SESQUILINEAR FORMS AND THE ORTHOGONALITY PRESERVING PROPERTY

S. M. S. NABAVI SALES

Communicated by Dan Timotin

We consider the orthogonality preserving property to study sesquilinear forms. We state some characterizations for Hermitian sesquilinear forms through this kind of approach. We, then, state a Wigner type equation and present some results in this regard. We investigate the superstability of the perturbation of such equation. Finally, we state some results in inner product C^*-modules concerning the issue.

AMS 2020 Subject Classification: 39B52,15A86.

Key words: sesquilinear forms, orthogonality preserving property, Wigner's equation.

1. INTRODUCTION

The notion of sesquilinear form may be regarded as a generalization of the notion of inner product in some sense. It plays an important role in the theory of linear operators on Hilbert spaces. There are several other objects in analysis that invoke intrinsically sesquilinear forms. For example, finite measures on σ-algebras, positive definite kernels on Banach spaces, positive linear functionals on (Banach)*-algebras and so on; see [12, 5] and references therein.

Let $\mathscr{X}$ be a vector space over field $\mathbb{C}$ and let $u : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ be a mapping which is linear in the first argument and conjugate linear (anti linear) in the second one. Then u is called a sesquilinear form on $\mathscr{X}$. An inner product defined on a vector space is an example of a sesquilinear form. However, there are other sesquilinear forms which are not inner products, necessarily. For example, if T is a bounded linear operator on a Hilbert space $(\mathscr{X}, \langle\cdot,\cdot\rangle)$, then u, defined by

$$u(x, y) = \langle Tx, y\rangle, \qquad (x, y \in \mathscr{X}), \tag{1}$$

is a sesquilinear form. Conversely, a sesquilinear form u, defined on a Hilbert space, determines uniquely a bounded linear operator T satisfying (1) by requiring u to be bounded which means that there exists a positive number M so

doi: 10.59277/mrar.2023.25.75.2.235

that $|u(x,y)| \leq M\|x\|\|y\|$ for $x, y \in \mathscr{X}$. For the cases when u is not necessarily bounded, finding such an operator, defined on the target space, which represents the sesquilinear form is an interesting problem; see [5]. The sesquilinear form u is an inner product if and only if T is a positive invertible operator. A sesquilinear form u is said to be positive semidefinite if $u(x,x) \geq 0$ for any $x \in \mathscr{X}$. Indeed, u defines an inner product if and only if $u(x,x) > 0$ for all $0 \neq x \in \mathscr{X}$. This is a special case of a greater class of sesquilinear forms, i.e. the class of Hermitian sesquilinear forms. A sesquilinear form u is called self-adjoint or Hermitian if $u(x,y) = \overline{u(y,x)}$ for all x, y. The polarisation identity which holds true for sesquilinear forms implies that a sesquilinear $u(\cdot,\cdot)$ is Hermitian if and only if $u(x,x) \in \mathbb{R}$ for any $x \in \mathscr{X}$.

Our main interest, in this paper, is to respond to the following problem:

P: let u and v be two sesquilinear forms and suppose that $u(x,y) = 0$ implies that $v(x,y) = 0$. Does $u = \lambda v$ for some $\lambda \in \mathbb{C}$?

This problem has been considered for the particular case when u and v define inner products; see for example [3, 4, 6, 9, 10, 13], and references therein. In this article, we provide an answer to this problem, on a more general level; see Theorem 2.3 and Theorem 2.1 below. However, the problem remains open in its most general sense. We also pay attention to Wigner's theorem related to analysis of symmetry properties of quantum systems. We express a Wigner type equation and present some results in this regard. We investigate the superstability of the perturbation of such equation, as well.

Finally, we restate some of our results in the framework of inner product C^*-modules. Let $\mathcal{A}$ be a C^*-algebra. A complex linear space $\mathcal{V}$ which is a right $\mathcal{A}$-module with a compatible scalar multiplication is an inner product $\mathcal{A}$-module if it is equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathcal{V} \times \mathcal{V} \to \mathcal{A}$ satisfying the following properties:

i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$,
ii) $\langle x, ya\rangle = \langle x, y\rangle a$,
iii) $\langle x, y\rangle^* = \langle y, x\rangle$,
iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,

for all $x, y, z \in \mathcal{V}, a \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. Note that $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ defines a norm on $\mathcal{V}$ due to Cauchy-Schwartz inequality. The interested reader is referred to [8] for further information of inner product C^*-modules.

