coefficients in M to the Chevalley-Eilenberg complex $(C_\bullet(\mathfrak{g}, M), d)$ of Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ with coefficients in M as follow.

For $x_1, \cdots, x_n \in \mathfrak{g}$, $m \in M$, define a mapping φ via

$$x_1 \wedge x_2 \wedge \cdots \wedge x_n \otimes m \mapsto \alpha(x_1) \wedge \alpha(x_2) \wedge \cdots \wedge \alpha(x_n) \otimes \alpha_M(m).$$

It is easy to check that φ is an isomorphism of complexes. Then it induces an isomorphism

$$H_n(\mathfrak{g}, (M, \alpha_M)) \cong H_n^{Lie}(\mathfrak{g}_L, M), \tag{9}$$

where $\mathfrak{g}_L$ acts on M by (8). If a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ is not regular, there are no such isomorphisms as the following example to explain.

Example 3.1. Let $n \geq 2$ and $\mathfrak{gl}(n, \mathbf{k})$ be the general linear Lie algebra of all $n \times n$ matrices over the field $\mathbf{k}$. Define $\alpha(x) = trace(x) I_n$, where I_n is the identity matrix. Since the image of α is in the center of $\mathfrak{gl}(n, \mathbf{k})$, the Hom-Jacobi identity holds. Thus $(\mathfrak{gl}(n, \mathbf{k}), [\cdot,\cdot], \alpha)$ is a multiplicative Hom-Lie algebra.

It is well known that

$$H_\bullet^{Lie}(\mathfrak{gl}(n, \mathbf{k}), \mathbf{k}) \cong \wedge[\theta_1, \theta_2, \cdots, \theta_n],$$

where $\wedge[\theta_1, \theta_2, \cdots, \theta_n]$ is the exterior algebra with generator θ_i of degree $2i-1$. However, from the definition of differential given by (8), the differential of the complex $C_\bullet(\mathfrak{gl}(n, \mathbf{k}), (\mathbf{k}, id))$ is zero. Thus $H_\bullet^{Lie}(\mathfrak{gl}(n, \mathbf{k}), \mathbf{k})$ is not isomorphic to $H_n(\mathfrak{gl}(n, \mathbf{k}), (\mathbf{k}, id))$.

Notice that the action (6) is invariant under the permuting factors in the tensor products. Thus $(C_n(\mathfrak{g}, M), \alpha_{n,M})$ is also a $\mathfrak{g}$-module with the action induced by (6). Explicitly, for $x_1, x_2, \cdots x_n \in \mathfrak{g}$ and $m \in M$,

$$\alpha_{n,M}(x_1 \wedge x_2 \wedge \cdots \wedge x_n \otimes m) = \alpha(x_1) \wedge \alpha(x_2) \wedge \cdots \wedge \alpha(x_n) \otimes \alpha_M(M).$$

In the following, we use the same notation for an endomorphism of a vector space V, its restriction to a subspace $W \subseteq V$ and the induced endomorphism of the quotient space V/U by its invariant subspace U.

Next, we recall the relative homology theory of Hom-Lie algebras which is called *Relative Hom-homology.* Suppose $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a multiplicative Hom-Lie algebra and (M, α_M) is a $\mathfrak{g}$-module. Let $\mathfrak{h}$ be a regular Hom-subalgebra of $\mathfrak{g}$. Then the group of relative Hom-n-chains is defined via

$$C_n(\mathfrak{g}, \mathfrak{h}, (M, \alpha_M)) = \frac{\wedge^n(\mathfrak{g}/\mathfrak{h}) \otimes M}{\mathfrak{h}.(\wedge^n(\mathfrak{g}/\mathfrak{h}) \otimes M)},$$

where the dot action of $\mathfrak{h}$ on $\wedge^n \mathfrak{g}/\mathfrak{h} \otimes M$ is defined via (6).

Lemma 3.1. *Suppose that $\mathfrak{h}$ is a Hom-ideal of a multiplicative Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ and (M, α_M) is a $\mathfrak{g}$-module. Then $\alpha(h) \cdot d\phi = d(h \cdot \phi)$ for any $\phi \in C_n(\mathfrak{h}, (M, \alpha_M))$ and $h \in \mathfrak{g}$.*

Proof. Since all the maps are **k**-linear, we may assume that ϕ is a monomial, i.e, $\phi = x_1 \wedge x_2 \wedge \cdots \wedge x_n \otimes m$, where $x_1, x_2, \cdots, x_n \in \mathfrak{h}$ and $m \in M$. For any $h \in \mathfrak{g}$, we have

