FINITENESS DIMENSIONS AND COFINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Let R be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, M a finite R-module, and n a non-negative integer. In this paper, for an arbitrary R-module X which is not necessarily finite, we prove the following results: (i) $f_{\mathfrak{a}}^n(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(M,X)$ is not an $\mathrm{FD}_{<n}$ R-module$\}$ if $\mathrm{Ext}_R^i(M/\mathfrak{a}M,X)$ is an $\mathrm{FD}_{<n}$ R-module for all i; (ii) $f_{\mathfrak{a}}^1(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(M,X)$ is not a minimax R-module$\}$ if $\mathrm{Ext}_R^i(M/\mathfrak{a}M,X)$ is finite for all i; (iii) $f_{\mathfrak{a}}^2(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(M,X)$ is not a weakly Laskerian R-module$\}$ if R is semi-local and $\mathrm{Ext}_R^i(M/\mathfrak{a}M,X)$ is finite for all i; (iv) $\mathrm{H}_{\mathfrak{a}}^i(M,X)$ is $\mathfrak{a}$-cofinite for all $i < f_{\mathfrak{a}}^2(M,X)$ and $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M,X)}(M,X))$ is finite if $\mathrm{Ext}_R^i(M/\mathfrak{a}M,X)$ is finite for all $i \leq f_{\mathfrak{a}}^2(M,X)$. Here, $f_{\mathfrak{a}}^n(M,X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}},X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R)$ and $\dim_R(R/\mathfrak{p}) \geq n\}$ is the nth finiteness dimension of M and X with respect to $\mathfrak{a}$ and $f_{\mathfrak{a}}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(M,X)$ is not a finite R-module$\}$ is the finiteness dimension of M and X with respect to $\mathfrak{a}$.

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1. INTRODUCTION

Throughout, R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of R, M is a finite (i.e., finitely generated) R-module, and n is a non-negative integer. For basic results, notations, and terminology not given in this paper, readers are referred to [10, 11].

An important problem in local cohomology is to investigate finiteness of local cohomology modules (see [22, Problem 2]). Let N be a finite R-module. The following theorem is an important result in local cohomology and known as Faltings' Local-global Principle for the finiteness of local cohomology modules (see [17, Satz 1] or [10, Theorem 9.6.1]).

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THEOREM 1.1. *Let t be a non-negative integer. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(N)$ *is a finite R-module for all $i \leq t$;*

(ii) $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}})$ *is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and all $i \leq t$.*

Another formulation of Faltings' Local-global Principle is in terms of the finiteness dimension $f_{\mathfrak{a}}(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(N) \text{ is not a finite } R\text{-module}\}$ of N with respect to $\mathfrak{a}$ with the usual convention that the infimum of the empty set is interpreted as ∞. In this formulation, Faltings' Local-global Principle says that $f_{\mathfrak{a}}(N) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R)\}$. Bahmanpour et al., in [7], introduced the notion of the nth finiteness dimension of N with respect to $\mathfrak{a}$ by $f^n_{\mathfrak{a}}(N) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$. Thus Faltings' Local-global Principle states that $f_{\mathfrak{a}}(N) = f^0_{\mathfrak{a}}(N)$, that is

$$f^0_{\mathfrak{a}}(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(N) \text{ is not a finite } R\text{-module}\}. \tag{1}$$

In [7, Corollary 2.4 and Proposition 3.7], the authors obtained that

$$f^1_{\mathfrak{a}}(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(N) \text{ is not a minimax } R\text{-module}\} \tag{2}$$

and if R is a semi-local ring, then

$$f^2_{\mathfrak{a}}(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(N) \text{ is not a weakly Laskerian } R\text{-module}\}. \tag{3}$$

Recall that an arbitrary R-module X is said to be minimax (resp. weakly Laskerian) if there exists a finite submodule X' of X such that X/X' is Artinian [33] (resp. the set of associated prime ideals of any quotient module of X is finite [16]). Mehrvarz et al., in [24, Theorem 2.10], generalized Faltings' Local-global Principle (1) and showed that

$$f^n_{\mathfrak{a}}(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(N) \text{ is not an } \mathrm{FD}_{<n} \ R\text{-module}\} \tag{4}$$

(see also [3, Theorem 2.5]). Recall that an arbitrary R-module X is said to be an $\mathrm{FD}_{<n}$ R-module if there exists a finite submodule X' of X such that $\dim_R(X/X') < n$ [2, 3]. Note that X is an $\mathrm{FD}_{<n}$ R-module if X is a finite R-module or $\dim_R(X) < n$, X is a finite R-module if and only if X is an $\mathrm{FD}_{<0}$ R-module, and X is an $\mathrm{FD}_{<1}$ (resp. $\mathrm{FD}_{<2}$) R-module if X is a minimax (resp. weakly Laskerian) R-module (see [2, Lemma 2.3]). The nth generalized local cohomology module

$$\mathrm{H}^n_{\mathfrak{a}}(X, Y) \cong \varinjlim_{i \in \mathbb{N}} \mathrm{Ext}^n_R(X/\mathfrak{a}^i X, Y)$$

of arbitrary R-modules X and Y with respect to $\mathfrak{a}$ was introduced by Herzog in [20]. It is clear that $\mathrm{H}^n_{\mathfrak{a}}(R, Y)$ is just the nth ordinary local cohomology module

$\mathrm{H}^n_{\mathfrak{a}}(Y)$ of arbitrary R-module Y with respect to $\mathfrak{a}$. In [21, Definition 2.3 and Theorem 2.4], Hoang introduced the notion of the nth finiteness dimension $f^n_{\mathfrak{a}}(M,N)$ of M and N with respect to $\mathfrak{a}$ by $f^n_{\mathfrak{a}}(M,N) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R)$ and $\dim_R(R/\mathfrak{p}) \geq n\}$, where $f_{\mathfrak{a}}(M,N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,N)$ is not a finite R-module$\}$, and generalized (4) by showing that

(5) $\quad f^n_{\mathfrak{a}}(M,N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,N)$ is not an $\mathrm{FD}_{<n}$ R-module$\}$.

