ON COMMUTATORS IN FINITE P-GROUPS OF ALMOST MAXIMAL CLASS

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Let $\Gamma(G)$ denote the set of commutators of a group G, $G' = \langle \Gamma(G) \rangle$ and $c(G)$ (or $cw(G)$) the minimal number such that every element of G' can be expressed as a product of at most $c(G)$ commutators. Recently, we proved that if G is a finite p-group of maximal class, then $\Gamma(G) = G'$. For finite p-groups of almost maximal class, the situation is more complicated. In this paper, we show that if $p = 2$ then $\Gamma(G) = G'$. If $p > 2$ we have two cases, according to the minimum number of generators of G, $d(G)$. We prove that if $d(G) = 3$, then $G' = \Gamma(G)$ and if $d(G) = 2$, $c(G) \leqslant 2$. As a consequence of this result, we prove that if $G = A \wr P$, in which A is a nontrivial finite abelian group and P is a 2-group of almost maximal class of order 2^n, $n \geq 5$, then the commutator length of G is equal to 2 or 3. Finally, we will provide various examples.

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1. INTRODUCTION

Let G be a group and G' its commutator subgroup. Denote by $c(G)$ (or $cw(G)$) the minimal number such that every element of G' can be expressed as a product of at most $c(G)$ commutators. A group G is called a c-group if $c(G)$ is finite. For any positive integer n, denote by c_n the class of groups with commutator length (or commutator width) equal to n. Denote by $\Gamma(G)$ the set of commutators in G.

In a group product of two commutators one may not be a commutator. Many examples of groups whose commutator subgroups contain a non commutator element are groups of prime power order. In [14], L. C. Kappe and R. F. More proved that for $p = 2$ the smallest integer n such that there exists a group of order 2^n in which $\Gamma(G) \neq G'$ is $n = 7$. And for any odd prime $n = 6$ is the smallest such number. In [3], Akhavan-Malayeri, used wreath product constructions to obtain, for any positive integer n, a solvable group of derived

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length n and commutator length equal to 1 *or* 2. Let $W = G \wr H$ be the wreath product of G by an n-generator abelian group H. In [2], she proved that every element of W' is a product of at most $n+2$ commutators, and every element of $W^2$is a product of at most $3n+4$ squares in W. This generalizes our previous result.

Throughout, p denotes a fixed prime and $cl(G)$, $d(G)$ denote the nilpotency class and the minimum number of generators of G, respectively. Recall that if $|G| = p^n$ and $cl(G) = c$, then the coclass of G is $cc(G) = n - c$. A non-abelian group of coclass 1 is called a p-group of maximal class and a group of coclass 2 is called a p-group of almost maximal class. Recently, Akhavan-Malayeri proved that if G is a p-group of maximal class, then $\Gamma(G) = G'$ (see [5]). For finite p-groups of almost maximal class, the situation is more complicated. In this paper, we show that if $p = 2$ then $\Gamma(G) = G'$ and if $p > 2$, then either $d(G) = 3$ and $c(G) = 1$ or $d(G) = 2$ and $c(G) = 1$ *or* 2. As a consequence of this result, we show that if P is a 2-group with $cc(P) < 3$ and $G = P \wr C_1 \wr \cdots \wr C_n$ where C_i is a finite cyclic group for $1 \leq i \leq n$, then $\Gamma(G) = G'$. Finally, let A be a non trivial finite abelian group and P be a 2-group of almost maximal class of order 2^n, $n \geq 5$. Let $G = A \wr P$. By using Guralnick's [9] result, we show that the commutator length of G is equal to 2 or 3. We also give a precise formula for expressing every element of G' as a product of two or three commutators.

2. MAIN RESULTS

Let G be a group and $x, y \in G$, then $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy$. By $Z(G)$, we denote the center of G. The i-th terms of the upper central series of G is denoted by $Z_i(G)$ and i-th terms of the lower central series of G for $i \geq 2$ is denoted by $\gamma_i(G)$. And $\gamma_2(G)$ is denoted by G'.