$$\begin{aligned}
& d(h \cdot \phi) \\
=& d(\sum_{i=1}^{n} \alpha(x_1) \wedge \cdots \wedge [h, x_i]_{\mathfrak{g}} \wedge \cdots \wedge \alpha(x_n) \otimes \alpha_M(m) \\
& + \alpha(x_1) \wedge \cdots \wedge \alpha(x_n) \otimes h.m) \\
=& \sum_{i=1}^{n} ((-1)^i \alpha^2(x_1) \wedge \cdots \widehat{[h.x_i]_{\mathfrak{g}}} \cdots \wedge \alpha^2(x_n) \otimes [h, x_i]_{\mathfrak{g}} \cdot \alpha_M(m) \\
& + \sum_{j=1}^{i-1} (-1)^{i+j+1} [[h, x_i]_{\mathfrak{g}}, \alpha(x_j)]_{\mathfrak{g}} \wedge \alpha^2(x_1) \cdots \widehat{\alpha(x_j)} \cdots \widehat{[h, x_i]_{\mathfrak{g}}} \cdots \wedge \\
& \alpha^2(x_n) \otimes \alpha_M^2(m) + \sum_{j=i+1}^{n} (-1)^{i+j} [[h, x_i]_{\mathfrak{g}}, \alpha(x_j)]_{\mathfrak{g}} \wedge \alpha^2(x_1) \wedge \cdots \\
& \widehat{[h, x_i]_{\mathfrak{g}}} \cdots \widehat{\alpha(x_j)} \cdots \wedge \alpha^2(x_n) \otimes \alpha_M^2(m) + \sum_{j=1, j \neq i}^{n} (-1)^j \alpha^2(x_1) \wedge \cdots \\
& \widehat{\alpha(x_j)} \cdots \wedge \alpha([h, x_i]_{\mathfrak{g}}) \wedge \cdots \wedge \alpha^2(x_n) \otimes \alpha(x_j).\alpha_M(m) \\
& + \sum_{1 \leq s < t \leq n, s,t \neq i} (-1)^{s+t} [\alpha(x_s), \alpha(x_t)]_{\mathfrak{g}} \wedge \alpha^2(x_1) \wedge \cdots \widehat{\alpha(x_s)} \cdots \widehat{\alpha(x_t)} \cdots \wedge \\
& \alpha^2(x_n) \otimes \alpha_M^2(m)) + d(\alpha(x_1) \wedge \cdots \wedge \alpha(x_n) \otimes h.m)
\end{aligned}$$

$$= \sum_{1=i<j\leq n} (-1)^{i+j}([[h,x_i]_{\mathfrak{g}},\alpha(x_j)]_{\mathfrak{g}} - [[h,x_j]_{\mathfrak{g}},\alpha(x_i)]_{\mathfrak{g}} - [\alpha(h),[x_i,x_j]_{\mathfrak{g}}]_{\mathfrak{g}})\wedge$$

$$\alpha^2(x_1)\wedge\alpha^2(x_2)\wedge\cdots\widehat{\alpha(x_i)}\cdots\widehat{\alpha(x_j)}\cdots\wedge\alpha^2(x_n)\otimes\alpha_M^2(m)) + \alpha(h)\cdot(d\phi).$$

Thus $\alpha(h)\cdot d\phi = d(h\cdot\phi)$ by Hom-Jacobi identity. $\square$

Let $\mathfrak{h}$ be a Hom-ideal of a multiplicative Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$. By Lemma 3.1,

$$d(\mathfrak{h}\cdot(\wedge^n(\mathfrak{g}/\mathfrak{h})\otimes M)) \subset \alpha(\mathfrak{h})\cdot(\wedge^n(\mathfrak{g}/\mathfrak{h})\otimes M) = \mathfrak{h}\cdot d(\wedge^n(\mathfrak{g}/\mathfrak{h})\otimes M).$$

Hence $(C_\bullet(\mathfrak{g},\mathfrak{h},(M,\alpha_M)),\overline{d})$ is a quotient complex of $(C_\bullet(\mathfrak{g},(M,\alpha_M)),d)$, where the differential $\overline{d}$ is induced by d. This quotient complex is called the relative Hom-complex of $\mathfrak{h}\subset\mathfrak{g}$. In addition, we use $H_n(\mathfrak{g},\mathfrak{h},(M,\alpha_M))$ to denote the nth homology group of relative Hom-complex of $\mathfrak{h}\subset\mathfrak{g}$. Moreover, we can see that $H_n(\mathfrak{g},\mathfrak{h},(M,\alpha_M))$ is a $\alpha(\mathfrak{h})$-module from the following proposition.

Proposition 3.1. *With the assumption as Lemma 3.1, we have that $H_n(\mathfrak{h},(M,\alpha_M))$ is an $\alpha(\mathfrak{g})$-module for any $n\in\mathbb{Z}_+$ via the action (6). Further, $\alpha(\mathfrak{h})$ acts trivially on $H_n(\mathfrak{h},(M,\alpha_M))$.*

Proof. For any $h\in\mathfrak{g}$, if ϕ is a Hom-n-cycle, then $d(\alpha(h)\cdot\phi) = \alpha^2(h)\cdot d\phi = 0$ by Lemma 3.1. Thus $H_n(\mathfrak{h},(M,\alpha_M))$ is a well-defined $\alpha(\mathfrak{g})$-module via the action (6). Furthermore, one can straightly check that $h\cdot\phi = -d(h\wedge\phi)$ for any $h\in\alpha(\mathfrak{h})$ and $\phi\in H_n(\mathfrak{h},(M,\alpha_M))$. $\square$

4. SPECTRAL SEQUENCE OF A HOM-LIE ALGEBRA

In this section, $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ always denotes a multiplicative Hom-Lie algebra. Let $\mathfrak{h}$ be a Hom-subalgebra of $\mathfrak{g}$ and (M,α_M) be a $\mathfrak{g}$-module. Suppose that d is always the differential of the complex $C_\bullet(\mathfrak{g},(M,\alpha_M))$.

