SHORTER PROOF OF THE RIESZ INTERPOLATION FORMULA FOR TRIGONOMETRIC POLYNOMIALS

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Communicated by Lucian Beznea

By means of partial fraction decompositions, a shorter proof is presented for the important interpolation formula of trigonometric polynomials discovered by Riesz (1914).

AMS 2020 Subject Classification: Primary 42A15; Secondary 65T40.

Key words: Riesz' interpolation formula, trigonometric polynomial, partial fraction decomposition.

1. INTRODUCTION AND OUTLINE

By a trigonometric polynomial of degree n, we mean a function of the form

$$F(x)=a_0+\sum_{k=1}^{n}\big(a_k\cos kx+b_k\sin kx\big),$$

where the coefficients a_k and b_k may be taken as complex numbers. One can think of such a function as a linear combination of several harmonics (described by sinusoidal functions $\sin kx$ and $\cos kx$) for a musical instrument. By making use of Euler's formula, the polynomial can be alternatively rewritten as

$$F(x)=\sum_{k=-n}^{n}c_ke^{kix},$$

where $a_0=c_0$ and the other coefficients are related by

$$\left.\begin{aligned}c_k&=\tfrac{a_k}{2}+\tfrac{b_k}{2i}\\ c_{-k}&=\tfrac{a_k}{2}-\tfrac{b_k}{2i}\end{aligned}\right\}\quad\text{and}\quad\left\{\begin{aligned}a_k&=c_k+c_{-k}\\ b_k&=i(c_k-c_{-k}).\end{aligned}\right.$$

Unlike the usual polynomial interpolations, Riesz [7, 1914] discovered the following interpolation formula for a trigonometric polynomial $F(x)$ of degree $\leq n$:

$$F'(x)=\frac{1}{4n}\sum_{k=1}^{2n}(-1)^{k-1}F(x+\alpha_k)\csc^2\frac{\alpha_k}{2}\quad\text{where}\quad\alpha_k=\frac{2k-1}{2n}\pi. \tag{1}$$

MATH. REPORTS **25(75)** (2023), *2*, 231–234
doi: 10.59277/mrar.2023.25.75.2.231

This formula has important applications to the inequalities of Bernstein [1] and Markov [5] (see also [4] and [8]). It has been reproduced, without proofs, in several standard texts on approximation and interpolation (see [6] and [9]–[11], just for example). Recently, Chu [3] found an elementary proof of this formula by decomposing trigonometric fractions into partial fractions. Here, we shall offer a more transparent one by applying the partial fraction approach (cf. [2]) directly to a rational function of e^{iy}. This will be done in the next section, where another significant result that appeared in Riesz' paper will also be recasted.

2. SHORTER PROOF OF RIESZ' FORMULA

For a natural number n, consider the trigonometric function $\cos ny = \frac{1+e^{2niy}}{2e^{niy}}$, that has $2n$ distinct zeros $\{\alpha_k = \frac{2k-1}{2n}\pi\}_{k=1}^{2n}$. Then $F(y)e^{niy}$ is a polynomial of degree $2n$ in e^{iy}. We have consequently the following partial fraction decomposition

$$\frac{F(y)}{\cos ny} = \frac{2F(y)e^{niy}}{1+e^{2niy}} = 2c_n + \sum_{k=1}^{2n} \frac{\lambda_k}{e^{iy}-e^{i\alpha_k}},$$

where the connection coefficients $\{\lambda_k\}_{k=1}^{2n}$ are determined by

$$\lambda_k = \lim_{y\to\alpha_k} F(y)\frac{e^{iy}-e^{i\alpha_k}}{\cos ny} = \frac{i}{n}(-1)^k F(\alpha_k)e^{i\alpha_k}.$$

Therefore, we have established the following formula

$$\frac{F(y)}{\cos ny} = 2c_n + \frac{i}{n}\sum_{k=1}^{2n}(-1)^k \frac{F(\alpha_k)e^{i\alpha_k}}{e^{iy}-e^{i\alpha_k}}. \tag{2}$$

Observing that

$$\frac{e^{i\alpha_k}}{e^{iy}-e^{i\alpha_k}} = \frac{e^{i(\alpha_k-y)/2}}{e^{i(y-\alpha_k)/2}-e^{i(\alpha_k-y)/2}} = \frac{1}{2i}\cot\frac{y-\alpha_k}{2} - \frac{1}{2}$$

we can reformulate (2) as follows:

$$\frac{F(y)}{\cos ny} = 2c_n + \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k F(\alpha_k)\cot\frac{y-\alpha_k}{2} - \frac{i}{2n}\sum_{k=1}^{2n}(-1)^k F(\alpha_k). \tag{3}$$

By means of the almost trivial sum below

$$\sum_{k=1}^{2n}(-1)^k e^{mi\alpha_k} = \begin{cases} 0, & -n < m < n; \\ 2ni, & m = -n; \\ -2ni, & m = n; \end{cases}$$

we can compute the following sum

$$\sum_{k=1}^{2n}(-1)^k F(\alpha_k) = 2nic_{-n} - 2nic_n.$$

Hence, the equality (3) can further be simplified into

$$\frac{F(y)}{\cos ny} = c_n + c_{-n} + \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k F(\alpha_k)\cot\frac{y-\alpha_k}{2} \tag{4}$$

which is equivalent to the equation (2) discovered by Riesz [7].

For this equation, its derivative with respect to y at $y = 0$ gives the equality

$$F'(0) = \frac{1}{4n}\sum_{k=1}^{2n}(-1)^{k-1} F(\alpha_k)\csc^2\frac{\alpha_k}{2}.$$

In place of $F(y)$, if we start with $F(x + y)$ (considered as a trigonometric polynomial in y) by adding an extra parameter x, we would confirm the interpolation formula for trigonometric polynomials anticipated in equation (1). Compared with the two known proofs by Riesz [7] and Chu [3], we believe that the present one is more accessible to the reader.

Instead, when $\mathcal{F}(x)$ is a trigonometric polynomial of degree $n - 1$, we would get analogously, from (4), the following equality without remainder term

$$\frac{\mathcal{F}(x+y)}{\cos ny} = \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k \mathcal{F}(x+\alpha_k)\cot\frac{y-\alpha_k}{2}.$$

Letting $y = 0$ in this equation leads us to another interpolation formula for trigonometric polynomials $\mathcal{F}(x)$ of degree $< n$

$$\mathcal{F}(x) = \frac{1}{2n}\sum_{k=1}^{2n}(-1)^{k-1} \mathcal{F}(x+\alpha_k)\cot\frac{\alpha_k}{2}. \tag{5}$$

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Received October 10, 2019

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