Let X be an arbitrary R-module which is not necessarily finite. Recently, in [1, Theorem 2.3], the authors generalized Faltings' Local-global Principle (1) and proved that if $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i, then $f^0_{\mathfrak{a}}(X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X)$ is not a finite R-module$\}$. We generalize and improve this result and the equality (5) by showing that the equality $f^n_{\mathfrak{a}}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X)$ is not an $\mathrm{FD}_{<n}$ R-module$\}$ holds if $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i. We also generalize and improve the equalities (2) and (3). We prove that if $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite R-module for all i, then $f^1_{\mathfrak{a}}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X)$ is not a minimax R-module$\}$ and, moreover, if R is a semi-local ring, then $f^2_{\mathfrak{a}}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X)$ is not a weakly Laskerian R-module$\}$.

Grothendieck, in [18], proposed the following conjecture.

Conjecture 1.2. $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(N))$ *is a finite R-module for all i.*

Hartshorne gave a counterexample to this conjecture in [19] and defined an R-module X to be $\mathfrak{a}$-cofinite if $\mathrm{Supp}_R(X) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i. He also asked the following question.

Question 1.3. When is $\mathrm{H}^i_{\mathfrak{a}}(N)$ an $\mathfrak{a}$-cofinite R-module for all i?

The following question is also an important problem in commutative algebra (see [22, Problem 4]).

Question 1.4. When is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(N))$ a finite set for all i?

As generalizations of Conjecture and Questions 1.2-1.4, we have the following questions (see [8, Question 1.1] and [29, Question 2.7]).

Question 1.5. When is $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M,N))$ a finite R-module for all i?

Question 1.6. When is $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ an $\mathfrak{a}$–cofinite R-module for all i?

Question 1.7. When is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M,N))$ a finite set for all i?

These questions were studied by several authors. In this direction, for an arbitrary R-module X which is not necessarily finite and for a non-negative integer t, we show that if $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite R-module for all $i \leq t$

and $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is an $\mathrm{FD}_{<2}$ R-module for all $i<t$, then $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is an $\mathfrak{a}$-cofinite R-module for all $i<t$, $\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}^t_{\mathfrak{a}}(M,X))$ is a finite R-module, and $\mathrm{Ass}_R(\mathrm{H}^t_{\mathfrak{a}}(M,X))$ is a finite set. This generalizes and improves all of the previous results concerning Conjecture and Questions 1.2-1.7 (see e.g., [14, 31, 23, 9, 30, 5, 6, 12, 26, 8, 25, 15, 7, 2, 13, 27]).

2. FINITENESS DIMENSIONS

The following lemmas are needed in the proof of the main result of this section. Note that, by [32, Theorem 2.3], the class of $\mathrm{FD}_{<n}$ R-modules forms a Serre subcategory of the category of R-modules (i.e., the class of R-modules which is closed under taking submodules, quotients, and extensions).

LEMMA 2.1. *Let M be a finite R-module, X an arbitrary R-module, and t a non-negative integer such that* $\mathrm{Ext}^i_R(M/\mathfrak{a}M,X)$ *is an* $\mathrm{FD}_{<n}$ *R-module for all $i\leq t$. Then* $\mathrm{Ext}^i_R(\mathrm{Tor}^R_j(R/\mathfrak{a},M),X)$ *is an* $\mathrm{FD}_{<n}$ *R-module for all $i\leq t$ and all j.*

Proof. The proof is similar to that of [28, Lemma 2.1 and Corollary 2.2] and left to the reader. □

LEMMA 2.2. *Let M be a finite R-module, X an arbitrary R-module, and t a non-negative integer such that*

(i) $\mathrm{Ext}^{t-i}_R(\mathrm{Tor}^R_i(R/\mathfrak{a},M),X)$ *is an* $\mathrm{FD}_{<n}$ *R-module for all $i\leq t$, and*

(ii) $\mathrm{Ext}^{t+1-i}_R(R/\mathfrak{a},\mathrm{H}^i_{\mathfrak{a}}(M,X))$ *is an* $\mathrm{FD}_{<n}$ *R-module for all $i<t$.*

Then $\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}^t_{\mathfrak{a}}(M,X))$ *is an* $\mathrm{FD}_{<n}$ *R-module.*

Proof. This is sufficiently similar to that of [28, Theorem 2.3] to be omitted. We leave the proof to the reader. □

LEMMA 2.3. *Let X be an $\mathfrak{a}$–torsion R-module such that $X_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p}\in\mathrm{Spec}(R)$ with $\dim_R(R/\mathfrak{p})\geq n$ and* $\mathrm{Hom}_R(R/\mathfrak{a},X)$ *is an* $\mathrm{FD}_{<n}$ *R-module. Then X is an* $\mathrm{FD}_{<n}$ *R-module.*

