UNITARY PARTS OF CONTRACTIVE LITTLE HANKEL OPERATORS ON THE BERGMAN SPACE

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In this paper, we consider little Hankel operators Γ_φ defined on the Bergman space $L^2_a(\mathbb{D})$ with symbol $\varphi \in H^\infty(\mathbb{D})$ that are contractions. Necessary and sufficient conditions are obtained for the existence of a nontrivial unitary part of these little Hankel operators. We also present an explicit description of this unitary part. This extends the results of Butz for Hankel operators defined on the Hardy space.

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1. INTRODUCTION

Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$ and $\mathrm{d}\theta$ be the arc-length measure on $\mathbb{T}$. For $1 \leq p < \infty$, $L^p(\mathbb{T})$ will denote the Lebesgue space of $\mathbb{T}$ induced by $\frac{\mathrm{d}\theta}{2\pi}$. Since $\mathrm{d}\theta$ is finite, $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$ for all $p \geq 1$. Given $f \in L^1(\mathbb{T})$, the Fourier coefficients of f are

$$l_n(f) = \hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\mathrm{d}\theta, \quad n \in \mathbb{Z},$$

where $\mathbb{Z}$ denotes the set of all integers. For $1 \leq p < \infty$, the Hardy space of $\mathbb{T}$, denoted by $\mathcal{H}^p(\mathbb{T})$, is the subspace of $L^p(\mathbb{T})$ consisting of functions f with $l_n(f) = 0$ for all negative integers n. Let $L^\infty(\mathbb{T})$ be the space of essentially bounded measurable functions on $\mathbb{T}$ with the essential supremum norm. Given $f \in L^1(\mathbb{T})$, the harmonic extension of f to $\mathbb{D}$ is denoted by $\hat{f}(z)$. The function $f \in H^p(\mathbb{D})$ if $f(z)$ is an analytic function on $\mathbb{D}$ with $\sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\mathrm{d}\theta\right\} < +\infty$. Fatou's theorem [7] implies that if $f \in H^p(\mathbb{D})$ then the limit $f(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ exists for almost every θ (with respect to $\mathrm{d}\theta$) and $f(e^{i\theta}) \in \mathcal{H}^p(\mathbb{T})$ and the harmonic extension of $f(e^{i\theta})$ to $\mathbb{D}$ is precisely $f(z)$. It is also not so important to distinguish [17] between $H^p(\mathbb{D})$ and $\mathcal{H}^p(\mathbb{T})$.

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Let $\varphi \in L^\infty(\mathbb{T})$ and $\left(\mathcal{H}^2(\mathbb{T})\right)^\perp = L^2(\mathbb{T}) \ominus \mathcal{H}^2(\mathbb{T})$, the orthogonal complement of $\mathcal{H}^2(\mathbb{T})$ in $L^2(\mathbb{T})$. The Hankel operator with symbol φ, denoted by H_φ, is the operator from $\mathcal{H}^2(\mathbb{T})$ into $\left(\mathcal{H}^2(\mathbb{T})\right)^\perp$ defined by $H_\varphi f = (I - P_+)(\varphi f)$. Here $(I - P_+)$ is the orthogonal projection from $L^2(\mathbb{T})$ onto $\left(\mathcal{H}^2(\mathbb{T})\right)^\perp$. Define other two operators S_φ and Γ_φ from $\mathcal{H}^2(\mathbb{T})$ into itself by $S_\varphi f = P_+J_+(\varphi f)$ and $\Gamma_\varphi f = P_+(\varphi J_+ f)$, where J_+ is a map from $L^2(\mathbb{T})$ into $L^2(\mathbb{T})$ defined as $J_+f(e^{it}) = f(e^{-it})$.

The matrix of H_φ with respect to the standard orthonormal basis of $\mathcal{H}^2(\mathbb{T})$ and $\left(\mathcal{H}^2(\mathbb{T})\right)^\perp$ is given by $\left\langle H_\varphi z^j, z^{-(i+1)}\right\rangle = \hat{\varphi}(-(i+j+1))$, the $(i+j+1)^{th}$ negative Fourier coefficient of φ, $i, j = 0, 1, 2, \cdots$. The matrices of H_φ, $S_{z\varphi}$ and $\Gamma_{J_+(z\varphi)}$ are same with respect to the standard orthonormal bases of $\mathcal{H}^2(\mathbb{T})$ and $\left(\mathcal{H}^2(\mathbb{T})\right)^\perp$. For the above reasons, all these three operators H_φ, S_φ and Γ_φ are referred to as Hankel operators [14] in the literature.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathrm{d}A(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$ be the Lebesgue area measure on $\mathbb{D}$, normalized so that the area measure of $\mathbb{D}$ is equal to 1. Let $L^2(\mathbb{D}, \mathrm{d}A)$ be the Hilbert space of complex-valued, absolutely square integrable, Lebesgue measurable functions f on $\mathbb{D}$ with the inner product

$$\langle f, g\rangle = \int f(z)\overline{g(z)}\mathrm{d}A(z).$$

The Bergman space $L_a^2(\mathbb{D})$ is the space of all analytic functions on $\mathbb{D}$ that are in $L^2(\mathbb{D}, \mathrm{d}A)$. The space $L_a^2(\mathbb{D})$ is a closed subspace [4] of $L^2(\mathbb{D}, \mathrm{d}A)$ and hence, there exists an orthogonal projection P from $L^2(\mathbb{D}, \mathrm{d}A)$ onto $L_a^2(\mathbb{D})$. Since point evaluation at $z \in \mathbb{D}$ is a bounded linear functional on the Hilbert space $L_a^2(\mathbb{D})$, the Riesz representation theorem implies that there exists a unique function K_z in $L_a^2(\mathbb{D})$ such that

$$f(z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)}\mathrm{d}A(w),$$

for all f in $L_a^2(\mathbb{D})$. Let $K(z,w)$ be the function on $\mathbb{D}\times\mathbb{D}$ defined by $K(z,w) = \overline{K_z(w)}$. The function $K(z,w)$ is called the Bergman kernel [17] of $\mathbb{D}$ or the reproducing kernel of $L_a^2(\mathbb{D})$ and $K(z,w) = \frac{1}{(1-z\bar{w})^2}$, for $z, w \in \mathbb{D}$. The sequence of functions $\{e_n(z)\} = \{\sqrt{(n+1)}z^n\}_{n=0}^\infty$ forms an orthonormal basis [17] for $L_a^2(\mathbb{D})$.