For each $n,p\in\mathbb{Z}_+$, let

$$F_pC_n(\mathfrak{g},(M,\alpha_M)) = span_{\mathbf{k}}\{x_1\wedge x_2\wedge\cdots\wedge x_n\otimes m | x_1,x_2,\cdots,x_{n-p}\in\mathfrak{h}\}.$$

It is easy to see that

$$d(F_pC_n(\mathfrak{g},(M,\alpha_M))) \subset F_pC_{n+1}(\mathfrak{g},(M,\alpha_M))$$

for any $p,n\in\mathbb{Z}_+$. Thus, we obtain the following filtration:

$$(10) F_0C_n(\mathfrak{g},(M,\alpha_M)) \subset\cdots\subset F_{n-1}C_n(\mathfrak{g},(M,\alpha_M)) \subset C_n(\mathfrak{g},(M,\alpha_M)).$$

This is a bounded filtration of complexes. Thus, there is a spectral sequence

$$(E^r_{pq}, d^r_{pq}: E^r_{pq}\to E^r_{p-r,q+r-1})$$

such that $E^0_{pq} = F_pC_{p+q}(\mathfrak{g},(M,\alpha_M))/F_{p-1}C_{p+q}(\mathfrak{g},(M,\alpha_M))$, where d^r_{pq} is induced by the differential of $C_\bullet(\mathfrak{g},(M,\alpha_M))$. This type of spectral sequence is called the Serre-Hochschild spectral sequence of $\mathfrak{h} \subset \mathfrak{g}$.

Lemma 4.1. *Let* $(E^r_{pq},\ d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1})$ *be the Serre-Hochschild spectral sequence of* $\mathfrak{h} \subset \mathfrak{g}$. *Then*

(i) $E^1_{pq} = H_q(\mathfrak{h},(\wedge^p\mathfrak{g}/\mathfrak{h} \otimes M, \alpha_{p,M}))$.

(ii) *If* $\mathfrak{h}$ *is a Hom-ideal of* $\mathfrak{g}$, *then*

$$H_q(\mathfrak{h},(\wedge^p\mathfrak{g}/\mathfrak{h} \otimes M, \alpha_{p,M})) \cong \frac{\ker\alpha_p \otimes \wedge^q\mathfrak{h} \otimes M + \wedge^p\mathfrak{g}/\mathfrak{h} \otimes \ker d_q}{\mathrm{Im}\alpha_p \otimes \mathrm{Im}d_{q+1}},$$

where $\alpha_p : \wedge^p\mathfrak{g}/\mathfrak{h} \to \wedge^p\mathfrak{g}/\mathfrak{h}$ *is defined via (6) and* d_i $(i = q, q+1)$ *is the differential of the Hom-i-chain* $C_i(\mathfrak{h},(M,\alpha_M))$.

(iii) *If* $\mathfrak{h}$ *is a regular Hom-subalgebra of* $\mathfrak{g}$, *then* $E^2_{p0} = H_p(\mathfrak{g},\mathfrak{h},(M,\alpha_M))$.

Proof. Define a linear map

$$\psi : \ F_pC_{p+q}(\mathfrak{g},(M,\alpha_M)) \to C_q(\mathfrak{h},(\wedge^p\mathfrak{g}/\mathfrak{h} \otimes M, \alpha_{p,M})) = \wedge^q h \otimes \wedge^p\mathfrak{g}/\mathfrak{h} \otimes M$$

via

$$x_1 \wedge x_2 \wedge \cdots \wedge x_{p+q} \otimes m \mapsto x_1 \wedge x_2 \wedge \cdots \wedge x_q \otimes \overline{x_{q+1}} \wedge \overline{x_{q+2}} \wedge \cdots \wedge \overline{x_{p+q}} \otimes m,$$

for $x_1, x_2, \cdots, x_q \in \mathfrak{h}$, $x_{q+1}, x_{q+2}, \cdots, x_{p+q} \in \mathfrak{g}$, $m \in M$, where $\overline{x}$ means the image of x in the quotient space $\mathfrak{g}/\mathfrak{h}$.

From the definition of the filtration (10), one knows that ψ is well-defined. In addition, ψ is surjective with the kernel $F_{p-1}C_{p+q}(\mathfrak{g},(M,\alpha_M))$. Thus, we get an isomorphism

$$\bar{\psi} : \quad F_pC_{p+q}(\mathfrak{g},(M,\alpha_M))/F_{p-1}C_{p+q}(\mathfrak{g},(M,\alpha_M)) \longrightarrow \wedge^q h \otimes \wedge^p\mathfrak{g}/\mathfrak{h} \otimes M.$$