2. RESULTS

The following theorem is a generalization of [3, Theorem 1] and [10, Theorem 2.1]. Modifying the proof of [3, Theorem 1] or [10, Theorem 2.1] one can express the proof of this theorem, thus it is unnecessary to state it.

THEOREM 2.1. *Let $(\mathscr{X}, \langle\cdot,\cdot\rangle)$ be a complex inner product space and let u be a sesquilinear form on $\mathscr{X}$. The following statements are equivalent;*

(i) *there exists a $\lambda \in \mathbb{C}$ so that $u(x,x) = \lambda\|x\|^2$, for all $x \in \mathscr{X}$,*

(ii) *there exists a $\lambda \in \mathbb{C}$ so that $u(x,y) = \lambda\langle x,y\rangle$, for any $x, y \in \mathscr{X}$,*

(iii) *if $\langle x,y\rangle = 0$, then $u(x,y) = 0$.*

COROLLARY 2.2. *Let T be a linear operator on a Hilbert space $\mathscr{X}$ and let $\alpha \in \mathbb{C}$. The following statements are equivalent*

(i) *there exists a $\lambda \in \mathbb{C}$ so that $\alpha\|T^2x\|^2 + \|Tx\|^2 = \lambda\|x\|^2$, for all $x \in \mathscr{X}$,*

(ii) *there exists a $\lambda \in \mathbb{C}$ so that $\alpha\langle T^2x, T^2y\rangle + \langle Tx, Ty\rangle = \lambda\langle x,y\rangle$, for any $x, y \in \mathscr{X}$,*

(iii) *$\langle x,y\rangle = 0$ implies that $\alpha\langle T^2x, T^2y\rangle + \langle Tx, Ty\rangle = 0$, for any $x, y \in \mathscr{X}$.*

Note that if we put $\alpha = -\frac{1}{2}$ and $\lambda = 1$ in this theorem, then T is a 2-isometry. One may express the same results for more general n-isometries [1] instead of 2-isometry. The next theorem, which is the main result in this part, states another generalization of [3, Theorem 1] and [10, Theorem 2.1].

THEOREM 2.3. *Let $u(\cdot,\cdot)$ be a sesquilinear form on a linear space $\mathscr{X}$ such that $u(x,y) = 0$ whenever $u(y,x) = 0$, for any $x, y \in \mathscr{X}$. Then there exists a $\lambda \in \mathbb{C}$ so that $|\lambda| = 1$ and λu is self-adjoint.*

Proof. Let x_0 be chosen in $\mathscr{X}$ such that $u(x_0,x_0) = r_0 e^{i\theta_0} \neq 0$ where $\theta_0 \in [0, 2\pi)$ and $r_0 > 0$, and let $y \in \mathscr{X}$. We want to show that

$$u(y,y) = \pm e^{i\theta_0}|u(y,y)|.$$

Assume that $u(y,y) = se^{i\eta}$ for some $s > 0$ and $\eta \in [0, 2\pi)$. The proof divides into two cases.

Case i. $u(x_0,y) = 0$. Thus, $u(y,x_0) = 0$. Let

$$v_1 = \frac{x_0}{\sqrt{r_0}e^{i\theta_0}}, \quad v_2 = \frac{x_0}{\sqrt{r_0}}, \quad w_1 = \frac{y}{\sqrt{s}e^{i\eta}}, \quad \text{and} \quad w_2 = \frac{y}{\sqrt{s}}.$$

Therefore $u(v_i, w_j) = u(w_j, v_i) = 0$ for $i, j \in \{1, 2\}$. This implies that $u(v_1 + w_1, v_2 - w_2) = 0$. Hence $u(v_2 - w_2, v_1 + w_1) = 0$ by our assumption which is $u(v_2, v_1) = u(w_2, w_1)$. Hence

$$\frac{u(x_0,x_0)}{r_0 e^{-i\theta_0}} = \frac{u(y,y)}{se^{-i\eta}}$$

that ensures $e^{2i\theta_0} = e^{2i\eta}$. Thus $\theta_0 = \eta, \eta + \pi$ which gives
$$u(y,y) = \pm e^{i\theta_0}|u(y,y)|.$$