Let G be a 2-group of almost maximal class of order 2^n, $n \geq 5$. First, we describe some notations which will be kept throughout. Following [8] and [12], by $\gamma_1(G)$ we mean the subgroup of G with the property that $\gamma_1(G)/\gamma_4(G)$ is the centraliser in $G/\gamma_4(G)$ of $G'/\gamma_4(G)$. Let $s \in G$ such that $s \notin \gamma_1(G)$ and $s \notin C_G(\gamma_{n-3}(G))$, and $s_1 \in \gamma_1(G) \setminus Z_{n-3}(G)$, we put

$$s_{i+1} = [s_i, s], \qquad i = 1, 2, \cdots .$$

If s, s_i are defined as above, then $|\gamma_i(G)/\gamma_{i+1}(G)| = 2$ and $\gamma_i(G) = \langle s_i, \gamma_{i+1}(G)\rangle$ where $i = 2, 3, ..., n-2$. Also $|G : \gamma_1(G)| = |\gamma_1(G) : Z_{n-3}(G)| = 2$ (see ([12, Theorem 3.4 and Theorem 3.1])).

In the rest of the paper, we use the above notations.

The main results of this paper are as follows.

Theorem 2.1. *Let G be a p-group of almost maximal class of order p^n, $n \geq 4$.*

(i) *If $p = 2$, then $c(G) = 1$. Also, for $n \geq 5$, every element of G' can be expressed as $[g, s]$ for suitable $g \in \gamma_1(G)$.*

(ii) *If $p > 2$, then either $d(G) = 3$ and $c(G) = 1$ or $d(G) = 2$ and $c(G) = 1$ or 2.*

To illustrate the applications of our results, the following consequences are given.

Corollary 2.1. *Let G be a 2-group of almost maximal class of order 2^n, $n \geq 5$ and $s \in G \setminus \gamma_1(G)$. Then*

(i) *If G/G' has exponent 2, then every element of G can be expressed in the form $s^i s_1^j t^k [g, s]$ in which $0 \leq i, j, k < 2$ and $g \in \gamma_1(G)$ and $t \in C_G(s) \setminus G'$.*

(ii) *If $\gamma_1(G)/G'$ has exponent 4, then every element of G can be expressed in the form $s^i s_1^j [g, s]$ in which $0 \leq i < 2$, $0 \leq j < 4$ and $g \in \gamma_1(G)$.*

(iii) *If G/G' has exponent 4 and $\gamma_1(G)/G'$ has exponent 2, then every element of G can be expressed in the form $s^i s_1^j [g, s]$ in which $0 \leq i < 4$, $0 \leq j < 2$ and $g \in \gamma_1(G)$.*

As a consequence of Theorem 2.1 and by repeated application of Rhemtulla's [15] result, we have

Corollary 2.2. *Let P a 2-group with $cc(P) < 3$. Suppose $G = P \wr C_1 \wr \cdots \wr C_n$ where C_i is a finite cyclic group for $1 \leq i \leq n$. Then $\Gamma(G) = G'$.*

One interesting result is indicated by the following theorem.

Theorem 2.2. *Let A be a nontrivial finite abelian group and P be a 2-group of almost maximal class of order 2^n with $n \geq 5$. Let $G = A \wr P$.*

(i) *If P/P' has exponent 4, then $c(G) = 2$. In particular, every element of G' is a product of at most two commutators $[b_1, s_1][s, g]^b$, for suitable $g \in G$ and b, b_1 of the base of group G.*

(ii) *If P/P' has exponent 2, then $c(G) = 2$ or 3. In particular, every element of G' is a product of at most three commutators $[b_2, t][b_1, s_1][s, g]^b$, for suitable $g \in G$ and b, b_1, b_2 of the base of group G.*

Finally, we will provide various examples to illustrate that in Theorem 2.1 (ii) both $c(G) = 1$ and $c(G) = 2$ can occur.