Proof. Suppose, on the contrary, that X is not an $\mathrm{FD}_{<n}$ R-module and seek a contradiction. Let $A_1=\{\mathfrak{p}\in\mathrm{Ass}_R(X):\dim_R(R/\mathfrak{p})\geq n\}$ and $\mathfrak{a}_1=\bigcap_{\mathfrak{p}\in A_1}\mathfrak{p}$. Since X is not an $\mathrm{FD}_{<n}$ R-module, $\dim_R(X)\geq n$. Thus A_1 is a non-empty and finite set because $\mathrm{Ass}_R(X)=\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a},X))$ and $\mathrm{Hom}_R(R/\mathfrak{a},X)$ is an $\mathrm{FD}_{<n}$ R-module. For all $\mathfrak{p}\in A_1$, $X_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$-module and so there exists a finite submodule $N(\mathfrak{p})$ of X such that $(N(\mathfrak{p}))_{\mathfrak{p}}=$

$X_{\mathfrak{p}}$. Let $N_1 = \sum_{\mathfrak{p}\in A_1} N(\mathfrak{p})$. Then N_1 is a finite submodule of X such that $A_1 \cap A_2 = \emptyset$ and $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$, where $A_2 = \{\mathfrak{p} \in \mathrm{Ass}_R(X/N_1) : \dim_R(R/\mathfrak{p}) \geq n\}$ and $\mathfrak{a}_2 = \bigcap_{\mathfrak{p}\in A_2} \mathfrak{p}$. Since X is not an $\mathrm{FD}_{<n}$ R-module, $\dim_R(X/N_1) \geq n$. Note that X/N_1 is an $\mathfrak{a}$–torsion R-module and $\mathrm{Hom}_R(R/\mathfrak{a}, X/N_1)$ is an $\mathrm{FD}_{<n}$ R-module from the exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{a}, X) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, X/N_1) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, N_1).$$

Thus A_2 is a non-empty and finite set, and so $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2$.

Therefore, using the above method on the R-module X/N_1, there is a finite submodule N_2 ($\supseteq N_1$) of X such that $A_2 \cap A_3 = \emptyset$ and $\mathfrak{a}_2 \subsetneq \mathfrak{a}_3$, where $A_3 = \{\mathfrak{p} \in \mathrm{Ass}_R(X/N_2) : \dim_R(R/\mathfrak{p}) \geq n\}$ and $\mathfrak{a}_3 = \bigcap_{\mathfrak{p}\in A_3} \mathfrak{p}$.

Proceeding in the same way, there is an ascending chain of ideals of Noetherian ring R,

$$\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \cdots \subsetneq \mathfrak{a}_i \subsetneq \cdots,$$

which is not stable. This contradiction shows that X is an $\mathrm{FD}_{<n}$ R-module, as we desired. □

Now we are prepared to state and prove the main result of this section which generalizes and improves [17, Satz 1], [10, Theorem 9.6.1 and 9.6.2], [3, Theorem 2.5], [24, Theorem 2.10], [21, Theorem 2.4], and [1, Theorems 1.1(i⇔ii) and 2.3].

THEOREM 2.4. *Let M be a finite R-module, X an arbitrary R-module, and t a non-negative integer such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$ (e.g., X is an $\mathrm{FD}_{<n}$ R-module). Then the following statements are equivalent:*

(i) *$\mathrm{H}^i_{\mathfrak{a}}(M, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$;*

(ii) *$\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}})$ is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \geq n$ and for all $i \leq t$.*

Proof. (i)⇒(ii). Let $\mathfrak{p}$ be a prime ideal of R with $\dim_R(R/\mathfrak{p}) \geq n$ and let $i \leq t$. Since $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ is an $\mathrm{FD}_{<n}$ R-module, there exists a finite submodule N_i of $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ such that $\dim_R(\mathrm{H}^i_{\mathfrak{a}}(M, X)/N_i) < n$. Thus $(\mathrm{H}^i_{\mathfrak{a}}(M, X)/N_i)_{\mathfrak{p}} = 0$ and so $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}}) \cong (\mathrm{H}^i_{\mathfrak{a}}(M, X))_{\mathfrak{p}} = (N_i)_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$-module.

(ii)⇒(i). We prove by using induction on t. Let $t = 0$. Since $\mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M, X))$ is an $\mathrm{FD}_{<n}$ R-module from Lemma 2.2, $\Gamma_{\mathfrak{a}}(M, X)$ is an $\mathrm{FD}_{<n}$ R-module by Lemma 2.3. Suppose that $t > 0$ and that $t-1$ is settled. It is enough to show that $\mathrm{H}^t_{\mathfrak{a}}(M, X)$ is an $\mathrm{FD}_{<n}$ R-module because $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t-1$ from the induction hypothesis

on $t-1$. Thus, by Lemmas 2.1 and 2.2, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M,X))$ is an $\mathrm{FD}_{<n}$ R-module. Hence $\mathrm{H}^t_{\mathfrak{a}}(M,X)$ is an $\mathrm{FD}_{<n}$ R-module from Lemma 2.3. □