Case ii. $u(x_0, y) \neq 0$. Thus $u(y, x_0) \neq 0$. Let $z = x_0 - \frac{y}{u(y,x_0)}u(x_0, x_0)$. Therefore, $u(z, x_0) = 0$ which implies that
$$0 = u(x_0, z) = u(x_0, x_0 - \frac{y}{u(y, x_0)}u(x_0, x_0)) = u(x_0, x_0) - \frac{\overline{u(x_0, x_0)}}{\overline{u(y, x_0)}}u(x_0, y).$$
It follows that
$$\frac{u(x_0, y)}{\overline{u(y, x_0)}} = \frac{u(x_0, x_0)}{\overline{u(x_0, x_0)}}. \tag{2}$$
The same argument shows that
$$\frac{u(y, x_0)}{\overline{u(x_0, y)}} = \frac{u(y, y)}{\overline{u(y, y)}}. \tag{3}$$
Comparing (2) and (3) we come to
$$\frac{u(x_0, x_0)}{\overline{u(x_0, x_0)}} = \frac{u(y, y)}{\overline{u(y, y)}}$$
which implies that $\theta_0 = \eta, \eta + \pi$ and consequently, in this case, we have
$$u(y,y) = \pm e^{i\theta_0}|u(y,y)|.$$
If $u(y,y) = 0$ then, obviously, $u(y,y) = e^{i\theta_0}|u(y,y)|$ holds true. Now define a sesquilinear form v to be $v := e^{-i\theta_0}u$ and note that $v(y,y) = \pm|u(y,y)| \in \mathbb{R}$ for all $y \in \mathscr{X}$. Therefore, v is self-adjoint on $\mathscr{X}$ and we are done. □

Remark 2.4.

1. If there exists an x_0 in $\mathscr{X}$ so that $0 \neq u(x_0, x_0) \in \mathbb{R}$ then u is, itself, Hermitian.

2. If u is a bounded sequilinear form on a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$, then we know that there exists a unique bounded linear operator T on $\mathscr{H}$ so that $u(x,y) = \langle Tx, y\rangle$ for all $x, y \in \mathscr{H}$. In fact, this theorem states that if $\langle Tx, y\rangle = 0$ implies $\langle Ty, x\rangle = 0$, then T is a self-adjoint operator multiplied by a modulus one constant.

3. In the proof of Theorem 2.3, we can conclude that $u(x,y) = e^{2i\theta_0}\overline{u(y,x)}$ for any $x, y \in \mathscr{X}$.

4. If for some $\lambda \in \mathbb{C}$ we have that $u(x,y) = \lambda\overline{u(y,x)}$, for any $x, y \in \mathscr{X}$, then $|\lambda| = 1$. In fact, in this case
$$u(x,y) = \lambda\overline{u(y,x)} = |\lambda|^2 u(x,y),$$
which implies that $|\lambda| = 1$.

2.1. **Wigner's Theorem**

We are now going to deal with the well-known Wigner's theorem. This theorem arises, originally, in quantum theory. It, indeed, provides a significant tool for presenting physical symmetries such as rotations and translations in the language of mathematics. It appeared firstly in 1931 ([15] p. 251), and then it engrossed a lot of attention; see [11, 4, 14] and references therein. Let $f : \mathscr{X} \to \mathscr{Y}$ be a mapping between inner product spaces. We say that f satisfies Wigner's equation if

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle|, \qquad (x, y \in \mathscr{X}).$$

Wigner [15] shows that if $f : \mathscr{X} \to \mathscr{Y}$ is a mapping between inner product spaces, satisfying this equation, then it is phase-equivalent to a linear or conjugate-linear isometry which means that there exists an isometry operator g which is either linear or anti-linear and there is a scalar-valued function σ with $|\sigma(x)| = 1$, for all $x \in \mathscr{X}$ so that $f = \sigma g$.