Let $L^\infty(\mathbb{D}, \mathrm{d}A)$ denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with $\|f\|_\infty = \text{ess sup}\{|f(z)| : z \in \mathbb{D}\} < \infty$. The space of all bounded analytic functions on $\mathbb{D}$ will be denoted by $H^\infty(\mathbb{D})$. For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ on $L_a^2(\mathbb{D})$ is defined by $T_\phi f = P(\phi f)$, $f \in L_a^2(\mathbb{D})$. The operator T_z is called the Bergman shift operator. The big Hankel operator H_ϕ

is a mapping from $L^2_a(\mathbb{D})$ into $(L^2_a(\mathbb{D}))^\perp$ defined by $H_\phi f = (I-P)(\phi f),\ f \in L^2_a(\mathbb{D})$. The function ϕ is called the symbol of the Hankel operator H_ϕ. The little Hankel operator h_ϕ from $L^2_a(\mathbb{D})$ into $\overline{L^2_a(\mathbb{D})} = \{\bar{f} : f \in L^2_a(\mathbb{D})\}$ is defined by $h_\phi = \bar{P}(\phi f), f \in L^2_a(\mathbb{D})$ where $\bar{P}$ is the projection from $L^2(\mathbb{D}, \mathrm{d}A)$ onto $\overline{L^2_a(\mathbb{D})}$. There are also many equivalent ways of defining little Hankel operators. For example, one can define the map S_ϕ from $L^2_a(\mathbb{D})$ into $L^2_a(\mathbb{D})$ by $S_\phi f = P(J(\phi f))$, where J is the mapping from $L^2(\mathbb{D})$ into $L^2(\mathbb{D})$ defined by $J(h(z)) = h(\bar{z})$. The map J is unitary and it is not difficult to see that $h_\phi = JS_\phi$. Let Γ_ϕ be the mapping from $L^2_a(\mathbb{D})$ into $L^2_a(\mathbb{D})$ defined by $\Gamma_\phi f = P(\phi J f)$. It is also not difficult to verify that $\Gamma_{J\phi} = S_\phi$. Thus the little Hankel operator h_ϕ is unitarily equivalent to the operator S_ψ for some $\psi \in L^\infty(\mathbb{D})$ and h_ϕ is unitarily equivalent to the operator Γ_θ for some $\theta \in L^\infty(\mathbb{D})$. Hence, we shall refer all these operators h_ϕ, S_ϕ and Γ_ϕ in the sequel as little Hankel operators on $L^2_a(\mathbb{D})$.

On the Hardy space of the disk, there is essentially one type of Hankel operators. In the Bergman space setting, there are two very different notions of Hankel operators, the big and little Hankel operators. The big Hankel operator H_ϕ maps into $(L^2_a(\mathbb{D}))^\perp$, while h_ϕ maps into the much smaller space $\overline{L^2_a(\mathbb{D})}$. The orthogonal complement of $H^2(\mathbb{T})$ in $L^2(\mathbb{T})$ differs by only one dimension from $\overline{\mathcal{H}^2(\mathbb{T})} = \{\bar{f} : f \in \mathcal{H}^2(\mathbb{T})\}$, whereas in the Bergman space setting, $(L^2_a(\mathbb{D}))^\perp = L^2(\mathbb{D}, \mathrm{d}A) \ominus L^2_a(\mathbb{D})$ is much bigger than $\overline{L^2_a(\mathbb{D})}$. On the Hardy space of the disk, some of the results on Hankel operators [14], [15] were obtained by examining the corresponding Hankel matrices, with respect to the standard orthonormal basis on $\mathcal{H}^2(\mathbb{T})$. On the Bergman space $L^2_a(\mathbb{D})$, the Hankel operators do not have nice matrices. However, there are many similarities between the theory of little Hankel operators on the Bergman space and the theory of Hankel operators on the Hardy space.

Let E be the unilateral shift on $\mathcal{H}^2(\mathbb{T})$. A necessary and sufficient condition for an operator $H \in \mathcal{L}(H^2(\mathbb{T}))$ to be a Hankel operator ([14], [15]) is that $E^*H = HE$. Hence, the kernel of a Hankel operator on the Hardy space is an invariant subspace of the unilateral shift on $\mathcal{H}^2(\mathbb{T})$ and it was shown by Kronecker [15] that the finite rank Hankel operators have symbols of the form $z\bar{u}h$ where u is a finite Blaschke product and $h \in \mathcal{H}^\infty(\mathbb{T})$. In this case, the rank of H is no greater than the number of zeroes of u (counted with multiplicity). Furthermore, the kernel of a Hankel operator is nonempty if the symbol is of the form $z\bar{u}h$ where u is inner and $h \in \mathcal{H}^\infty(\mathbb{T})$.

Little Hankel operators on the Bergman space behave more like Hankel operators on the Hardy space. It was shown by N. S. Faour [8] that if T_z is the Bergman shift operator then an operator S from $L^2_a(\mathbb{D})$ into itself is a little Hankel operator if and only if $T_z^* S = S T_z$. Thus the kernel of a little Hankel operator on the Bergman space is an invariant subspace of the Bergman shift operator. It was shown in [5] that if S_ϕ is a finite rank little Hankel operator on $L^2_a(\mathbb{D})$ then $\ker S_\phi = \Theta L^2_a(\mathbb{D})$ for some inner function [11] $\Theta \in L^2_a(\mathbb{D})$ such that the following conditions hold: (i) Θ vanishes on $\mathbf{a}= \{a_j\}_{j=1}^N$, a finite sequence of points in $\mathbb{D}$. (ii) $\|\Theta\|_{L^2} = 1$. (iii) Θ is equal to a constant plus a linear combination of the Bergman kernel functions $K(z,a_1), K(z,a_2), \cdots, K(z,a_n)$ and certain of their derivatives. Further, if S_ϕ is finite rank then $\text{rank} S_\phi =$ number of zeroes of Θ counting multiplicities. Moreover, it is easy to check that $S_\phi = 0$ if and only if $\phi \in (\overline{L^2_a})^\perp$ and if ψ is in $L^\infty(\mathbb{D})$ and S_ψ is a finite rank little Hankel operator on $L^2_a(\mathbb{D})$ then $\psi = \phi + \chi$ where $\chi \in (\overline{L^2_a})^\perp \cap L^\infty(\mathbb{D})$ and $\bar{\phi}$ is a linear combination of the Bergman kernels and some of their derivatives. Thus the little Hankel operator on the Bergman space does share some features with the Hankel operators on the Hardy space.

It is well known [17], that the Hankel operator $H_{\bar{\phi}}$, $\phi \in \mathcal{H}^2(\mathbb{T})$ is bounded if and only if $\phi \in BMOA = P_+ L^\infty(\mathbb{T})$ and $H_{\bar{\phi}}$ is compact if and only if $\phi \in VMOA = P_+ C(\mathbb{T})$. If ϕ is analytic in $\mathbb{D}$, then the little Hankel operator $h_{\bar{\phi}}$ defined on $L^2_a(\mathbb{D})$ is bounded if and only if $\phi \in \mathcal{B} = PL^\infty(\mathbb{D})$, the Bloch space. Similarly, if ϕ is analytic in $\mathbb{D}$, then the little Hankel operator $h_{\bar{\phi}}$ defined on $L^2_a(\mathbb{D})$ is compact if and only if $\phi \in \mathcal{B}_0 = PC(\overline{\mathbb{D}})$, the little Bloch space. There exists a best compact approximant to a Hankel operator on the Hardy space [14],[15] which is also a Hankel operator. In [9], Ghatage has shown that given ϕ belonging the Bloch space $\mathcal{B}$, there exists ϕ_0 belonging to the little Bloch space $\mathcal{B}_0$, such that $\|S_{\bar{\phi}}\|_e = \|S_{\bar{\phi}} - S_{\bar{\phi}_0}\|$. Thus, a bounded little Hankel operator on the Bergman space always has a best compact approximant that is also a little Hankel operator on the Bergman space. For details, see [9]. In this paper, we obtain necessary and sufficient conditions for the existence of a nontrivial unitary part of little Hankel operators defined on the Bergman space $L^2_a(\mathbb{D})$ with symbol $\varphi \in H^\infty(\mathbb{D})$ that are contractions. We also present an explicit description of this unitary part. This extends the results of Butz [2] for Hankel operators defined on the Hardy space.