Definition 2.5. (cf. [21, Definition 2.3]) Let M be a finite R-module, X an arbitrary R-module (not necessarily finite), and n a non-negative integer. We set

$$f_{\mathfrak{a}}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not a finite } R\text{-module}\}$$

and

$$f^n_{\mathfrak{a}}(M,X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$$

which are called finiteness dimension and nth finiteness dimension of M and X with respect to $\mathfrak{a}$, respectively. When $M = R$, we write $f_{\mathfrak{a}}(X) = f_{\mathfrak{a}}(R,X)$ and $f^n_{\mathfrak{a}}(X) = f^n_{\mathfrak{a}}(R,X)$ which are called finiteness dimension and nth finiteness dimension of X with respect to $\mathfrak{a}$, respectively. Thus

$$f_{\mathfrak{a}}(X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X) \text{ is not a finite } R\text{-module}\}$$

and

$$f^n_{\mathfrak{a}}(X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}.$$

Corollary 2.6. *Let M be a finite R-module and let X be an arbitrary R-module such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i (in fact, for all $i \leq f^n_{\mathfrak{a}}(M,X)$). Then*

$$f^n_{\mathfrak{a}}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \textit{ is not an } \mathrm{FD}_{<n} \ R\textit{-module}\}.$$

Proof. This follows from Theorem 2.4. □

Corollary 2.7. *Let M be a finite R-module, X an arbitrary R-module, and t a non-negative integer such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite R-module for all $i \leq t$. Then the following statements are equivalent:*

(i) *$\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is a finite R-module for all $i \leq t$;*

(ii) *$\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}})$ is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and for all $i \leq t$.*

Proof. Apply Theorem 2.4 with $n = 0$. □

Corollary 2.8. *Let M be a finite R-module and let X be an arbitrary R-module such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite R-module for all i (in fact, for all $i \leq f^0_{\mathfrak{a}}(M,X)$). Then*

$$f_{\mathfrak{a}}^0(M, X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(M, X) \text{ is not a finite } R\text{-module}\},$$

that is

$$f_{\mathfrak{a}}(M, X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

Proof. Take $n = 0$ in Corollary 2.6. □

We have the following corollaries for the ordinary local cohomology modules.

Corollary 2.9. *Let X be an arbitrary R-module and let t be a non-negative integer such that $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$. Then the following statements are equivalent:*

(i) *$\mathrm{H}_{\mathfrak{a}}^i(X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$;*

(ii) *$\mathrm{H}_{\mathfrak{a}R_{\mathfrak{p}}}^i(X_{\mathfrak{p}})$ is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \geq n$ and for all $i \leq t$.*

Corollary 2.10. *Let X be an arbitrary R-module such that $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i (in fact, for all $i \leq f_{\mathfrak{a}}^n(X)$). Then*

$$f_{\mathfrak{a}}^n(X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(X) \text{ is not an } \mathrm{FD}_{<n} \text{ } R\text{-module}\}.$$

Corollary 2.11. *Let X be an arbitrary R-module and let t be a non-negative integer such that $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq t$. Then the following statements are equivalent:*

(i) *$\mathrm{H}_{\mathfrak{a}}^i(X)$ is a finite R-module for all $i \leq t$;*

(ii) *$\mathrm{H}_{\mathfrak{a}R_{\mathfrak{p}}}^i(X_{\mathfrak{p}})$ is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and for all $i \leq t$.*

Corollary 2.12. *Let X be an arbitrary R-module such that $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is a finite R-module for all i (in fact, for all $i \leq f_{\mathfrak{a}}^0(X)$). Then*

$$f_{\mathfrak{a}}^0(X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(X) \text{ is not a finite } R\text{-module}\},$$

that is

$$f_{\mathfrak{a}}(X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

3. COFINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

Lemma 3.1. *Let M be a finite R-module, X an arbitrary R-module, and s, t non-negative integers such that*

(i) $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ *is a finite R-module for all $t \leq i \leq s+t+1$,*

(ii) $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ *is an $\mathfrak{a}$–cofinite R-module for all $i < t$, and*

(iii) $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ *is an* $\mathrm{FD}_{<2}$ *R-module for all $t \leq i \leq s+t$.*
Then $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ is an $\mathfrak{a}$–cofinite R-module for all $i \leq s+t$.

Proof. We prove the lemma by induction on s. Let $s = 0$. Since $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M, X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M, X))$ are finite from [28, Corollary 2.2, Theorem 2.3, Theorem 2.7, and Corollary 2.14], $\mathrm{H}^t_{\mathfrak{a}}(M, X)$ is $\mathfrak{a}$–cofinite by [2, Theorem 3.1].

Suppose that $s > 0$ and that $s-1$ is settled. It is enough to show that $\mathrm{H}^{s+t}_{\mathfrak{a}}(M, X)$ is $\mathfrak{a}$–cofinite because $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ is $\mathfrak{a}$–cofinite for all $i \leq s+t-1$ from the induction hypothesis on $s-1$. By [28, Corollary 2.2, Theorem 2.3, Theorem 2.7, and Corollary 2.14], $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{s+t}_{\mathfrak{a}}(M, X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^{s+t}_{\mathfrak{a}}(M, X))$ are finite. Thus $\mathrm{H}^{s+t}_{\mathfrak{a}}(M, X)$ is $\mathfrak{a}$–cofinite from [2, Theorem 3.1]. □

We prove the main result of this section which generalizes and improves all of the previous results concerning Conjecture and Questions 1.2-1.7 (see e.g., [14, Theorem 1], [31, Theorem 1.1], [23, Theorem B], [9, Theorem 2.2], [30, Theorem 2.1], [5, Theorem 2.5], [6, Theorem 2.6], [12, Theorem 2.5], [26, Theorem 3.2], [8, Theorem 3.6], [25, Theorem 2.10], [15, Theorem 2.5], [7, Theorems 2.3 and 3.2], [2, Theorem 3.4], and [13, Theorem 1.2]).