3. PROOFS

To prove Theorem 2.1, we need the following results. The third is a result of Peter Stroud [17].

THEOREM 3.1 ([5, Theorem 1]). *If G is a p-group of maximal class, then $\Gamma(G) = G'$.*

In the following theorem, we state various sufficient conditions implying that $\Gamma(G) = G'$.

THEOREM 3.2. *The following conditions on a group G imply $\Gamma(G) = G'$:*

(i) *If G is nilpotent and G' is cyclic (see* [16, Corollary p. 642]*).*

(ii) *Let G be a finite p-group with G' elementary abelian of rank less than or equal to 3 (see* [14, Theorem 2.4]*).*

(iii) *G' is an abelian p-group for $p > 3$ and $d(G') \leq 3$ (see* [10, Theorem B]*).*

The following lemma is a result of Peter Stroud [17].

LEMMA 3.1. *Let $G = \langle x_1, x_2, ..., x_n \rangle$ be a nilpotent group. Then every element of G' is a product of n commutators $[x_1, g_1]...[x_n, g_n]$, for suitable g_i in G.*

We shall use the following well known identities for groups which are nilpotent of class 3.

LEMMA 3.2. *Let G be a nilpotent group of class 3 and let x,y be elements of G. Then, for all integers r,s the following hold:*

$$[x^r, y] = [x, y]^r [x, y, x]^{r(r-1)/2},$$

$$[x^r, y^s] = [x, y]^{rs} [x, y, x]^{rs(r-1)/2} [x, y, y]^{rs(s-1)/2}.$$

Now, we turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose G is a group of almost maximal class of order p^n. If $n = 4$, then $|G'| = p$ and $|G/G'| = p^3$. By Theorem 3.2 (i), $\Gamma(G) = G'$. We also include a direct proof. It is clear that the minimum number of generators of G, $d(G)$, equals to 2 or 3. Suppose $G = \langle a, b \rangle$. Since $G' \leq Z(G)$, we have $G' = \langle [a, b] \rangle = \{1, [a, b], \cdots, [a^{p-1}, b]\} = \Gamma(G)$. Now, if $G = \langle a, b, c \rangle$, then $G' = \langle [a, b], [a, c], [b, c] \rangle$. Since $|G'| = p$, we may assume $G' = \langle [a, b] \rangle = \{1, [a, b], \cdots, [a^{p-1}, b]\} = \Gamma(G)$.

In the rest of the proof, we may assume $n \geq 5$.

(i) For $n \geq 5$, we will use induction on n. If $n = 5$, then $G' = \langle s_2, s_3 \rangle$. Since $|G'| = 4$ and $|\gamma_i(G)/\gamma_{i+1}(G)| = 2$ for $i = 2, 3$, every element of G' has the form $[s_1^i, s][s_2^j, s] = [s_1^i s_2^j, s]$ where $0 \leq i, j < 2$.

Next, let $n > 5$ and G be a 2-group of almost maximal class of order 2^n. Let also $\overline{G} = G/\gamma_{n-2}(G)$. It is easy to prove that $\overline{G}$ is a 2-group of almost maximal class of order 2^{n-1} and $\gamma_i(\overline{G}) = \overline{\gamma_i(G)}$ for $1 \leq i \leq n-2$.

It is clear that $\overline{s} \notin \gamma_1(\overline{G})$. We claim that $\overline{s} \notin C_{\overline{G}}(\gamma_{n-4}(\overline{G}))$. Suppose instead $\overline{s} \in C_{\overline{G}}(\gamma_{n-4}(\overline{G}))$. Therefore $1 = [\overline{s_{n-4}}, \overline{s}] = \overline{s_{n-3}}$, hence $s_{n-3} \in \gamma_{n-2}(G)$ and so $\gamma_{n-3}(G) = \langle s_{n-3}, \gamma_{n-2}(G) \rangle = \gamma_{n-2}(G)$, a contradiction. Therefore, $\overline{s} \notin C_{\overline{G}}(\gamma_{n-4}(\overline{G}))$.