Let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from the Hilbert space $\mathcal{H}$ into the Hilbert space $\mathcal{K}$ and let $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H},\mathcal{H})$.

Definition 1.1. A maximizing vector for $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is a nonzero vector $x \in \mathcal{H}$ such that $\|Tx\| = \|T\|\|x\|$.

Thus a maximizing vector for T is one at which T attains its norm. Compact operators on Hilbert spaces do have maximizing vectors.

Definition 1.2. An operator $R \in \mathcal{L}(\mathcal{H})$ is called a contraction if $\|R\| \leq 1$. A contraction $R \in \mathcal{L}(\mathcal{H})$ is completely non-unitary if R has no nontrivial reducing subspace M such that the restriction of R to M is unitary.

It is known [16] that for any contraction R on $\mathcal{H}$, we can find a unique orthogonal decomposition $\mathcal{H} = G \bigoplus K$ such that $R\big|_G$ is unitary and $R\big|_K$ is completely non-unitary. We also include the possibility that either G or K may be the subspace $\{0\}$. The subspace G is given by $G = \{f \in \mathcal{H} : \|R^n f\| = \|f\| = \|R^{*^n} f\|, \quad n = 1, 2, \dots\}$ and is called the unitary subspace of R. The operator $R\big|_G$(the restriction of R to G) is called the **unitary part** of R.

In 1972, Goor [10] established that if T_ψ is a Toeplitz operator defined on the Hardy space $H^2(\mathbb{T})$ with symbol $\psi \in L^\infty(\mathbb{T})$, and $\|T_\psi\| \leq 1$, then T_ψ is completely non-unitary unless ψ is a constant. In 1977, Butz [2], using the result of Goor obtained necessary and sufficient conditions for the existence of a nontrivial unitary part of a Hankel contraction. In this paper, we extend the result of Butz [2] and obtained necessary and sufficient conditions for the existence of a nontrivial unitary part of a little Hankel operator defined on the Bergman space which is also a contraction.

The organizations of this paper is as follows: In Section 2, we derive certain elementary properties of little Hankel operators $\Gamma_\varphi, \varphi \in H^\infty(\mathbb{D})$. We show that $\|\Gamma_\varphi\| = \text{dist}_{L^\infty(\mathbb{D})}(\varphi, \overline{\bar{z}H^\infty(\mathbb{D})})$. In Section 3, we find conditions on Γ_φ that guarantees it has a nontrivial unitary subspace G and discuss where such result finds its use in the form of few corollaries. Finally, we end up with a section which first concludes the article and then provides a discussion which may lead to several new problems in this area.

2. PRELIMINARIES

In this section, we show that if $\varphi \in H^\infty(\mathbb{D})$, then

$$\|\Gamma_\varphi\| = \text{dist}_{L^\infty(\mathbb{D})}(\varphi, \overline{\bar{z}H^\infty(\mathbb{D})})$$

and $\Gamma^*_\varphi = \Gamma_{\phi^+}$, where $\varphi^+(z) = \overline{\varphi(\bar{z})}$. Further, given $\varphi \in H^\infty(\mathbb{D})$, there exists a $\psi \in L^\infty(\mathbb{D})$ such that $\Gamma_\varphi = \Gamma_{J\psi}$ and $\|\psi\|_\infty = \|\Gamma_\varphi\|$. The characterization of bounded Hankel operator on the Hardy space is due to Nehari [12]. He showed that a Hankel matrix corresponds to a bounded operator $H \in \mathcal{L}(\mathcal{H}^2(\mathbb{T}))$ if and only if there exists a function φ in $L^\infty(\mathbb{T})$ such that $H = S_\varphi$. Moreover, φ can be chosen so that $\|S_\varphi\| = \|\varphi\|_\infty$. In Theorem 2.4, we show that similar characterization is also possible for little Hankel operators defined on the Bergman space. To prove Theorem 2.4, we need to establish the results of Lemma 2.1 and Lemma 2.2 and Lemma 2.3.

Lemma 2.1. *Let* $\varphi \in H^\infty(\mathbb{D})$*. Then* $\|\Gamma_\varphi\| \leq \text{dist}_{L^\infty(\mathbb{D})}(\varphi, \overline{\bar{z}H^\infty(\mathbb{D})})$.

Proof. Notice that $\Gamma_\varphi = 0$ if $\varphi \in \overline{\bar{z}H^\infty(\mathbb{D})}$. This can be verified as follows: Let $f \in L^2_a(\mathbb{D})$ and $\varphi \in \overline{\bar{z}H^\infty(\mathbb{D})}$. Then $\Gamma_\varphi f = P(\varphi J f) = 0$ since

$\varphi Jf \in \overline{L_a^2(\mathbb{D})}$. Now $\|\Gamma_\varphi f\| = \|P(\varphi Jf)\| \leq \|\varphi\|_\infty \|Jf\| = \|\varphi\|_\infty \|f\|$ for $f \in L_a^2(\mathbb{D})$. Thus $\|\Gamma_\varphi\| \leq \|\varphi\|_\infty$. Further, since $\Gamma_\varphi f = \Gamma_{\varphi - \bar{z}\bar{g}} f$ for all $g \in H^\infty(\mathbb{D})$, hence $\|\Gamma_\varphi\| = \|\Gamma_{\varphi-\bar{z}\bar{g}}\| \leq \|\varphi - \bar{z}\bar{g}\|_\infty$ for all $g \in H^\infty(\mathbb{D})$. It follows therefore that $\|\Gamma_\varphi\| \leq \text{dist}_{L^\infty(\mathbb{D})}(\varphi, \overline{\bar{z}H^\infty(\mathbb{D})})$. □

Let $\{u_k\}_{k=1}^\infty$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Let $A \in \mathcal{L}(\mathcal{H})$ and assume $A = (a_{jk})_{j,k=1}^\infty$ is the matrix representation of A with respect to the orthonormal basis $\{u_k\}_{k=1}^\infty$. Put $Au_k = g_k$, $k = 1, 2, 3, \ldots$. Since $a_{jk} = \langle Au_k, u_j \rangle$, $j, k = 1, 2, 3, \ldots$, hence $g_k = \sum_{j=1}^\infty a_{jk} u_j, k = 1, 2, 3, \ldots$ and $\sum_{j=1}^\infty |a_{jk}|^2 < \infty$, $k = 1, 2, 3, \ldots$. Thus the kth column of A are the components of the vector into which A maps the kth element of the basis. We shall use this idea in proving the next lemma.