Theorem 3.2. *Let M be a finite R-module, X an arbitrary R-module, and t a non-negative integer such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite R-module for all $i \leq t$ and $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ is an* $\mathrm{FD}_{<2}$ *R-module for all $i < t$. Then the following statements hold true:*

(i) *Y_i and $\mathrm{H}^i_{\mathfrak{a}}(M, X)/Y_i$ are $\mathfrak{a}$-cofinite R-modules for all $i < t$ and every* $\mathrm{FD}_{<1}$ *R-submodule Y_i of $\mathrm{H}^i_{\mathfrak{a}}(M, X)$. In particular, $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ is an $\mathfrak{a}$-cofinite R-module for all $i < t$;*

(ii) *Let N be a finite R-module. Then $\mathrm{Ext}^j_R(N, Y_i)$, $\mathrm{Tor}^R_j(N, Y_i)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(M, X)/Y_i)$, and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(M, X)/Y_i)$ are $\mathfrak{a}$-cofinite R-modules for all $i < t$, all j, and every* $\mathrm{FD}_{<1}$ *R-submodule Y_i of $\mathrm{H}^i_{\mathfrak{a}}(M, X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(M, X))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(M, X))$ are $\mathfrak{a}$-cofinite R-modules for all $i < t$ and all j;*

(iii) $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^t_{\mathfrak{a}}(M,X)/Y)$ *is a finite R-module for every* $\operatorname{FD}_{<1}$ *R-submodule Y of* $\operatorname{H}^t_{\mathfrak{a}}(M,X)$. *In particular,* $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^t_{\mathfrak{a}}(M,X))$ *is a finite R-module;*

(iv) $\operatorname{Ass}_R(\operatorname{H}^t_{\mathfrak{a}}(M,X)/Y)$ *is a finite set for every* $\operatorname{FD}_{<1}$ *R-submodule Y of* $\operatorname{H}^t_{\mathfrak{a}}(M,X)$. *In particular,* $\operatorname{Ass}_R(\operatorname{H}^t_{\mathfrak{a}}(M,X))$ *is a finite set;*

(v) *Assume that* $\operatorname{Ext}^{t+1}_R(M/\mathfrak{a}M, X)$ *is finite. Then*

$$\operatorname{Ext}^1_R(R/\mathfrak{a}, \operatorname{H}^t_{\mathfrak{a}}(M,X)/Y)$$

is a finite R-module for every $\operatorname{FD}_{<1}$ *R-submodule Y of* $\operatorname{H}^t_{\mathfrak{a}}(M,X)$. *In particular,* $\operatorname{Ext}^1_R(R/\mathfrak{a}, \operatorname{H}^t_{\mathfrak{a}}(M,X))$ *is a finite R-module.*

Proof. (i) Since $\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M,X))$ and $\operatorname{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M,X))$ are finite by [28, Corollary 2.2, Theorem 2.3, and Theorem 2.7], $\Gamma_{\mathfrak{a}}(M,X)$ is $\mathfrak{a}$-cofinite from [2, Theorem 3.1], and so $\operatorname{H}^i_{\mathfrak{a}}(M,X)$ is $\mathfrak{a}$-cofinite for all $i < t$ by Lemma 3.1. Let $i < t$ and let Y_i be an $\operatorname{FD}_{<1}$ R-submodule of $\operatorname{H}^i_{\mathfrak{a}}(M,X)$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, Y_i)$ is finite and so Y_i is $\mathfrak{a}$-cofinite from [2, Lemma 3.3]. Thus $\operatorname{H}^i_{\mathfrak{a}}(M,X)/Y_i$ is $\mathfrak{a}$-cofinite by the short exact sequence

$$0 \longrightarrow Y_i \longrightarrow \operatorname{H}^i_{\mathfrak{a}}(M,X) \longrightarrow \operatorname{H}^i_{\mathfrak{a}}(M,X)/Y_i \longrightarrow 0.$$

(ii) It follows from the first part and [2, Theorem 3.7].

(iii) Let Y be an $\operatorname{FD}_{<1}$ R-submodule of $\operatorname{H}^t_{\mathfrak{a}}(M,X)$. From the first part and [28, Corollary 2.2 and Theorem 2.3], $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^t_{\mathfrak{a}}(M,X))$ is finite. Thus $\operatorname{Hom}_R(R/\mathfrak{a}, Y)$ is finite and so Y is $\mathfrak{a}$-cofinite by [2, Lemma 3.3]. Hence, from the exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^t_{\mathfrak{a}}(M,X)) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^t_{\mathfrak{a}}(M,X)/Y) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, Y),$$

$\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}^t_{\mathfrak{a}}(M,X)/Y)$ is finite.

(iv) Follows by the third part and [11, Exercise 1.2.28].