We claim that the following diagram commutes

$$\begin{array}{ccc}
\frac{F_pC_{p+q}(\mathfrak{g},(M,\alpha_M))}{F_{p-1}C_{p+q}(\mathfrak{g},(M,\alpha_M))} & \xrightarrow{d^0_{pq}} & \frac{F_pC_{p+q-1}(\mathfrak{g},(M,\alpha_M))}{F_{p-1}C_{p+q-1}(\mathfrak{g},(M,\alpha_M))} \\
\downarrow \bar{\psi} & & \downarrow \bar{\psi} \\
\wedge^q h \otimes \wedge^p\mathfrak{g}/\mathfrak{h} \otimes M & \xrightarrow{d'} & \wedge^{q-1} h \otimes \wedge^p\mathfrak{g}/\mathfrak{h} \otimes M
\end{array}$$

where d' is the differential of the complex $C_\bullet(\mathfrak{h},(\wedge^p\mathfrak{g}/\mathfrak{h} \otimes M, \alpha_{p,M}))$. Indeed, for $x_1, x_2, \cdots, x_q \in \mathfrak{h}$ and $x_{q+1}, x_{q+2}, \cdots, x_{q+p} \in \mathfrak{g}$, we have

$$\begin{aligned}
&d^0_{pq}(x_1 \wedge x_2 \wedge \cdots \wedge x_{p+q} \otimes m)\\
=&\sum_{1\leq i<j\leq q} (-1)^{i+j}[x_i, x_j]_{\mathfrak{g}} \wedge \alpha(x_1) \wedge \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \wedge \alpha(x_{p+q}) \otimes \alpha_M(m) + \mu\\
&+\sum_{i=1}^{q}\sum_{j=q+1}^{p+q} (-1)^{i+j}[x_i, x_j]_{\mathfrak{g}} \wedge \alpha(x_1) \wedge \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \wedge \alpha(x_{p+q}) \otimes \alpha_M(m)\\
&+\sum_{i=1}^{q}(-1)^i \alpha(x_1) \wedge \alpha(x_2) \wedge \cdots \widehat{x_i} \cdots \wedge \alpha(x_{p+q}) \otimes x_i.m\\
=&\sum_{1\leq i<j\leq q} (-1)^{i+j}[x_i, x_j]_{\mathfrak{g}} \wedge \alpha(x_1) \wedge \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \wedge \alpha(x_q) \otimes \alpha_{p,M}(\overline{x_{q+1}}\\
&\wedge \overline{x_{q+2}} \wedge \cdots \wedge \overline{x_{p+q}} \otimes m) + \sum_{i=1}^{q}(-1)^i \alpha(x_1) \wedge \cdots \widehat{\alpha(x_i)} \cdots \wedge \alpha(x_q) \otimes x_i.(\overline{x_{q+1}}\\
&\wedge \cdots \wedge \overline{x_{p+q}} \otimes m) + \mu\\
=&d'(x_1 \wedge \cdots \wedge x_q \otimes \overline{x_{q+1}} \wedge \overline{x_{q+2}} \wedge \cdots \wedge \overline{x_{q+p}} \otimes m) + \mu,
\end{aligned}$$

where

$$\begin{aligned}
\mu =& \sum_{1+q\leq i<j\leq p+q} (-1)^{i+j} \alpha(x_1) \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge \alpha(x_{p+q}) \otimes \alpha_M(m)\\
&+ \sum_{i=1+q}^{p+q} (-1)^i \alpha(x_1) \wedge \cdots \widehat{x_i} \cdots \wedge \alpha(x_{p+q}) \otimes x_i.m.
\end{aligned}$$

It is clear that $\mu \in F_{p-1}C_{p+q-1}(\mathfrak{g}, (M, \alpha_M))$. Thus $\overline{\psi} d_0^{pq} = d'\overline{\psi}$ and the first claim of proposition holds. For the second one, if $\mathfrak{h}$ is a Hom-ideal of $\mathfrak{g}$, $\mathfrak{h}$ acts trivially on $\wedge^p \mathfrak{g}/\mathfrak{h}$. Thus, it is not hard to check that we have the following commutative diagram.

$$\begin{array}{ccc}
E^0_{pq} & \xrightarrow{d^0_{pq}} & E^0_{p,q-1}\\
\Big\downarrow \bar{\psi} & & \Big\downarrow \bar{\psi}\\
\wedge^p \mathfrak{g}/\mathfrak{h} \otimes \wedge^q \mathfrak{h} \otimes M & \xrightarrow{\alpha_p \otimes d_q} & \wedge^p \mathfrak{g}/\mathfrak{h} \otimes \wedge^{q-1} \mathfrak{h} \otimes M,
\end{array}$$

where $\alpha_p : \wedge^p \mathfrak{g}/\mathfrak{h} \to \wedge^p \mathfrak{g}/\mathfrak{h}$ is defined via (6) and d_q is the differential of Hom-q-chain $C_q(\mathfrak{h}, (M, \alpha_M))$. Furthermore, we have $\ker(\alpha_p \otimes d_q) = \ker \alpha_p \otimes \wedge^q \mathfrak{h} \otimes M + \wedge^p \mathfrak{g}/\mathfrak{h} \otimes \ker d_q$, $\mathrm{Im}(\alpha_p \otimes d_q) = \mathrm{Im}(\alpha_p) \otimes Im(d_q)$. This finishes the proof of the second claim.