One could regard the following equation as a generalization of Wigner's equation,

$$|\langle f(x), y\rangle| = |\langle f(y), x\rangle|, \qquad (x, y \in \mathscr{X}). \tag{4}$$

Note that if $f : \mathscr{X} \to \mathscr{X}$ is a mapping on an inner product space satisfying $\langle f(x), y\rangle = \overline{\langle f(y), x\rangle}$, for all $x, y \in \mathscr{X}$ then f is linear. In fact, if x, y and z are in $\mathscr{X}$ and $\alpha \in \mathbb{C}$, then

$$\langle f(\alpha x + y), z\rangle = \overline{\langle f(z), \alpha x + y\rangle} = \alpha\overline{\langle f(z), x\rangle} + \overline{\langle f(z), y\rangle} = \langle \alpha f(x) + f(y), z\rangle.$$

Thus f is a self-adjoint operator whenever $\mathscr{X}$ is a Hilbert space and f is continuous. It is easy to see that f is injective if it is surjective.

Let f be a mapping on an inner product space $\mathscr{X}$ satisfying

$$\mathrm{Re}(\langle f(x), y\rangle) = \mathrm{Re}(\langle f(y), x\rangle), \qquad (x, y \in \mathscr{X}).$$

If $f(\mathrm{i}x) = \mathrm{i}f(x)$, then f is linear and $\langle f(x), y\rangle = \overline{\langle f(y), x\rangle}$ for all $x, y \in \mathscr{X}$. In fact, in this case, we have that

$$\mathrm{Im}(\langle f(x), y\rangle) = -\mathrm{Im}(\langle f(y), x\rangle), \qquad (x, y \in \mathscr{X}), \tag{5}$$

because

$$\begin{aligned}\mathrm{Im}(\langle f(x), y\rangle) &= \mathrm{Re}(-\mathrm{i}\langle f(x), y\rangle) = \mathrm{Re}(\langle f(x), \mathrm{i}y\rangle) \\ &= \mathrm{Re}(\langle f(\mathrm{i}y), x\rangle) = -\mathrm{Im}(\langle f(y), x\rangle).\end{aligned}$$

Similarly, we can see that if $f(\mathrm{i}x) = -\mathrm{i}f(x)$, for all $x \in \mathscr{X}$, then $\langle f(x), y\rangle = \langle f(y), x\rangle$, for all $x, y \in \mathscr{X}$ and, in this case, f is anti-linear. We say that the mapping f is self-orthogonal at x if $\langle f(x), x\rangle = 0$ but $f(x) \neq 0$.

Lemma 2.5 ([7]). *Let $\mathscr{X}$ be an inner product space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. If $x, y \in \mathscr{X}$ satisfy*
$$|\langle x, v\rangle| = |\langle y, v\rangle|, \qquad (v \in \mathscr{X}),$$
then there is a modulus one $\lambda \in \mathbb{F}$ such that $y = \lambda x$.

Theorem 2.6. *Let $f : \mathscr{X} \to \mathscr{X}$ be a mapping on a complex inner product space $\mathscr{X}$ satisfying (4) and (5) and let f not be self-orthogonal at any point x of $\mathscr{X}$. Then*
$$\langle f(x), y\rangle = \overline{\langle f(y), x\rangle}, \qquad (x, y \in \mathscr{X}).$$

Proof. First, we note that

$$|\langle f(\lambda x), y\rangle| = |\langle f(y), \lambda x\rangle| = |\lambda||\langle f(y), x\rangle| = |\lambda||\langle f(x), y\rangle| = |\langle \lambda f(x), y\rangle|. \tag{6}$$