(v) This is similar to the proof of the third part. □

Corollary 3.3. *Let M be a finite R-module and let X be an arbitrary R-module such that* $\operatorname{Ext}^i_R(M/\mathfrak{a}M, X)$ *is a finite R-module for all* $i \le f^2_{\mathfrak{a}}(M,X)$. *Then the following statements hold true:*

(i) Y_i *and* $\operatorname{H}^i_{\mathfrak{a}}(M,X)/Y_i$ *are* $\mathfrak{a}$*-cofinite R-modules for all* $i < f^2_{\mathfrak{a}}(M,X)$ *and every* $\operatorname{FD}_{<1}$ *R-submodule* Y_i *of* $\operatorname{H}^i_{\mathfrak{a}}(M,X)$. *In particular,* $\operatorname{H}^i_{\mathfrak{a}}(M,X)$ *is an* $\mathfrak{a}$*-cofinite R-module for all* $i < f^2_{\mathfrak{a}}(M,X)$*;*

(ii) *Let N be a finite R-module. Then* $\operatorname{Ext}^j_R(N, Y_i)$, $\operatorname{Tor}^R_j(N, Y_i)$, $\operatorname{Ext}^j_R(N, \operatorname{H}^i_{\mathfrak{a}}(M,X)/Y_i)$, *and* $\operatorname{Tor}^R_j(N, \operatorname{H}^i_{\mathfrak{a}}(M,X)/Y_i)$ *are* $\mathfrak{a}$*-cofinite R-modules* *for all* $i < f_{\mathfrak{a}}^2(M,X)$*, all* j*, and every* $\mathrm{FD}_{<1}$ R*-submodule* Y_i *of* $\mathrm{H}_{\mathfrak{a}}^i(M,X)$*. In particular,* $\mathrm{Ext}_R^j(N,\mathrm{H}_{\mathfrak{a}}^i(M,X))$ *and* $\mathrm{Tor}_j^R(N,\mathrm{H}_{\mathfrak{a}}^i(M,X))$ *are* $\mathfrak{a}$*-cofinite* R*-modules for all* $i < f_{\mathfrak{a}}^2(M,X)$ *and all* j*;*

(iii) $\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M,X)}(M,X)/Y)$ *is finite for every* $\mathrm{FD}_{<1}$ R*-submodule* Y *of* $\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M,X)}(M,X)$*. In particular,* $\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M,X)}(M,X))$ *is a finite* R*-module;*

(iv) $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M,X)}(M,X)/Y)$ *is a finite set for every* $\mathrm{FD}_{<1}$ R*-submodule* Y *of* $\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M,X)}(M,X)$*. In particular,* $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M,X)}(M,X))$ *is a finite set;*

(v) *Assume that* $\mathrm{Ext}_R^{f_{\mathfrak{a}}^2(M,X)+1}(M/\mathfrak{a}M,X)$ *is finite. Then*

$$\mathrm{Ext}_R^1(R/\mathfrak{a},\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M,X)}(M,X)/Y)$$

is finite for every $\mathrm{FD}_{<1}$ R*-submodule* Y *of* $\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M,X)}(M,X)$*. In particular, the* R*-module* $\mathrm{Ext}_R^1(R/\mathfrak{a},\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(M,X)}(M,X))$ *is finite.*

Proof. This follows from Corollary 2.6 and Theorem 3.2. □

COROLLARY 3.4. *Let* M *and* X *be finite* R*-modules such that*

$$\dim_R((M\otimes_R X)/\mathfrak{a}(M\otimes_R X))\leq 1$$

(e.g., $\dim(R/\mathfrak{a})\leq 1$*). Then the following statements hold true:*

(i) Y_i *and* $\mathrm{H}_{\mathfrak{a}}^i(M,X)/Y_i$ *are* $\mathfrak{a}$*-cofinite* R*-modules for all* i *and every* $\mathrm{FD}_{<1}$ R*-submodule* Y_i *of* $\mathrm{H}_{\mathfrak{a}}^i(M,X)$*. In particular,* $\mathrm{H}_{\mathfrak{a}}^i(M,X)$ *is an* $\mathfrak{a}$*-cofinite* R*-module for all* i*;*

(ii) *Let* N *be a finite* R*-module. Then* $\mathrm{Ext}_R^j(N,Y_i)$*,* $\mathrm{Tor}_j^R(N,Y_i)$*,* $\mathrm{Ext}_R^j(N,\mathrm{H}_{\mathfrak{a}}^i(M,X)/Y_i)$*, and* $\mathrm{Tor}_j^R(N,\mathrm{H}_{\mathfrak{a}}^i(M,X)/Y_i)$ *are* $\mathfrak{a}$*-cofinite* R*-modules for all* i*, all* j*, and every* $\mathrm{FD}_{<1}$ R*-submodule* Y_i *of* $\mathrm{H}_{\mathfrak{a}}^i(M,X)$*. In particular,* $\mathrm{Ext}_R^j(N,\mathrm{H}_{\mathfrak{a}}^i(M,X))$ *and* $\mathrm{Tor}_j^R(N,\mathrm{H}_{\mathfrak{a}}^i(M,X))$ *are* $\mathfrak{a}$*-cofinite* R*-modules for all* i *and all* j*;*

(iii) $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(M,X)/Y_i)$ *is a finite set for all* i *and every* $\mathrm{FD}_{<1}$ R*-submodule* Y_i *of* $\mathrm{H}_{\mathfrak{a}}^i(M,X)$*. In particular,* $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(M,X))$ *is a finite set for all* i*.*

Proof. It follows from Theorem 3.2. □

For the ordinary local cohomology modules, we have the following results.