Lemma 2.2. *Consider the infinite matrix $R = [r_{ij}]_{i,j=-\infty}^\infty$ and let R_{ts} be the sub matrix of R given by $R_{ts} = [r_{ij}]_{i=t,j=s}^\infty$ where $t, s \in \mathbb{Z}$. If R_{ts} defines a contraction on l^2 for all $t, s \in \mathbb{Z}$ then R defines a contraction on $l^2(\mathbb{Z})$.*

Proof. Let D be the matrix of a contraction on $l^2(\mathbb{Z})$. Then any column of D consists of the components of the image of a basis vector and so is an l^2 sequence of norm less than or equal to 1. Similarly, every row of D (applying this idea to D^*) is an l^2 sequence of norm less than or equal to 1. Let $D = R_{ts}$. Then $\sum_{j=s}^\infty |r_{tj}|^2 \leq 1$ for all $t, s \in \mathbb{Z}$. Hence $\sum_{j=-\infty}^\infty |r_{tj}|^2 \leq 1, t \in \mathbb{Z}$. Thus if $z = (z_n)_{n=-\infty}^\infty \in l^2(\mathbb{Z})$, then the series $y_t = \sum_{j=-\infty}^\infty r_{tj} z_j$ converges. Let $Rz = y = (y_t)_{t=-\infty}^\infty$. Suppose $\|z\| = 1$ in $l^2(\mathbb{Z})$. Then $z^s = (z_s, z_{s+1}, z_{s+2}, \ldots)$ has norm at most 1 in l^2, and so $\|R_{ts} z^s\| \leq 1$. That is, $\sum_{i=t}^\infty |\sum_{j=s}^\infty r_{ij} z_j|^2 \leq 1$. Letting $s \to -\infty$, we obtain $\sum_{i=t}^\infty |y_i|^2 \leq 1, t \in \mathbb{Z}$, and then letting $t \to -\infty$, we obtain $\sum_{i=-\infty}^\infty |y_i|^2 \leq 1$. Hence $y \in l^2(\mathbb{Z})$ and $\|y\| \leq 1$ and R defines a contraction on $l^2(\mathbb{Z})$. □

Let $L_{\tilde{\phi}}$ denotes an operator in $\mathcal{L}(L_a^2(\mathbb{D}))$ with a classical Toeplitz matrix with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$ with symbol $\widetilde{\phi}$ in $L^\infty(\mathbb{T})$, i.e., $\langle L_{\tilde{\phi}} e_j, e_i \rangle = \widehat{\tilde{\phi}}(i-j)$ where $\widehat{\tilde{\phi}}(k)$ is the k^{th} Fourier coefficient of $\tilde{\phi}$. Similarly, $B_{\tilde{\phi}}$ denotes an operator in $\mathcal{L}(L_a^2(D))$ with a classical Hankel matrix with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$ with symbol $\tilde{\phi}$ in $L^\infty(\mathbb{T})$, i.e., $\langle B_{\tilde{\phi}} e_j, e_i\rangle = \hat{\tilde{\phi}}(i+j)$ where $\hat{\tilde{\phi}}(k)$ is the k^{th} Fourier coefficient of $\tilde{\phi}$. It is important to note that if $\tilde{\phi}$ and $\tilde{\psi}$ belong to $L^\infty(\mathbb{T})$ then $L_{\widetilde{\phi\psi}} = B_{\tilde{\phi}^+}B_{\tilde{\psi}} + L_{\tilde{\phi}}L_{\tilde{\psi}}$ and $B_{\widetilde{\phi\psi}} = L_{\tilde{\phi}^+}B_{\tilde{\psi}} + B_{\tilde{\phi}}L_{\tilde{\psi}}$ where $\tilde{\phi}^+(z) = \tilde{\phi}(\bar{z})$. Moreover, if $\phi = \sum_{k=0}^\infty \hat{\phi}(k)z^k$ belongs to $L^\infty(\mathbb{D})$ then $\tilde{\phi}$ will denote the function $\sum_{k=0}^\infty \hat{\phi}(k)e^{ik\theta}$ in $L^\infty(\mathbb{T})$.

Notice also that if $\phi \in H^\infty(\mathbb{D})$ then $T_\phi = D_1 L_{\tilde{\phi}} D_2$ where D_1 is the operator on $L_a^2(\mathbb{D})$ given by $D_1 e_j = \frac{1}{\sqrt{j+1}} e_j$ and D_2 the operator on $L_a^2(\mathbb{D})$ given by $D_2 e_j = \sqrt{j+1}e_j$. The operator D_1 is bounded but D_2 is an unbounded operator and $D_2 D_1 = I$, the identity operator. Similarly if $\phi \in \overline{H^\infty(\mathbb{D})}$ then $T_\phi = D_2 L_{\tilde{\phi}} D_1$. Let S_ϕ be the little Hankel operator on $L_a^2(\mathbb{D})$. If $\phi \in H^\infty(\mathbb{D})$ then $S_\phi = 0$. Now consider the operator $W : L_a^2(\mathbb{D}) \longrightarrow L_a^2(\mathbb{D})$ such that $We_j = e_{j+1}, j \geq 0$. It can easily be checked that $(D_1 W D_2)^n = D_1 W^n D_2$ and $\|D_1 W^n D_2\| = 1$. Similarly, one can also check that

$$\|D_1 W^{*n} D_2\| = \|(D_1 W^* D_2)^n\| = \sqrt{n+1}.$$

Let $\mathcal{R} : L_a^2(\mathbb{D}) \longrightarrow L_a^2(\mathbb{D})$ such that $\mathcal{R}f(z) = \frac{f(z)-f(0)}{z}$. Let $S = T_z$, the Bergman shift operator. It is to be noted that $\mathcal{R}S = I, \mathcal{R} = D_1 W^* D_2$ and $S = D_1 W D_2$. Further observe that $\mathcal{R} = T_{\bar{z}} + K$ where K is a compact operator on $L_a^2(\mathbb{D})$.

Lemma 2.3. *If $\varphi \in L^\infty(\mathbb{D})$, then the following hold:*

(i) $\Gamma_\varphi^* = \Gamma_{\phi^+}$, *where* $\varphi^+(z) = \overline{\varphi(\bar{z})}$.

(ii) *If $\phi \in \overline{H^\infty(\mathbb{D})}$, then $S_\phi = D_2 B_{\tilde{\psi}} D_2$ where $\tilde{\psi}(e^{i\theta}) = \sum_{k=0}^\infty \frac{1}{k+1}\hat{\phi}(-k)e^{-ik\theta}$ i.e., $\tilde{\psi}$ is the convolution on the circle of $\tilde{\phi} = \sum_{k=0}^\infty \hat{\phi}(-k)e^{-ik\theta}$ with the function $\tilde{\phi}_1 = \sum_{k=0}^\infty \frac{1}{k+1}e^{-ik\theta}$.*

Proof. Let $f, g \in L_a^2(\mathbb{D})$. Then

$$\begin{aligned}\langle \Gamma_\varphi^* f, g\rangle &= \langle f, \Gamma_\varphi g\rangle = \langle f, P(\varphi J g)\rangle = \langle f, \varphi J g\rangle = \int_{\mathbb{D}} f(z)\overline{\varphi(z)(Jg)(z)}\mathrm{d}A(z)\\ &= \int_{\mathbb{D}} f(\bar{z})\varphi^+(z)\overline{g(z)}\mathrm{d}A(z) = \int_{\mathbb{D}} (Jf)(z)\varphi^+(z)\overline{g(z)}\mathrm{d}A(z)\\ &= \langle P(\varphi^+ Jf), g\rangle = \langle \Gamma_{\varphi^+} f, g\rangle.\end{aligned}$$

Hence $\Gamma_\varphi^* = \Gamma_{\phi^+}$ for $\varphi \in L^\infty(\mathbb{D})$. The proof of (i) follows.