Therefore, $f(\lambda x) = \lambda' f(x)$ for some λ' with $|\lambda'| = |\lambda|$ by Lemma 2.5. Condition (5) implies that

$$\mathrm{Im}(\langle f(x), x\rangle) = 0, \qquad (x \in \mathscr{X}).$$

Now let $x \in \mathscr{X}$ and $f(\mathrm{i}x) = \lambda(x) f(x)$ in which $|\lambda(x)| = 1$. From

$$\mathrm{Re}(\lambda(x)\langle f(x), x\rangle) = \mathrm{Re}(\langle f(\mathrm{i}x), x\rangle) = \mathrm{Im}(\mathrm{i}\langle f(\mathrm{i}x), x\rangle) = -\mathrm{Im}(\langle f(\mathrm{i}x), \mathrm{i}x\rangle) = 0$$

and

$$\begin{aligned} \mathrm{Im}(\lambda(x)\langle f(x), x\rangle) &= \mathrm{Im}(\langle f(\mathrm{i}x), x\rangle) = -\mathrm{Im}(\langle f(x), \mathrm{i}x\rangle) \\ = \mathrm{Im}(\mathrm{i}\langle f(x), x\rangle) &= \mathrm{Re}(\langle f(x), x\rangle) \end{aligned}$$

we have that $\lambda(x) = \mathrm{i}$ that means $f(\mathrm{i}x) = \mathrm{i}f(x)$ for any $x \in \mathscr{X}$. Now for $x, y \in \mathscr{X}$ we have that

$$\begin{aligned} \mathrm{Re}(\langle f(x), y\rangle) &= \mathrm{Im}(\mathrm{i}\langle f(x), y\rangle) = -\mathrm{Im}(\langle f(x), \mathrm{i}y\rangle) \\ = \mathrm{Im}(\langle f(\mathrm{i}y), x\rangle) &= \mathrm{Im}(\mathrm{i}\langle f(y), x\rangle) = \mathrm{Re}(\langle f(y), x\rangle) \end{aligned}$$

and we are done. $\square$

In light of Wigner's theorem, our conjecture here is that f is phase-equivalent to a Hermitian linear operator g whenever f is continuous and satisfies (4). In the following, we make some observations in this regard which may be useful. Let $f : \mathscr{X} \to \mathscr{X}$ be a mapping on a complex inner product space $\mathscr{X}$ satisfying (4). Then

(O1) for $\lambda \in \mathbb{C}$ and $x \in \mathscr{X}$ we have that $f(\lambda x) = \lambda' f(x)$ for some λ' with $|\lambda'| = |\lambda|$. This is concluded immediately due to Lemma 2.5 and (6).

(O2) for $x, y \in \mathscr{X}$ there exist λ and λ' with $|\lambda| = |\lambda'| = 1$ such that

$$f(x+y) = \lambda f(x) + \lambda' f(y).$$

Indeed for $x, y, z \in \mathscr{X}$ there exist modulus one numbers $\eta, \alpha, \alpha', \beta$ and β' so that

$$\begin{aligned}
|\langle f(x+y), z\rangle| &= |\langle f(z), x+y\rangle| \\
&= |\langle f(z), x\rangle + \langle f(z), y\rangle| \\
&= \eta(\langle f(z), x\rangle + \langle f(z), y\rangle) \\
&= \eta\alpha|\langle f(z), x\rangle| + \eta\alpha'|\langle f(z), y\rangle| \\
&= \eta\alpha|\langle f(x), z\rangle| + \eta\alpha'|\langle f(y), z\rangle| \\
&= \eta\alpha\beta\langle f(x), z\rangle + \eta\alpha'\beta'\langle f(y), z\rangle \\
&= \langle \gamma f(x) + \gamma' f(y), z\rangle = |\langle \gamma f(x) + \gamma' f(y), z\rangle|
\end{aligned}$$

where $\gamma = \eta\alpha\beta$ and $\gamma' = \eta\alpha'\beta'$. Now, the result is concluded from Lemma 2.5.

(O3) there exists a $T : \mathscr{X} \to \mathscr{X}$ which satisfies (4), is phase-equivalent to f, and $T(\lambda x) = \lambda T(x)$ for all $\lambda \in \mathbb{C}$ and $x \in \mathscr{X}$. To see this, let $x \in \mathscr{X}$. There exists a number $\gamma(x)$ with $|\gamma(x)| = 1$ so that

$$\langle \gamma(x) f(x), x\rangle = |\langle f(x), x\rangle|, \qquad (x \in \mathscr{X}).$$