COROLLARY 3.5. *Let X be an arbitrary R-module and let t be a non-negative integer such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq t$ and $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<2}$ R-module for all $i < t$. Then the following statements hold true:*

(i) *Y_i and $\mathrm{H}^i_{\mathfrak{a}}(X)/Y_i$ are $\mathfrak{a}$-cofinite R-modules for all $i < t$ and every $\mathrm{FD}_{<1}$ R-submodule Y_i of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-cofinite R-module for all $i < t$;*

(ii) *Let N be a finite R-module. Then $\mathrm{Ext}^j_R(N, Y_i)$, $\mathrm{Tor}^R_j(N, Y_i)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$, and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$ are $\mathfrak{a}$-cofinite R-modules for all $i < t$, all j, and every $\mathrm{FD}_{<1}$ R-submodule Y_i of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ are $\mathfrak{a}$-cofinite R-modules for all $i < t$ and all j;*

(iii) *$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ is a finite R-module for every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^t_{\mathfrak{a}}(X)$. In particular, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X))$ is a finite R-module;*

(iv) *$\mathrm{Ass}_R(\mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ is a finite set for every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^t_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ass}_R(\mathrm{H}^t_{\mathfrak{a}}(X))$ is a finite set;*

(v) *Assume that $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, X)$ is finite. Then $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ is a finite R-module for every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^t_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X))$ is a finite R-module.*

COROLLARY 3.6. *Let X be an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq f^2_{\mathfrak{a}}(X)$. Then the following statements hold true:*

(i) *Y_i and $\mathrm{H}^i_{\mathfrak{a}}(X)/Y_i$ are $\mathfrak{a}$-cofinite R-modules for all $i < f^2_{\mathfrak{a}}(X)$ and every $\mathrm{FD}_{<1}$ R-submodule Y_i of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-cofinite R-module for all $i < f^2_{\mathfrak{a}}(X)$;*

(ii) *Let N be a finite R-module. Then $\mathrm{Ext}^j_R(N, Y_i)$, $\mathrm{Tor}^R_j(N, Y_i)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$, and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$ are $\mathfrak{a}$-cofinite R-modules for all $i < f^2_{\mathfrak{a}}(X)$, all j, and every $\mathrm{FD}_{<1}$ R-submodule Y_i of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ are $\mathfrak{a}$-cofinite R-modules for all $i < f^2_{\mathfrak{a}}(X)$ and all j;*

(iii) *$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)/Y)$ is finite for every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)$. In particular, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X))$ is a finite R-module;*

(iv) *$\mathrm{Ass}_R(\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)/Y)$ is a finite set for every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ass}_R(\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X))$ is a finite set;*

(v) *Assume that* $\mathrm{Ext}_R^{f_{\mathfrak{a}}^2(X)+1}(R/\mathfrak{a}, X)$ *is finite. Then*

$$\mathrm{Ext}_R^1(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(X)}(X)/Y)$$

is finite for every $\mathrm{FD}_{<1}$ *R-submodule* Y *of* $\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(X)}(X)$. *In particular, the R-module* $\mathrm{Ext}_R^1(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(X)}(X))$ *is finite.*

Corollary 3.7. *Let* X *be a finite R-module such that* $\dim_R(X/\mathfrak{a}X) \leq 1$. *Then the following statements hold true:*

(i) Y_i *and* $\mathrm{H}_{\mathfrak{a}}^i(X)/Y_i$ *are* $\mathfrak{a}$*-cofinite R-modules for all* i *and every* $\mathrm{FD}_{<1}$ *R-submodule* Y_i *of* $\mathrm{H}_{\mathfrak{a}}^i(X)$. *In particular,* $\mathrm{H}_{\mathfrak{a}}^i(X)$ *is an* $\mathfrak{a}$*-cofinite R-module for all* i*;*

(ii) *Let* N *be a finite R-module. Then* $\mathrm{Ext}_R^j(N, Y_i)$, $\mathrm{Tor}_j^R(N, Y_i)$, $\mathrm{Ext}_R^j(N, \mathrm{H}_{\mathfrak{a}}^i(X)/Y_i)$, *and* $\mathrm{Tor}_j^R(N, \mathrm{H}_{\mathfrak{a}}^i(X)/Y_i)$ *are* $\mathfrak{a}$*-cofinite R-modules for all* i, *all* j, *and every* $\mathrm{FD}_{<1}$ *R-submodule* Y_i *of* $\mathrm{H}_{\mathfrak{a}}^i(X)$. *In particular,* $\mathrm{Ext}_R^j(N, \mathrm{H}_{\mathfrak{a}}^i(X))$ *and* $\mathrm{Tor}_j^R(N, \mathrm{H}_{\mathfrak{a}}^i(X))$ *are* $\mathfrak{a}$*-cofinite R-modules for all* i *and all* j*;*

(iii) $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X)/Y_i)$ *is a finite set for all* i *and every* $\mathrm{FD}_{<1}$ *R-submodule* Y_i *of* $\mathrm{H}_{\mathfrak{a}}^i(X)$. *In particular,* $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X))$ *is a finite set for all* i.

4. THE FIRST AND THE SECOND FINITENESS DIMENSIONS

In the first main result of this section, we generalize and improve [7, Proposition 2.2 and Corollary 2.4].