If $\phi \in \overline{H^\infty(\mathbb{D})}$, then with respect to the basis $\{z^n\sqrt{n+1}, n \geq 0\}$, the matrix of $S_\phi \in \mathcal{L}(L_a^2(\mathbb{D}))$ is $(m_{ij}a_{i+j})$ where (a_{i+j}) is the matrix of $H_\phi \in$

$\mathcal{L}(\mathcal{H}^2)$ with respect to the basis $\{e^{in\theta}, n \geq 0\}$ and $m_{ij} = \frac{\sqrt{i+1}\sqrt{j+1}}{i+j+1}$. This brings us to the map Φ which sends bounded Hankel operators on H^2 into bounded Bergman little Hankel operators on $L^2_a(\mathbb{D})$ via Schur multiplication defined by $\Phi(A) = (m_{ij}a_{ij})$ where $A = (a_{ij})$ with respect to the standard basis $\{e^{in\theta}, n \geq 0\}$ of H^2 and m_{ij} is the multiplier defined above. If $A = (a_{ij})$, then $\Phi(A) = (m_{ij}a_{ij})$ with $m_{ij} > 0$ for all i, j. The operators S_ϕ and $D_2 B_{\tilde{\psi}} D_2$ belong to $\mathcal{L}(L^2_a(\mathbb{D}))$. The proof of (ii) follows since the matrices of these operators with respect to the standard orthonormal basis for $L^2_a(\mathbb{D})$ are same where

$$\tilde{\psi}(e^{i\theta}) = \sum_{k=0}^{\infty} \frac{1}{k+1} \hat{\phi}(-k) e^{-ik\theta} \tag{2.1}$$

i.e., $\tilde{\psi}$ is the convolution on the circle of $\tilde{\phi} = \sum_{k=0}^{\infty} \hat{\phi}(-k)e^{-ik\theta}$ with the function $\tilde{\phi}_1 = \sum_{k=0}^{\infty} \frac{1}{k+1} e^{-ik\theta}$. □

Theorem 2.4. *Let $\varphi \in H^\infty(\mathbb{D})$. Then $\|\Gamma_\varphi\| = \operatorname{dist}_{L^\infty(\mathbb{D})}(\varphi, \overline{\bar{z}H^\infty(\mathbb{D})})$. Further, there exists $\psi \in L^\infty(\mathbb{D})$ such that $\Gamma_\varphi = \Gamma_{J\psi}$ and $\|\psi\|_\infty = \|\Gamma_\varphi\|$.*

Proof. From Lemma 2.1, it follows that $\|\Gamma_\varphi\| \leq \operatorname{dist}_{L^\infty(\mathbb{D})}(\varphi, \overline{\bar{z}H^\infty(\mathbb{D})})$. We shall prove the opposite inequality. Without loss of generality, we shall assume $\|\Gamma_\varphi\| = 1$. This is so, because we may replace φ by $\lambda\varphi$ for some suitable scalar λ. Now from Lemma 2.3, it follows that $\Gamma_\varphi f = P(\varphi J f) = PJ((J\varphi)f) = S_{J\varphi} f$ and $S_{J\varphi} = D_2 B_{\widetilde{J\varphi} * \widetilde{\varphi}_1} D_2$ where $D_2 e_j = \sqrt{j+1} e_j$, $j = 0, 1, 2, \ldots$ and $\widetilde{\Xi} = \widetilde{J\varphi} * \widetilde{\varphi}_1$ is the convolution on the circle of $\widetilde{J\varphi}$ with $\widetilde{\varphi}_1$, where $\widetilde{\varphi}_1(e^{i\theta}) = \sum_{k=0}^{\infty} \frac{1}{k+1} e^{-ik\theta}$ and $B_{\widetilde{\Xi}}$ is the operator in $\mathcal{L}(L^2_a(\mathbb{D}))$ having a classical Hankel matrix with symbol $\widetilde{\Xi} \in L^\infty(\mathbb{T})$.

Let $\widehat{\widetilde{J\varphi}}(-n) = a_{-n}$, $n \in \mathbb{Z}_+$. Let

$$H_0 = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

be the matrix of $S^{H^2}_{\widetilde{J\varphi}}$ (Hankel operator defined on $\mathcal{H}^2(\mathbb{T})$ with symbol $\widetilde{J\varphi}$) with respect to the orthonormal basis $\{\frac{e^{int}}{\sqrt{2\pi}}\}_{n=0}^{\infty}$ of $\mathcal{H}^2(\mathbb{T})$ and it has norm 1 as an operator on l^2. Construct $a_1, a_2, \cdots \in \mathbb{C}$ inductively as follows. Suppose that a_j, $1 \leq j < k$ are such that the infinite Hankel matrix

$$H_k = \begin{bmatrix} a_k & a_{k-1} & a_{k-2} & \cdots & a_1 & a_0 & a_{-1} & \cdots \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_0 & a_{-1} & a_{-2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

is a contraction, $k \in \mathbb{N} \bigcup \{0\}$. Notice that this is so when $k = 0$. We shall adjoin a new first column so as to preserve the Hankel structure and the property of being a contraction. The Hankel pattern fixes all the entries of the new first column except its first entry, which is to be adjusted.

Let

$$H_k(p) = \begin{bmatrix} p & a_k & a_{k-1} & a_{k-2} & \cdots \\ a_k & a_{k-1} & a_{k-2} & a_{k-3} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} p & Q \\ R & S \end{bmatrix},$$

where Q, R and S are suitable matrices of types $1 \times \infty, \infty \times 1$ and $\infty \times \infty$, respectively. Further, we have $\begin{bmatrix} R & S \end{bmatrix} = H_k = \begin{bmatrix} Q \\ S \end{bmatrix}$, and H_k is a contraction, by the inductive hypothesis. Using Parrott's theorem [13], we guarantee the existence of an element $p \in \mathbb{C}$ such that $H_k(p)$ is a contraction. Let $a_{k+1} = p$. Now $H_{k+1} = H_k(a_{k+1})$ and is a contraction. By induction the sequence $(a_k)_{k=1}^{\infty}$ has the property that $\|H_k\| \leq 1$ for all $k \geq 1$. Consider the infinite Hankel matrix $R = (a_{-i-j+1})_{i,j=-\infty}^{\infty}$. Let

$$R_{ts} = \begin{bmatrix} a_{-t-s+1} & a_{-t-s} & \cdots \\ a_{-t-s} & a_{-t-s-1} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

which is either H_{-t-s+1} or (if $t + s > 1$) a submatrix of H_0. In either case it is a contraction, and so R is itself a contraction by Lemma 2.2. From [1], the function ψ with Fourier series $\widetilde{\psi}(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$ belongs to $L^\infty(\mathbb{T})$ and satisfies $\|\widetilde{\psi}\| = \|R\| \leq 1$. Let $e^{i\theta} g(e^{i\theta}) = \widetilde{J\varphi} - \widetilde{\psi}$. Since $\widetilde{J\varphi}$ and $\widetilde{\psi}$ have the same non-positive Fourier coefficients $a_{-n}, n \geq 0$, hence $e^{i\theta} g(e^{i\theta}) \in e^{i\theta} H^\infty(\mathbb{T})$. Let $zg, J\varphi, \psi$ be the harmonic extensions of $e^{i\theta} g(e^{i\theta}), \widetilde{J\varphi}$ and $\widetilde{\psi}$, respectively. Then $zg = J\varphi - \psi \in zH^\infty(\mathbb{D})$. Hence

$$J(zg) \in \bar{z}\overline{H^\infty(\mathbb{D})}$$

and

$$0 = \Gamma_{J(zg)} = S_{zg} = S_{J\varphi} - S_\psi = \Gamma_\varphi - \Gamma_{J\psi}.$$

Thus $\Gamma_\varphi = \Gamma_{J\psi}$ and

$$\|\Gamma_\varphi\| \leq \operatorname{dist}_{L^\infty(\mathbb{D})}(\varphi, \bar{z}\overline{H^\infty(\mathbb{D})}) \leq \|\varphi - J(zg)\|_\infty = \|J\psi\|_\infty = \|\psi\|_\infty \leq 1$$
$$= \|\Gamma_\varphi\|.$$

This proves that $\|\Gamma_\varphi\| = \operatorname{dist}_{L^\infty(\mathbb{D})}(\varphi, \bar{z}\overline{H^\infty(\mathbb{D})})$.