Define T to be $T(x) = \gamma(x) f(x)$ for all $x \in \mathscr{X}$. Thus $\langle T(x), x\rangle \geq 0$ for any x. Now, we want to show that $T(\lambda x) = \lambda T(x)$ for $\lambda \in \mathbb{C}$ and $x \in \mathscr{X}$. Let $\lambda = r\mathrm{e}^{\mathrm{i}\theta} \neq 0$ and $x \in \mathscr{X}$ be chosen to be fixed. Then $T(rx) = tT(x)$ for some number t with $|t| = r$ by (O1). But

$$0 \leq \langle T(rx), rx\rangle = tr\langle T(x), x\rangle.$$

Since $r\langle T(x), x\rangle \geq 0$ we have that $t \geq 0$ so $t = r$. Now assume that $T(\mathrm{e}^{\mathrm{i}\theta} x) = \mathrm{e}^{\mathrm{i}\eta} T(x)$ for some η. Therefore,

$$0 \leq \langle T(\mathrm{e}^{\mathrm{i}\theta} x), \mathrm{e}^{\mathrm{i}\theta} x\rangle = \mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\eta}\langle T(x), x\rangle.$$

Thus $\mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\eta} \geq 0$ because $\langle T(x), x\rangle \geq 0$. Hence, $\eta = \theta$ and we are done.

These observations seem to be useful concerning the conjecture stated above. Because if we could show that T is additive, then according to Theorem 2.3 we get the desired result. But T is close to being additive due to (O2).

2.2. **Superstability**

In this part, we pay attention to a phenomenon concerning the perturbation of the equation (4) called superstability.

Theorem 2.7. *Let $\mathscr{X}$ be a Hilbert space and consider a control function $\phi : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ satisfying*

$$\lim_{n\to\infty} c^n \phi(c^{-n}x, y) = 0, \qquad (x, y \in \mathscr{X}) \tag{7}$$

for some positive constant $c \neq 1$. Now suppose that a mapping $f : \mathscr{X} \to \mathscr{X}$ satisfies

$$||\langle f(x), y\rangle| - |\langle f(y), x\rangle|| \leq \phi(x, y), \tag{8}$$

for any $x, y \in \mathscr{X}$. Then f satisfies (4).

Proof. Let $x \in \mathscr{X}$ be chosen to be fixed. For $n \in \mathbb{N}$ let f_n define to be $f_n(x) := c^n f(c^{-n}x)$. Inequality (8) implies that

$$||\langle f_n(x), y\rangle| - |\langle f(y), x\rangle|| \leq c^n \phi(c^{-n}x, y), \qquad (n \in \mathbb{N},\ y \in \mathscr{X}). \tag{9}$$

Therefore the sequence $\{|\langle f_n(x), y\rangle|\}_{n\in\mathbb{N}}$ converges for any $y \in \mathscr{X}$. Now, the well-known Banach–Steinhaus theorem implies that $\{f_n(x)\}_n$ is bounded. Hence, there exists a subsequence of which converging weakly to an element of $\mathscr{X}$, $\tilde{f}(x)$ say. Without loss of generality, we may assume that

$$\langle f_n(x), y\rangle \to \langle \tilde{f}(x), y\rangle, \qquad (y \in \mathscr{X}).$$

Thus, in this way, we come to a mapping $\tilde{f} : \mathscr{X} \to \mathscr{X}$. From (9) and the assumption (7) we have that

$$|\langle \tilde{f}(x), y\rangle| = |\langle f(y), x\rangle|, \qquad (x, y \in \mathscr{X}). \tag{10}$$

Let $x, y \in \mathscr{X}$. From (8) we have that

$$||\langle f_n(x), y\rangle| - |\langle f_m(y), x\rangle|| \leq c^{m+n} \phi(c^{-n}x, c^{-m}y), \qquad (m, n \in \mathbb{N}).$$

Therefore,

$$|\langle \tilde{f}(x), y\rangle| = |\langle f_m(y), x\rangle|, \qquad (m \in \mathbb{N}).$$

Letting $m \to \infty$ we observe that

$$|\langle \tilde{f}(x), y\rangle| = |\langle \tilde{f}(y), x\rangle|, \qquad (x, y \in \mathscr{X}). \tag{11}$$