Theorem 4.1. *Let* M *be a finite R-module,* X *an arbitrary R-module, and* t *a non-negative integer such that* $\mathrm{Ext}_R^i(M/\mathfrak{a}M, X)$ *is a finite R-module for all* $i \leq t+1$. *Then the following statements are equivalent:*

(i) $\mathrm{H}_{\mathfrak{a}}^i(M, X)$ *is a minimax R-module for all* $i \leq t$*;*

(ii) $\mathrm{H}_{\mathfrak{a}}^i(M, X)$ *is an* $\mathrm{FD}_{<1}$ *R-module for all* $i \leq t$*;*

(iii) $\mathrm{H}_{\mathfrak{a}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, X_{\mathfrak{p}})$ *is a finite* $R_{\mathfrak{p}}$*-module for all* $\mathfrak{p} \in \mathrm{Spec}(R)$ *with* $\dim_R(R/\mathfrak{p}) \geq 1$ *and for all* $i \leq t$.

Proof. (i)⇒(ii). This is clear.

(ii)⇒(i). Let $i \leq t$. Since $\mathrm{H}_{\mathfrak{a}}^i(M, X)$ is an $\mathrm{FD}_{<1}$ R-module, there exists a finite submodule N_i of $\mathrm{H}_{\mathfrak{a}}^i(M, X)$ such that $\dim_R(\mathrm{H}_{\mathfrak{a}}^i(M, X)/N_i) < 1$. On the other hand, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^i(M, X)/N_i)$ is finite by Theorem 3.2. Now, since $\dim_R(\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^i(M, X)/N_i)) < 1$, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^i(M, X)/N_i)$ is Artinian.

Thus $\mathrm{H}^i_{\mathfrak{a}}(M,X)/N_i$ is Artinian by [10, Theorem 7.1.2], and so $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is minimax.

(ii)⇔(iii). Follows from Theorem 2.4. □

Corollary 4.2. *Let M be a finite R-module and let X be an arbitrary R-module such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M,X)$ is a finite R-module for all i (in fact, for all $i \le f^1_{\mathfrak{a}}(M,X)+1$). Then*

$$\begin{aligned} f^1_{\mathfrak{a}}(M,X) &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not a minimax } R\text{-module}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not an } \mathrm{FD}_{<1} \ R\text{-module}\}. \end{aligned}$$

Proof. This follows from Theorem 4.1. □

The following theorem is the second main result of this section which generalizes and improves [7, Proposition 3.7].

Theorem 4.3. *Let R be a semi-local ring, M a finite R-module, X an arbitrary R-module, and t a non-negative integer such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M,X)$ is a finite R-module for all $i \le t+1$. Then the following statements are equivalent:*

(i) *$\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is a weakly Laskerian R-module for all $i \le t$;*

(ii) *$\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is an $\mathrm{FD}_{<2}$ R-module for all $i \le t$;*

(iii) *$\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}},X_{\mathfrak{p}})$ is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \ge 2$ and for all $i \le t$.*

Proof. (i)⇒(ii). This is clear from [4, Theorem 3.3].

(ii)⇒(i). Let $i \le t$. Since $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is an $\mathrm{FD}_{<2}$ R-module, there is a finite submodule N_i of $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ such that $\dim_R(\mathrm{H}^i_{\mathfrak{a}}(M,X)/N_i) < 2$. On the other hand, by Theorem 3.2, the set $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M,X)/N_i)$ is finite. Thus $\mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M,X)/N_i)$ is a finite set. Hence, by [4, Theorem 3.3], $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is weakly Laskerian.

(ii)⇔(iii). It follows from Theorem 2.4. □

Corollary 4.4. *Let R be a semi-local ring, M a finite R-module, and X an arbitrary R-module such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M,X)$ is a finite R-module for all i (in fact, for all $i \le f^2_{\mathfrak{a}}(M,X)+1$). Then*

$$\begin{aligned} f^2_{\mathfrak{a}}(M,X) &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not a weakly Laskerian } R\text{-module}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not an } \mathrm{FD}_{<2} \ R\text{-module}\}. \end{aligned}$$

Proof. Follows from Theorem 4.3. □

We have the following results for the ordinary local cohomology modules.

Corollary 4.5. *Let X be an arbitrary R-module and let t be a non-negative integer such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq t+1$. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is a minimax R-module for all $i \leq t$;*

(ii) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an $\mathrm{FD}_{<1}$ R-module for all $i \leq t$;*

(iii) $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ *is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \geq 1$ and for all $i \leq t$.*

Corollary 4.6. *Let X be an arbitrary R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i (in fact, for all $i \leq f^1_{\mathfrak{a}}(X)+1$). Then*

$$\begin{aligned} f^1_{\mathfrak{a}}(X) &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X) \text{ is not a minimax } R\text{-module}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X) \text{ is not an } \mathrm{FD}_{<1} \ R\text{-module}\}. \end{aligned}$$

Corollary 4.7. *Let R be a semi-local ring, X an arbitrary R-module, and t a non-negative integer such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \leq t+1$. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is a weakly Laskerian R-module for all $i \leq t$;*

(ii) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an $\mathrm{FD}_{<2}$ R-module for all $i \leq t$;*

(iii) $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ *is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \geq 2$ and for all $i \leq t$.*

Corollary 4.8. *Let R be a semi-local ring and let X be an arbitrary R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i (in fact, for all $i \leq f^2_{\mathfrak{a}}(X)+1$). Then*

$$\begin{aligned} f^2_{\mathfrak{a}}(X) &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X) \text{ is not a weakly Laskerian } R\text{-module}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X) \text{ is not an } \mathrm{FD}_{<2} \ R\text{-module}\}. \end{aligned}$$

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