For (iii), by (i) and Proposition 3.1, $E^1_{p0} = \frac{\wedge^p \mathfrak{g}/\mathfrak{h}\otimes M}{\mathfrak{h}.\wedge^p \mathfrak{g}/\mathfrak{h}\otimes M} = C_p(\mathfrak{g},\mathfrak{h},(M,\alpha_M))$. Since both d^1_{p0} and $\overline{d}$ are induced by the initial differential of $C_\bullet(\mathfrak{g},(M,\alpha_M))$, we have $d^1_{p0} = \overline{d}$. As a consequence, $E^2_{p0} = H_p(\mathfrak{g},\mathfrak{h},(M,\alpha_M))$. $\square$

Suppose that $\mathfrak{h}$ is a Hom-ideal of a regular Hom-Lie algebra. Then we can obtain the following proposition from Lemma 4.1

Proposition 4.1. *Suppose* $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ *is a regular Hom-Lie algebra and* (M,α_M) *is a* $\mathfrak{g}$*-module. Let* $\mathfrak{h}$ *be a Hom-ideal of* $\mathfrak{g}$*. Then*

$$E^2_{pq} \cong H_p(\mathfrak{g}/\mathfrak{h},(H_q(\mathfrak{h},(M,\alpha_M)),\alpha_{q,M})).$$

Proof. From Lemma 4.1, we obtain that

$$E^1_{pq} \cong \frac{\wedge^p(\mathfrak{g}/\mathfrak{h})\otimes \ker d_q}{\mathrm{Im}\alpha_p \otimes \mathrm{Im} d_{q+1}} \cong \wedge^p(\mathfrak{g}/\mathfrak{h})\otimes H_q(\mathfrak{h},(M,\alpha_M)).$$

Since $\mathfrak{h}$ is a Hom-ideal of $\mathfrak{g}$, $H_q(\mathfrak{h},(M,\alpha_M))$ is a $\mathfrak{g}$-module for any $q \geq 0$ by Proposition 3.1. Thus, to complete our proof, it suffices to check that the following diagram is commutative

$$(11)\qquad \begin{array}{ccc} E^1_{pq} & \xrightarrow{d^1_{pq}} & E^1_{p-1,q} \\ \downarrow \cong & & \downarrow \cong \\ \wedge^p(\mathfrak{g}/\mathfrak{h})\otimes H_q(\mathfrak{h},(M,\alpha_M)) & \xrightarrow{d_1} & \wedge^{p-1}(\mathfrak{g}/\mathfrak{h})\otimes H_q(\mathfrak{h},(M,\alpha_M)) \end{array}$$

where d_1 is the differential of complex $C_\bullet(\mathfrak{g}/\mathfrak{h},(H_q(\mathfrak{h},(M,\alpha_M)),\alpha_{q,M}))$. Recall that d^1_{pq} is induced by the differential d of the complex $C_\bullet(\mathfrak{g},(M,\alpha_M))$.

Suppose $x_1,x_2,\cdots,x_q \in \mathfrak{h}$, $x_{q+1},x_{q+2},\cdots,x_{p+q} \in \mathfrak{g}$ and $m \in M$. Let $d = \mu_1 + \mu_2$, where

$$\begin{aligned} &\mu_1(x_1\wedge\cdots\wedge x_{p+q}\otimes m)\\ &= \sum_{1\leq i<j\leq q}(-1)^{i+j}[x_i,x_j]\wedge\alpha(x_1)\wedge\alpha(x_2)\wedge\cdots\widehat{x_i}\cdots\widehat{x_j}\cdots\wedge\alpha(x_{p+q})\otimes\alpha_M(m)\\ &\quad+\sum_{i=1}^{q}(-1)^i\alpha(x_1)\wedge\alpha(x_2)\wedge\cdots\widehat{x_i}\cdots\wedge\alpha(x_{p+q})\otimes x_i.m, \end{aligned}$$

and

$$
\begin{aligned}
&\mu_2(x_1 \wedge \cdots \wedge x_{p+q} \otimes m) \\
= &\sum_{1+q \le i < j \le p+q} (-1)^{i+j}[x_i, x_j] \wedge \alpha(x_1) \wedge \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \wedge \alpha(x_{p+q}) \otimes \alpha_M(m) \\
&+ \sum_{i=1}^{q} \sum_{j=q+1}^{p+q} (-1)^{i+j}[x_i, x_j] \wedge \alpha(x_1) \wedge \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots \wedge \alpha(x_{p+q}) \otimes \alpha_M(m) \\
&+ \sum_{i=1+q}^{q+p} (-1)^i \alpha(x_1) \wedge \cdots \widehat{x_i} \cdots \wedge \alpha(x_{p+q}) \otimes x_i.m.
\end{aligned}
$$

Let $\xi' \in C_{p+q}(\mathfrak{g}, (M, \alpha_M))$ be a preimage of

$$\xi \in E^1_{pq} = \wedge^p(\mathfrak{g}/\mathfrak{h}) \otimes H_q(\mathfrak{h}, (M, \alpha_M)).$$