Now (10) and (11) ensures that f satisfies (4), and we are done. □

2.3. Inner product C^*-moduls

At this point, we are going to restate Theorem 2.3 in the setting of inner product C^*-modules. We prove the main theorem using some ideas of [6, Theorem 3.1]. First some preliminaries from [6]; see also [2]. Throughout this section, we assume that $\mathbb{K}(\mathscr{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathscr{H})$ where $\mathbb{B}(\mathscr{H})$ and $\mathbb{K}(\mathscr{H})$ stand for the C^*-algebra of all bounded linear operators and the C^*-algebra of all compact operators on a Hilbert space $(\mathscr{H}, (.,.))$, respectively. Let $\xi, \eta \in \mathscr{H}$. By $\xi \otimes \eta$ we mean the rank one operator defined to be $(\xi \otimes \eta)\nu = (\nu, \eta)\xi$. Thus $\xi \otimes \xi$ is a projection onto the one dimensional subspace generated by ξ, whenever $\|\xi\| = 1$. Hence when $\mathbb{K}(\mathscr{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathscr{H})$, then $\mathcal{A}$ contains all of such minimal projections. Now let $e = \xi \otimes \xi$ with $\xi \in \mathscr{H}, \|\xi\| = 1$ be a minimal projection and let $\mathcal{V}$ be an inner product $\mathcal{A}$-module. Thus

$$\mathcal{V}_e = \{xe; x \in \mathcal{V}\}$$

is a complex inner product space contained in $\mathcal{V}$ in which inner product is defined to be

$$(x, y) = \operatorname{tr}(\langle x, y \rangle) \qquad (x, y \in \mathcal{V}_e).$$

Note that for $x = eu$ and $y = ev$, where $u, v \in \mathcal{V}$,

$$\langle x, y \rangle = e\langle u, v \rangle e = (\langle u, v \rangle \xi, \xi)\, \xi \otimes \xi$$

and

$$(x, y) = \operatorname{tr}(\langle x, y \rangle) = (\langle u, v \rangle \xi, \xi),$$

thus

$$\langle x, y \rangle = (x, y)e.$$

Hence the following assertions are obtained immediately; see [6],

(i) $x, y \in \mathcal{V}_e$ are orthogonal in $(\mathcal{V}_e, (\cdot, \cdot))$ if and only if they are orthogonal in $(\mathcal{V}, \langle \cdot, \cdot \rangle)$,

(ii) if $x \in \mathcal{V}_e$, then the norm of x induced from $(\mathcal{V}_e, (\cdot, \cdot))$ is the same as that induced by $(\mathcal{V}, \langle \cdot, \cdot \rangle)$,

THEOREM 2.8. *Let $\mathcal{A}$ be a C^*-algebra so that $\mathbb{K}(\mathscr{H}) \subset \mathcal{A} \subset \mathbb{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, and let T be a nonzero mapping on an inner product $\mathcal{A}$-module $\mathcal{W}$. Then the following statements are equivalent;*

(i) *T is $\mathcal{A}$-linear and $\langle Tx, y \rangle = 0$ implies that $\langle Ty, x \rangle = 0$ for any $x, y \in \mathcal{W}$,*

(ii) *there exists a complex number λ with $|\lambda| = 1$ so that $\langle Tx, y \rangle = \lambda \langle Ty, x \rangle^*$ for any $x, y \in \mathcal{W}$.*

Proof. let $e = \xi \otimes \xi$ be a minimal projection and consider linear operator $T_e := T|_{\mathcal{W}_e}$ defined on $\mathcal{W}_e$. T_e satisfies the following requirement;

$$(T_e(ex), ey) = 0 \implies (T_e(ey), ex) = 0, \qquad (x, y \in \mathcal{W}).$$

Therefore Theorem 2.3 ensures that there exists a modulus one number λ_e so that

$$(T(ex), ey) = \lambda_e (T(ey), ex)^*,$$

for any $x, y \in \mathcal{W}$. This yields

$$(T(ex), ey)e = \lambda_e (T(ey), ex)^* e,$$

and hence

$$\langle T(ex), ey\rangle = \lambda_e \langle T(ey), ex\rangle^*$$

which implies that

$$e\langle T(x), y\rangle e = \lambda_e e\langle T(y), x\rangle^* e. \tag{12}$$

Now let $f = \eta \otimes \eta$ be another minimal projection in $\mathcal{A}$ and let $u = \xi \otimes \eta$. Thus $e = ufu^*$ and there exists a modulus one number λ_f so that