Further $\bar{z}\bar{g} \in \bar{z}\overline{H^\infty(\mathbb{D})}$, $\Gamma_{J\psi} = \Gamma_{\varphi - J(zg)} = \Gamma_\varphi$ and

$$\|\psi\|_\infty = \|J\psi\|_\infty = \|\varphi - J(zg)\|_\infty = \|J\psi\|_\infty \leq \operatorname{dist}_{L^\infty(\mathbb{D})}(\varphi, \bar{z}\overline{H^\infty(\mathbb{D})}).$$

Hence $\|\psi\|_\infty = \|J\psi\|_\infty = \mathrm{dist}_{L^\infty(\mathbb{D})}(\varphi, \bar{z}\overline{H^\infty(\mathbb{D})}) = \|\Gamma_\varphi\|$. $\square$

Definition 2.5. Let $\varphi \in H^\infty(\mathbb{D})$. If there exists $\psi \in L^\infty(\mathbb{D})$ such that $\Gamma_\varphi = \Gamma_{J\psi}$ and $\|\psi\|_\infty = \|\Gamma_\varphi\|$ then the function $J\psi$ is called the minifunction of Γ_φ.

3. BEST APPROXIMATION AND THE MINIFUNCTION OF Γ_φ

In this section, we show that if $\varphi \in H^\infty(\mathbb{D})$ and Γ_φ has a maximizing vector f, then there is a unique best approximation $\bar{z}\bar{g} \in \bar{z}\overline{H^\infty(\mathbb{D})}$ to φ in the $L^\infty(\mathbb{D})$ norm and $J(zg) = \varphi - \frac{\Gamma_{J\psi} f}{f}$, where $J\psi, \psi \in L^\infty(\mathbb{D})$ is the minifunction of Γ_φ. Further, we find conditions on Γ_φ that guarantees it has a nontrivial unitary subspace G and discussed the applications of these results in the form of few corollaries. The techniques used in Theorem 3.1 and Theorem 3.2 is similar to those used in the Hardy space case. For details, see Partington [14] and Butz [2].

THEOREM 3.1. *Let $\varphi \in H^\infty(\mathbb{D})$. Suppose Γ_φ has a maximizing vector f. Then there is a unique best approximation $\bar{z}\bar{g} \in \bar{z}\overline{H^\infty(\mathbb{D})}$ to φ in the $L^\infty(\mathbb{D})$ norm such that $\|\varphi - \bar{z}\bar{g}\|_\infty = \mathrm{dist}_{L^\infty(\mathbb{D})}(\varphi, \bar{z}\overline{H^\infty(\mathbb{D})})$. Further, $J(zg) = \varphi - \frac{\Gamma_{J\psi} f}{f}$, where $J\psi, \psi \in L^\infty(\mathbb{D})$ is the minifunction of Γ_φ.*

Proof. Without loss of generality, assume $\|f\| = 1$. By Theorem 2.4, there exists $\psi \in L^\infty(\mathbb{D})$ such that $\Gamma_\varphi = \Gamma_{J\psi}$ and $\|\Gamma_\varphi\| = \|\psi\|_\infty$. Now

$$\begin{aligned}\|\Gamma_\varphi\| = \|\Gamma_\varphi f\| = \|\Gamma_{J\psi} f\| &= \|P(J\psi J f)\| = \|P(J(\psi f))\| = \|JPJ(\psi f)\| \\ &= \|\bar{P}(\psi f)\| \leq \|\psi f\| \leq \|\psi\|_\infty \|f\| = \|\psi\|_\infty = \|\Gamma_\varphi\|.\end{aligned}$$

Thus $\|\bar{P}(\psi f)\| = \|JPJ(\psi f)\| = \|\psi f\|$. But $\psi f = \bar{P}(\psi f) + (I - \bar{P})(\psi f)$. Hence $\|\psi f\|^2 = \|\bar{P}(\psi f)\|^2 + \|(I - \bar{P})(\psi f)\|^2$. Therefore, $\|(I - \bar{P})(\psi f)\| = 0$. That is, $(I - \bar{P})(\psi f) = 0$.

Thus we obtain $\psi f = \bar{P}(\psi f) = h_\psi f = JPJ(\psi f) = J\Gamma_{J\psi} f = J\Gamma_\varphi f$ for all $f \in L^2_a(\mathbb{D})$, the domain of Γ_φ. Now f is non zero almost everywhere on $\mathbb{D}$ as $\|f\| = 1$. Let $\psi(z) = \frac{(J\Gamma_{J\psi} f)(z)}{f(z)}$ almost everywhere. Let

$$J(zg) = \varphi - J\psi = \varphi - \frac{\Gamma_{J\psi} f}{f}.$$

Since $\Gamma_\varphi = \Gamma_{J\psi}$, we obtain $J(zg) \in \bar{z}\overline{H^\infty(\mathbb{D})}$ and $\|\varphi - J(zg)\|_\infty = \|J\psi\|_\infty = \|\psi\|_\infty = \|\Gamma_\varphi\| = \mathrm{dist}_{L^\infty(\mathbb{D})}(\varphi, \bar{z}\overline{H^\infty(\mathbb{D})})$.

To prove the uniqueness, let $h \in H^\infty(\mathbb{D})$ be such that

$$\|\varphi - J(zh)\|_\infty = \mathrm{dist}_{L^\infty(\mathbb{D})}(\varphi, \bar{z}\overline{H^\infty(\mathbb{D})}).$$

Let $J\psi = \varphi - J(zh)$. Then $\Gamma_\varphi = \Gamma_{J\psi}$ and $\|\psi\|_\infty = \|J\psi\|_\infty = \|\Gamma_\varphi\|$. Then we proceed as before to show that $\psi(z) = \frac{(J\Gamma_{J\psi} f)(z)}{f(z)}$. $\square$

THEOREM 3.2. *Let $\varphi \in H^\infty(\mathbb{D})$ and Γ_φ be the little Hankel operator defined on the Bergman space with symbol φ. Then Γ_φ will have a non trivial unitary subspace G only when there exists a minifunction $\psi(z)$ for Γ_φ such that*

(i) $|\psi(z)| = 1$ *a.e on* $\mathbb{D}$.