Then the image of $d(\xi')$ in the quotient E^1_{pq} does not depend on the choice of ξ', which is denoted by $\overline{d(\xi')}$. At this point, one can see that $\overline{\mu_1(\xi')} = d_0(\xi) = 0$, where d_0 is the differential of complex $C_\bullet(\mathfrak{h}, (\wedge^p(\mathfrak{g}/\mathfrak{h}) \otimes M, \alpha_{p,M}))$. Similarly, we have $\overline{\mu_2(\xi')} = d_1(\xi)$. This implies that the diagram 11 is commutative. □

Suppose $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a multiplicative Hom-Lie algebra and $\mathfrak{g}_A$ is its abelian Hom-Lie algebra. Then $\mathfrak{g}_A$ is also a Hom-Lie algebra which shares the same linear map α with $\mathfrak{g}$. Let $(M, 0)$ be a $\mathfrak{g}$-module. Then $(M, 0)$ is also a $\mathfrak{g}_A$-module with the same action as $\mathfrak{g}$. Furthermore, $dC_n(\mathfrak{g}, (M, 0)) = dC_n(\mathfrak{g}_A, (M, 0))$ for any $n \ge 0$. This implies that

$$H_n(\mathfrak{g}, (M, 0)) \cong H_n(\mathfrak{g}_A, (M, 0)). \tag{12}$$

If $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a regular abelian Hom-Lie algebra, then every α-invariant subspace is a Hom-Lie ideal. Then we can compute the homological groups of a finite dimensional regular abelian Hom-Lie algebra by using Proposition 4.1. Explicitly, we have the following corollary.

Corollary 4.1. *Let $(L_n, [\cdot,\cdot]_n, \alpha)$ be a regular abelian Hom-Lie algebra with a basis $\{e_1, e_2, \cdots, e_n\}$. Suppose that L_{n-1} is the Hom-ideal generated by $\{e_1, e_2, \cdots, e_{n-1}\}$ and $\alpha(e_i) = \sum_{1 \le k \le i} a_{ik} e_k$, where $a_{ik} \in \mathbf{k}$ and $a_{ii} \ne 0$. Then, for any $p \in \mathbb{Z}_+$,*

$$H_p(\mathfrak{g}, (M, \alpha_M)) \cong H_p(L_{n-1}, (M, \alpha_M))^{e_n} \oplus H_{p-1}(L_{n-1}, (M, \alpha_M))^{e_n},$$

where $H_s(L_{n-1}, (M, \alpha_M))^{e_n} = \{v \in H_s(L_{n-1}, (M, \alpha_M)) | e_n \cdot v = 0\}$, for any $s \ge 0$. In particular,

$$dim H_p(\mathfrak{g}, (M, \alpha_M)) \le dim H_p(L_{n-1}, (M, \alpha_M)) + dim H_{p-1}(L_{n-1}, (M, \alpha_M)).$$

Proof. From the Serre-Hochschild spectral sequence of $L_{n-1} \subset L_n$, we get

$$E^2_{pq} = H_p(e_n, (H_q(L_{n-1}, (M, \alpha_M)), \alpha_{q,M}))$$

by Proposition 4.1. Thus $E^2_{pq} = 0$ unless $p = 0$ or $p = 1$. Since the differential d^r_{pq} of spectral sequence has degree $(-r, -1+r)$. It implies that $d^r_{pq} = 0$ for $r \geq 2$. So $E^\infty_{pq} = E^2_{pq}$. By now, using $E^2_{1q} = E^2_{0q} = H_q(L_{n-1}, (M, \alpha_M))^{e_n}$, one can easily complete the proof. □

As an application of Proposition 4.1, we obtain the homological groups of a finite-dimensional multiplicative Hom-Lie algebra with coefficient in M as follow.

THEOREM 4.1. *Suppose $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a finite dimensional multiplicative Hom-Lie algebra with $n(\alpha) = s$ and (M, α_M) is a $\mathfrak{g}$-module. Then for any $n \geq 0$,*

$$H_n(\mathfrak{g}, (M, \alpha_M)) \cong \bigoplus_{p+q=n} H_p(\mathfrak{g}/\ker\alpha^s, (H_q(\ker\alpha^s, (M, \alpha_M)), \alpha_{q,M})). \tag{13}$$

Proof. Since $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a finite dimensional multiplicative Lie algebra with $n(\alpha) = s$, $\mathfrak{g} = R \ltimes \ker\alpha^s$ by Lemma 2.1, where $R \simeq \mathfrak{g}/\ker(\alpha^s)$ is a regular Hom-subalgebra of $\mathfrak{g}$. Using the Serre-Hochschild spectral sequence of $\ker\alpha^s \subset \mathfrak{g}$, one can obtain that

$$E^1_{pq} = \wedge^p R \otimes H_q(\ker\alpha^s, (M, \alpha_M))$$

by Proposition 4.1. As $H_q(\ker\alpha^s, (M, \alpha_M))$ is a well-defined R-module with action (6), $E^2_{pq} = H_p(R, (H_q(\ker\alpha^s, (M, \alpha_M)), \alpha_{q,M}))$ by Proposition 4.1. It is easy to see that $d^r_{pq} = 0$ for $r \geq 2$. Thus $E^\infty_{pq} = E^2_{pq}$. □

To compute homological groups of a finite dimensional multiplicative Hom-Lie algebra, one need only to compute that of some regular Hom-Lie algebras by Theorem 4.1. About the homological groups of a regular finite dimensional Hom-Lie algebra, we have the following.