$$f\langle T(x), y\rangle f = \lambda_f f\langle T(y), x\rangle^* f. \tag{13}$$

From (12) and the fact that $ufu^* = e$ we have that

$$ufu^*\langle T(x), y\rangle ufu^* = \lambda_e e\langle T(y), x\rangle^* e.$$

Whence,

$$uf\langle T(u^*x), u^*y\rangle fu^* = \lambda_e e\langle T(y), x\rangle^* e$$

which in conjunction with (13) brings us to

$$\lambda_f uf\langle T(u^*y), u^*x\rangle^* fu^* = \lambda_e e\langle T(y), x\rangle^* e.$$

It follows that

$$\lambda_f e\langle T(y), x\rangle^* e = \lambda_e e\langle T(y), x\rangle^* e.$$

Now assume that $e\langle T(y), x\rangle^* e \neq 0$. This implies that $\lambda_f = \lambda_e = \lambda$. Thus

$$e\langle T(x), y\rangle e = \lambda e\langle T(y), x\rangle^* e$$

for any minimal projection $e \in \mathcal{A}$ and any $x, y \in \mathcal{W}$. This implies that $\langle T(x), y\rangle = \lambda\langle T(y), x\rangle^*$ for any $x, y \in \mathcal{W}$ and we are done. □

Using this method, one can restate Corollary 2.2 in the setting of inner product C^*-modules, very routinely.

Acknowledgments. The author wishes to thank the referees for their comments which helped improve the paper.

REFERENCES

[1] J. Agler and M. Stankus, *m-isometric transformations of Hilbert space I.* Integral Equations Operator Theory **21** (1995), 383–429.

[2] D. Bakić and B. Guljaš, *Hilbert C^*-moduls over C^*-algebras of compact operators.* Acta Sci. Math. (Szeged) **68** (2002), 249–269.

[3] J. Chmieliński, *Linear mapping approximately preserving orthogonality.* J. Math. Anal. Appl. **304** (2005), 158–169.

[4] J. Chmieliński, *Orthogonality equation with two unknown functions.* Aequationes Math. **90** (2016), 11–23.

[5] R. Corso and C. Trapani, *Representation theorems for solvable sesquilinear forms.* Integral Equations Operator Theory **89** (2017), 43–68.

[6] D. Ilišević and A. Turnšek, *Approximately orthogonality preserving mappings on C^*-modules.* J. Math. Anal. Appl. **341** (2007), 298–308.

[7] D. Ilišević and A. Turnšek, *On superstability of the Wigner equation.* Linear Algebra Appl. **407** (2005), 189–195.

[8] E.C. Lance, *Hilbert C^*-Modules. A Toolkit for Operator Algebraists.* London Math. Soc. Lecture Note Ser. **210**, Cambridge Univ. Press, Cambridge, 1995.

[9] S.M.S. Nabavi Sales, *On mappings which approximately preserve angles.* Aequationes Math. **92** (2018), 1079–1090.

[10] J. Rätz, *Comparison of inner products.* Aequationes Math. **57** (1999), 312–321.

[11] J. Rätz, *On Wigner's theorem: remarks, complements, comments, and corollaries.* Aequationes Math. **52** (1996), 1–9.

[12] T. Titkos, *On means of nonnegative sesquilinear forms.* Acta Math. Hungar. **143** (2014), *2*, 515–533.

[13] A. Turnšek, *On mapping approximately preserving orthogonality.* J. Math. Anal. Appl. **336** (2007), 625–631.

[14] A. Turnšek, *A variant of Wigner's functional equation.* Aequationes Math. **89** (2015), 949–956.

[15] E.P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren.* Friedr. Vieweg und Sohn Akt.-Ges., 1931.

Received October 16, 2019

Hakim Sabzevari University
Department of Mathematics and Computer Sciences
P.O. Box 397, Sabzevar, Iran
sadegh.nabavi@gmail.com; sadegh.nabavi@hsu.ac.ir