(ii) $\psi(z)J\psi(z) = c^2$ *a.e. for some constant c of modulus 1. In this case, the minifunction ψ is unique and G is given by the following three equivalent expressions.*

(iii) $G = \left(z\psi L_a^2(\mathbb{D})\right)^\perp \cap L_a^2(\mathbb{D})$ *where* $\left(z\psi L_a^2(\mathbb{D})\right)^\perp$ *is the orthogonal complement of* $z\psi L_a^2(\mathbb{D})$ *with respect to* $L^2(\mathbb{D}, dA)$.

(iv) $G = \left\{f \in L_a^2(\mathbb{D}) : \Gamma_\varphi^* \Gamma_\varphi f = f\right\}$.

(v) $G = \left\{f \in L_a^2(\mathbb{D}) : \bar{c}\Gamma_\varphi f = f\right\} \bigoplus \left\{f \in L_a^2(\mathbb{D}) : \bar{c}\Gamma_\varphi f = -f\right\}$.

Proof. Let $\varphi \in H^\infty(\mathbb{D})$. Then by Theorem 2.4, there exists $\psi_1 \in L^\infty(\mathbb{D})$ such that $\Gamma_\varphi = \Gamma_{\psi_1}$ and $\|\psi_1\|_\infty = \|\Gamma_\varphi\| = \|\Gamma_{\psi_1}\|$. Then $\|\Gamma_{\psi_1}\| = \|\Gamma_\varphi\| \leq 1$ and suppose $\Gamma_\varphi = \Gamma_{\psi_1}$ has a nontrivial unitary subspace. By Lemma 2.3, it follows that there exists $f \neq 0$ in $L_a^2(\mathbb{D})$ such that $\|\Gamma_{\psi_1}^n f\| = \|f\| = \|\Gamma_{\psi_1^+}^n f\|$ for $n = 1, 2, \ldots$. If $n = 1$, this gives that

$$\|f\| = \|\Gamma_{\psi_1} f\| = \|P(\psi_1 Jf)\| \leq \|\psi_1 Jf\| \leq \|\psi_1\|_\infty \|Jf\| = \|Jf\| = \|f\|.$$

Hence, we obtain $P(\psi_1 Jf) = \psi_1 Jf$ and therefore $\Gamma_{\psi_1} f = \psi_1 Jf \in L_a^2(\mathbb{D})$. It follows from Theorem 3.1, that the minifunction ψ_1 is unique. Now since $\|\psi_1 Jf\| = \|Jf\|$, we obtain

$$\int_{\mathbb{D}} |\psi_1(z)|^2 |Jf(z)|^2 \mathrm{d}A(z) = \int_{\mathbb{D}} |Jf(z)|^2 \mathrm{d}A(z).$$

Hence $\displaystyle\int_{\mathbb{D}} (|\psi_1(z)|^2 - 1)|Jf(z)|^2 \mathrm{d}A(z) = 0$. This implies $|\psi_1(z)| = 1$ a.e on $\mathbb{D}$. This establishes (i).

Taking now $n = 2$, we obtain

$$\|f\| = \|\Gamma_{\psi_1}^2 f\| = \|\Gamma_{\psi_1}(\psi_1 Jf)\| \leq \|\psi_1(J\psi_1)f\| \leq \|\psi_1\|_\infty \|J\psi_1\|_\infty \|f\| \leq \|f\|$$

and therefore $\Gamma_{\psi_1}^2 f = \psi_1(J\psi_1)f = T_\theta f$ where $\theta = \psi_1(J\psi_1)$. Proceeding similarly, we obtain

$$\Gamma_{\psi_1}^{2n+1} f = \psi_1^{n+1}(J\psi_1^n)Jf, \quad n = 0, 1, 2, \ldots. \tag{3.1}$$

and

$$\Gamma_{\psi_1}^{2n} f = [\psi_1 J\psi_1]^n f, \quad n = 1, 2, 3, \ldots. \tag{3.2}$$

Similarly, we can also verify that $\Gamma_{\psi_1^+}^{2n} f = [\psi_1^+ J\psi_1^+]^n f, \quad n = 1, 2, 3, \ldots$ where $\psi_1^+(z) = \overline{\psi_1(\bar{z})} = \overline{J\psi_1(z)}$. Thus

$$\Gamma_{\psi_1^+}^2 f = [\psi_1^+ J\psi_1^+] f = T_{\psi_1^+ \overline{\psi_1}} f = T_{\overline{\psi_1 J\psi_1}} f = T_\theta^* f.$$

Thus the Toeplitz operator T_θ has a nontrivial unitary part. Therefore, it follows from [6] that $(\psi_1 J\psi_1)(z) = c^2$ a.e. on $\mathbb{D}$ for some constant $c \in \mathbb{C}$, $|c| = 1$. Hence $\bar{c}\psi_1(z) = c\psi_1^+(z), z \in \mathbb{D}$. Formulas (3.1) and (3.2) now reduce to

$$\Gamma_{\psi_1}^{2n+1} f = c^{2n}\Gamma_{\psi_1} f, \quad n = 0, 1, 2, \ldots.$$

$$\Gamma_{\psi_1}^{2n} f = c^{2n} f, \quad n = 1, 2, 3, \ldots.$$

valid for all f in the unitary subspace of Γ_{ψ_1}. Similar expressions can be obtained for $\Gamma_{\psi_1^+}^{2n+1} f$ and $\Gamma_{\psi_1^+}^{2n} f$. The maximal subspace on which Γ_φ is unitary now becomes

$$\begin{aligned} G &= \{f \in L_a^2(\mathbb{D}) : \|\Gamma_\varphi f\| = \|f\|\} = \{f \in L_a^2(\mathbb{D}) : \|\Gamma_{\psi_1} f\| = \|f\|\} \\ &= \{f \in L_a^2(\mathbb{D}) : \Gamma_{\psi_1}^* \Gamma_{\psi_1} f = f\} = \{f \in L_a^2(\mathbb{D}) : \Gamma_\varphi^* \Gamma_\varphi f = f\} \end{aligned}$$

which establishes (iv).

Again we have $f \in G$ if and only if $\psi_1 Jf \in L_a^2(\mathbb{D})$ if and only if $(J\psi_1)f \perp zL_a^2(\mathbb{D})$ if and only if $f \in (z\psi_1^+ L_a^2(\mathbb{D}))^\perp \cap L_a^2(\mathbb{D})$ (the orthogonal complement of $(z\psi_1^+ L_a^2(\mathbb{D}))$ with respect to $L^2(\mathbb{D}, \mathrm{d}A)$). Now $f \in (z\psi_1^+ L_a^2(\mathbb{D}))^\perp \cap L_a^2(\mathbb{D})$ if and only if $f \in (z\psi_1 L_a^2(\mathbb{D}))^\perp \cap L_a^2(\mathbb{D})$ (the orthogonal complement with respect to $L^2(\mathbb{D}, \mathrm{d}A)$), since $J\psi_1 = c^2\overline{\psi_1}$. This establishes (iii). Finally, since $|c| = 1$, we have $(\bar{c}\Gamma_{\psi_1})^* = \bar{c}\Gamma_{\psi_1}$. Hence $\bar{c}\Gamma_{\psi_1}$ is self-adjoint with $\|\bar{c}\Gamma_{\psi_1}\| = 1$. This establishes (v) and the proof is complete. □

Let Ω be a bounded symmetric domain in $\mathbb{C}$. We assume that Ω is in its standard (Harish-Chandra) realization so that $0 \in \Omega$ and Ω is circular. The domain Ω is also starlike; i.e., $z \in \Omega$ implies that $tz \in \Omega$ for all $t \in [0, 1]$. Let $\mathrm{Aut}(\Omega)$ be the Lie group of all automorphisms (biholomorphic mappings) of Ω, and G_0 the isotropy subgroup at 0; i.e., $G_0 = \{\Psi \in \mathrm{Aut}(\Omega) : \Psi(0) = 0\}$. Since Ω is bounded symmetric, we can canonically define for each a in Ω an automorphism ϕ_a in $\mathrm{Aut}(\Omega)$ such that

(i) $\phi_a \circ \phi_a(z) \equiv z$;

(ii) $\phi_a(0) = a, \phi_a(a) = 0$;

(iii) ϕ_a has a unique fixed point in Ω.