PROPOSITION 4.2. *Suppose $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a regular Hom-Lie algebra of Lie type of $\mathfrak{g}_L$ and (M, α_M) is a finite dimensional $\mathfrak{g}$-module with $n(\alpha_M) = t$, then*

$$H_n(\mathfrak{g}, (M, \alpha_M)) \cong H_n(\mathfrak{g}, (\ker\alpha^t_M, \alpha_M)) \oplus H^{Lie}_n(\mathfrak{g}_L, M/\ker\alpha^t_M). \tag{14}$$

Proof. Since $\alpha^t_M(x \cdot m) = \alpha^t(x) \cdot \alpha^t_M(m) = 0$ for any $x \in \mathfrak{g}$ and $m \in \ker\alpha^t_M$, $\ker\alpha^t_M$ is a $\mathfrak{g}$-submodule of M. With similar analysis of Lemma 2.1, one get the following short splitting exact sequence of $\mathfrak{g}$-modules

$$0 \to \ker\alpha^t_M \to M \to M/\ker\alpha^t_M \to 0.$$

Thus, for any $n \geq 0$,

$$\begin{aligned} H_n(\mathfrak{g},(M,\alpha_M)) \cong & H_n(\mathfrak{g},(\ker\alpha_M^t \oplus M/\ker\alpha_M^t,\alpha_M)) \\ \cong & H_n(\mathfrak{g},(\ker\alpha_M^t,\alpha_M)) \oplus H_n(\mathfrak{g},(M/\ker\alpha_M^t,\alpha_M)) \\ \cong & H_n(\mathfrak{g},(\ker\alpha_M^t,\alpha_M)) \oplus H_n^{Lie}(\mathfrak{g}_L, M/\ker\alpha_M^t). \end{aligned}$$

by (9). □

In addition, about the homological group $H_n(\mathfrak{g},(\ker\alpha_M^t,\alpha_M))$, we have the following result.

COROLLARY 4.2. *Suppose* $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ *is a regular Lie algebra and* (M,α_M) *is a* $\mathfrak{g}$*-module with* $n(\alpha_M)=t$. *Then*

$$dimH_n(\mathfrak{g},(\ker\alpha_M^t,\alpha_M)) \leq \sum_{i=1}^{t} dimH_n(\mathfrak{g}_A,(\ker\alpha_M^i/\ker\alpha_M^{i-1},0)). \tag{15}$$

Proof. Consider the bounded filtration of $\mathfrak{g}$-modules,

$$0 \subset (\ker\alpha_M,\alpha_M) \subset (\ker\alpha_M^2,\alpha_M) \subset \cdots \subset (\ker\alpha_M^t,\alpha_M).$$

For any $s,n \in \mathbb{Z}_+$, define a subcomplex by

$$F_sC_n(\mathfrak{g},(\ker\alpha_M^t,\alpha_M)) = C_n(\mathfrak{g},(\ker\alpha_M^s,\alpha_M)).$$

Then we have a spectral sequence with

$$E_{pq}^1 \cong H_{p+q}(\mathfrak{g},(\ker\alpha_M^p/\ker\alpha_M^{p-1},\alpha_M)).$$

Since α_M acts trivially on $\ker\alpha_M^p/\ker\alpha_M^{p-1}$,

$$E_{pq}^1 \cong H_{p+q}(\mathfrak{g}_A,(\ker\alpha_M^p/\ker\alpha_M^{p-1},0))$$

by (12). Hence (15) is established. □

Finally, let us assume that $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ is a finite dimensional non-regular multiplicative Hom-Lie algebra and (M,α_M) is a finite dimensional $\mathfrak{g}$-module, where $s=n(\alpha)$, $t=n(\alpha_M)$. Let $k=\max(s,t)$. Then $\mathfrak{g}=R \ltimes \ker\alpha^s$ according to Lemma 2.1, where R is a regular Hom-subalgebra of Lie type of R_L. Consequently, by (13), (14), and (15), we have

$$
\begin{aligned}
&dimH_n(\mathfrak{g},(M,\alpha_M))\\
&\quad=\sum_{p+q=n}\dim H_p(R,(H_q(\ker\alpha^s,(M,\alpha_M)),\alpha_{q,M}))\\
&\quad=\sum_{p+q=n}\dim(H_p^{Lie}(R_L,\frac{H_q(\ker\alpha^s,(M,\alpha_M))}{\ker\alpha^k_{q,M}})\\
&\qquad\qquad+H_p(R,(\ker\alpha^k_{q,M},\alpha_{q,M})))\\
&\quad\leq\sum_{p+q=n}\dim(H_p^{Lie}(R_L,\frac{H_q(\ker\alpha^s,(M,\alpha_M))}{\ker\alpha^k_{q,M}})\\
&\qquad\qquad+\sum_{j=1}^{k}H_p(R_A,(\frac{\ker\alpha^j_{q,M}}{\ker\alpha^{j-1}_{q,M}},0))).
\end{aligned}
\tag{16}
$$

As a consequence, we believe that the abelian Hom-Lie algebras are very important in the homology theory of multiplicative Hom-Lie algebras.