Actually, the above three conditions completely characterize the $\phi_a's$ as the set of all (holomorphic) geodesic symmetries of Ω. When $\Omega = \mathbb{D}$, we have noted that

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z},$$

for all a and z in $\mathbb{D}$. They are involutive Möbius transformations on $\mathbb{D}$.

Let $\mathrm{d}A$ be the normalized Lebesgue measure on Ω. We consider the Bergman space $L^2_a(\Omega)$ of holomorphic functions in $L^2(\Omega, \mathrm{d}A)$. The reproducing kernel $K(z, w)$ of $L^2_a(\Omega, \mathrm{d}A)$ is holomorphic in z and anti-holomorphic in w, and

$$\int_\Omega |K(z, w)|^2 \mathrm{d}A(w) = K(z, z) > 0,$$

for all z in Ω. Thus we can define for each $\lambda \in \Omega$ a unit vector k_λ in $L^2_a(\Omega)$ by $k_\lambda(z) = \frac{K(z,\lambda)}{\sqrt{K(\lambda,\lambda)}}$. For any $\Psi \in \mathrm{Aut}(\Omega)$, we denote by $J_\Psi(z)$ the complex Jacobian determinant of the mapping $\Psi : \Omega \to \Omega$. If $a \in \Omega$, then there exists a unimodular constant $\theta(a)$ such that

$$J_{\phi_a}(z) = \theta(a) k_a(z),$$

for all $z \in \Omega$. In the simplest case $\Omega = \mathbb{D}$, we have $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ and

$$J_{\phi_a}(z) = \phi'_a(z) = -k_a(z),$$

thus $\theta(a) = -1$ is independent of a.

The results of Theorem 3.1 and Theorem 3.2 can be extended to this setting. We shall define little Hankel operator Γ_ϕ on $L^2_a(\Omega)$ with symbol ϕ in $L^\infty(\Omega)$ and one can establish a version of Theorem 3.1 as the matrices of little Hankels in this case also have special forms. With the same conditions cited in Theorem 3.1, one can guarantee the existence of a minifunction. Further, Theorem 3.2 will be valid, since in this case proceeding similarly one can show that the Toeplitz operator $T_\theta = T_{\psi_1 J \psi_1}$ defined on $L^2_a(\Omega)$ has a nontrivial unitary part and hence it follows from [6] that $\psi_1 J \psi_1 \equiv c^2$ a.e. on Ω for some constant $c \in \mathbb{C}$ with $|c| = 1$ where ψ_1 is the minifunction of Γ_ϕ.

We shall now present a characterization of those little Hankel operators Γ_φ, with $\varphi \in H^\infty(\mathbb{D})$ such that it has a nontrivial unitary subspace.

Corollary 3.3. *Let Γ_φ be a little Hankel operator on $L^2_a(\mathbb{D})$ with symbol $\varphi \in H^\infty(\mathbb{D})$. Suppose Γ_φ is a contraction. Then a necessary and sufficient condition that Γ_φ have a nontrivial unitary subspace is that there exists a constant $c \in \mathbb{C}, |c| = 1$ such that $\bar{c}\Gamma_{\psi_1}$ is self-adjoint where ψ_1 is the minifunction of Γ_φ and that*

$$G = \{f \in L^2_a(\mathbb{D}) : \Gamma_{\psi_1} f = cf\} \bigoplus \{f \in L^2_a(\mathbb{D}) : \Gamma_{\psi_1} f = -cf\} \neq \{0\}.$$

Proof. The corollary follows from Theorem 3.2. $\square$

Using a result of [16] for completely nonunitary contractions, we can define a functional calculus for functions $v \in H^\infty(\mathbb{D})$.

Corollary 3.4. *Let Γ be a little Hankel operator on $L_a^2(\mathbb{D})$ which is also a contraction. If either $\|\Gamma\| < 1$ or $c\Gamma$ is not real for all $c \in \mathbb{C}$, then the map $v \to v(\Gamma)$ from $H^\infty(\mathbb{D})$ into $\mathcal{L}(L_a^2(\mathbb{D}))$ defined by*

$$v(\Gamma) = \text{strong} \lim_{r\to 1} \sum_{k=0}^{\infty} a_k r^k \Gamma^k$$

where $v(z) = \sum_{k=0}^{\infty} a_k z^k$ is a contractive homomorphism of the algebra $H^\infty(\mathbb{D})$ into $\mathcal{L}(L_a^2(\mathbb{D}))$.

Proof. The result follows from [16] and Theorem 3.2. □

Theorem 3.2 can be also be used to establish a property of the point spectrum of a bounded little Hankel operator.

Corollary 3.5. *Let Γ_φ be a bounded little Hankel operator on $L_a^2(\mathbb{D})$ with symbol $\varphi \in H^\infty(\mathbb{D})$. If $\mu = \|\Gamma_\varphi\|$ is an eigenvalue of Γ_φ, then Γ_φ is self-adjoint. Hence, if $\alpha\Gamma_\varphi$ is not self-adjoint for any nonzero $\alpha \in \mathbb{C}$, then $|\mu| < \|\Gamma_\varphi\|$ holds for all eigenvalue μ of Γ_φ (if any).*

Proof. Without loss of generality, we shall assume $\|\Gamma_\varphi\| = 1$. This implies $G = \{f \in L_a^2(\mathbb{D}) : \Gamma_\varphi f = f\}$ is a reducing subspace of Γ_φ (for example, see [16]). If $f \neq 0$ satisfies $\Gamma_\varphi f = f$, then by the proof of Theorem 3.2, $\Gamma_\varphi^* = \bar{\alpha}\Gamma_\varphi$ and $\Gamma_\varphi^2 f = \alpha f$ for some $\alpha \in \mathbb{C}, |\alpha| = 1$. But since $\Gamma_\varphi^2 f = f$, we get $\alpha = 1$, and hence $\Gamma_\varphi^* = \bar{\alpha}\Gamma_\varphi = \Gamma_\varphi$ which shows that Γ_φ is self-adjoint. □

4. CONCLUDING REMARKS

Butz [3] obtained characterization for the existence of a nontrivial unitary part of a generalized Toeplitz operator acting on a Hilbert space with unilateral shift. In this contest, it will be of interest to obtain characterizations for the existence of a nontrivial unitary part of a Toeplitz operator defined on the Bergman space over any bounded symmetric domain.

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