Corollary 4.3. *Keep the notations as above. If* $\max(n(\alpha),n(\alpha_M))\leq 1$, *then*

$$H_n(\mathfrak{g},(M,\alpha_M))\cong\bigoplus_{p+q=n}H_p^{Lie}(R_L,\frac{H_q(\ker\alpha,(M,\alpha_M))}{\ker\alpha_{q,M}})\oplus H_p(R_A,(\ker\alpha_{q,M},0)).$$

Example 4.2. Suppose $\mathcal{H}$ is vector space with basis $\{x_1,x_2,x_3\}$. The operation of $\mathcal{H}$ is determined by the following brackets:

$$[x_1,x_2]_{\mathcal{H}}=x_3,\ [x_1,x_3]_{\mathcal{H}}=0,\ [x_2,x_3]_{\mathcal{H}}=0.$$

Define an endomorphism α of the vector space $\mathcal{H}$ by

$$\alpha(x_1)=x_1,\ \alpha(x_2)=x_3,\ \alpha(x_3)=0.$$

Then $(\mathcal{H},[\cdot,\cdot]_{\mathcal{H}},\alpha)$ is a multiplicative Hom-Lie algebra with $\ker\alpha^2=\mathbf{k}x_3\oplus\mathbf{k}x_2$. Suppose $(\mathbf{k},id)$ is the trivial $\mathfrak{g}$-module. Consider the Serre-Hochschild spectral sequence of $\ker\alpha^2\subset\mathcal{H}$. By straight computation, we can obtain that $E^2_{00}=\mathbf{k}$, $E^2_{01}=\mathbf{k}x_2, E^2_{11}=\mathbf{k}(x_1\otimes x_3)$, $E^2_{12}=\mathbf{k}(x_1\otimes x_2\wedge x_3)$, $E^2_{02}=\mathbf{k}(x_2\wedge x_3)$, $E^2_{10}=\mathbf{k}x_1$. Thus

$$\dim H_n(\mathcal{H},(\mathbf{k},id))\equiv\begin{cases}1, & \text{for } n=0\\ 2, & \text{for } n=1.\\ 2, & \text{for } n=2.\\ 1, & \text{for } n=3.\\ 0, & \text{otherwise.}\end{cases}$$

Proposition 4.3. *Suppose* $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ *is a multiplicative Hom-Lie algebra of Lie type of a Lie algebra* $\mathfrak{g}_L$. *Assume that* (M,α_M) *is a* $\mathfrak{g}$*-module.*

Consider the Serre-Hochschild spectral sequence of $\ker\alpha \subset \mathfrak{g}$*. Then* $d^r_{pq} = 0$ *for* $r > 2$ *and* $d^2_{pq} = 0$ *for* $q \neq 0$*.*

Proof. We abbreviate $F_pC_n(\mathfrak{g}, (M, \alpha_M))$ as F_pC_n for any $n, p \in \mathbb{Z}_+$. Notice that $[\ker\alpha, \mathfrak{g}]_{\mathfrak{g}} = 0$. Thus the differential d acts trivially on the subspace F_pC_{p+q} for $q > 1$. Since d^r_{pq} is induced by the differential d, $d^r_{pq} = 0$ for $q > 1$ and $r \geq 0$. By definition, one can see that the differential vanishes on the set

$$\{\phi \in F_{p-1}C_p \setminus F_{p-2}C_p | d(c) \in F_{p-r}C_{p-1} \text{ for } r \geq 2\}.$$

It implies that $d^r_{p1} = 0$ for $r \geq 2$. Similarly, we have $d^r_{p0} = 0$ for $r > 2$. □

In general, d^2_{p0} in Proposition 4.3 may be non-trivial.

Example 4.3. Let $\mathfrak{g}_L$ be a four dimensional Lie algebra. Suppose that $\{x_1, x_2, x_3, x_4\}$ is a basis of $\mathfrak{g}_L$. The non-trivial brackets of $\mathfrak{g}_L$ given by

$$[x_1, x_2] = x_3, \ [x_2, x_3] = x_4.$$

and define $\alpha(x_1) = x_2$, $\alpha(x_2) = x_3$, $\alpha(x_3) = x_4$, $\alpha(x_4) = 0$. Thus $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ is a multiplicative Hom-Lie algebra with $\ker\alpha = \mathbf{k}x_4$ and non-trivial bracket: $[x_1, x_2]_{\mathfrak{g}} = [\alpha(x_1), \alpha(x_2)] = x_4$. Let $(\mathbf{k}, id)$ be the trivial module. Now consider the Serre-Hochschild spectral sequence of $\ker\alpha \subset \mathfrak{g}$. It is easy to see that $x_1 \wedge x_2 \in E^2_{2,0} = H_2(\mathfrak{g}/\ker\alpha, (\mathbf{k}, id))$ and $x_4 \in E^2_{01} = H_0(\mathfrak{g}/\ker\alpha, (\ker\alpha, 0))$. Furthermore, we have $d^2_{20}(x_1 \wedge x_2) = -x_4$.

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