We know $s_1 \in \gamma_1(G) \backslash Z_{n-3}(G)$. We claim that $\overline{s_1} \notin Z_{n-4}(\overline{G})$. Suppose instead $\overline{s_1} \in Z_{n-4}(\overline{G})$, therefore for all $g_1, ..., g_{n-4} \in G$, we have $[\overline{s_1}, \overline{g_1}, ..., \overline{g_{n-4}}] = 1$, so $[s_1, g_1, ..., g_{n-4}] \in \gamma_{n-2}(G) \leq Z(G)$. Therefore $s_1 \in Z_{n-3}(G)$, a contradiction. By induction on n, $\overline{G}' = \{[\overline{g}, \overline{s}] | g \in \gamma_1(G)\}$. Hence if $\gamma \in G'$, then $\overline{\gamma} = [\overline{g}, \overline{s}]$ with $g \in \gamma_1(G)$. Thus $\gamma = [g, s]s_{n-2}^k = [g, s][s_{n-3}^k, s] = [gs_{n-3}^k, s]$ for $0 \leq k < 2$. This shows $\Gamma(G) = G'$ and completes the proof.

(ii) If $n = 5$, then G/G' is a group of order p^2 or p^3. Suppose $|G/G'| = p^3$. Then by Theorem 3.2, $\Gamma(G) = G'$.

Suppose $|G/G'| = p^2$. If $G = \langle a, b \rangle$, then $G' = \langle [a, b], [a, b, a], [a, b, b] \rangle$. We have $1 = [a, b, b^p] = [a, b, b]^p$ and $1 = [a, b, a^p] = [a, b, a]^p$ since $a^p, b^p \in G'$ and $cl(G) = 3$. Therefore $\gamma_3(G)$ has exponent p. By Lemma 3.2, we have $[b^p, a] = [b, a]^p [b, a, b]^{p(p-1)/2} = [b, a]^p$. So $|[a, b]| = p$ *or* p^2. We claim that $|[a, b]| = p$. Suppose instead, $|[a, b]| = p^2$. So there exist $x \in \{a, b\}$ such that $[a, b, x] \notin \langle [a, b] \rangle$, since otherwise $G' = \langle [a, b] \rangle$, a contradiction. Now, we have two cases:

(a) If $x = a$, then $G' = \langle [a, b] \rangle \times \langle [a, b, a] \rangle$. Now, $a^p \in G'$, therefore $a^p = [a, b]^i [a, b, a]^j$ for $0 \leq i < p^2$ and $0 \leq j < p$. Since G' is an abelian group, we have $1 = [a^p, a] = [[a, b]^i [a, b, a]^j, a] = [a, b, a]^i$. So $p|i$ and $[a^p, b] = [[a, b]^i, b] = [a, b, b]^i = 1$. But $[a, b]^p = [a^p, b] = 1$, a contradiction.

(b) If $x = b$, then by a similar argument, we will have the same contradiction as case (a). So G' is an elementary abelian group and by Theorem 3.2, $\Gamma(G) = G'$.

If $n \geq 6$ and d(G)=3, by ([8, p. 65]), G contains a subgroup H of maximal class in which $\gamma_i(H) = \gamma_i(G)$ for $2 \leq i \leq n-2$. Therefore by Theorem 3.1, $c(G) = 1$. If $d(G) = 2$, by Lemma 3.1, $c(G) \leq 2$. □

We will provide various examples to illustrate that in Theorem 2.1(ii) both $c(G) = 1$ and $c(G) = 2$ can occur.

Remark 3.3 ([6, Remark. 3.2]). Note that in a 2-group G of almost maximal class of order 2^n, $n \geq 5$, we have $|Z_{n-3}(G) : G'| = 2$ and $|G : G'| = 8$.

So $G/G' \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $G/G' \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

We use the following lemma, to prove Corollary 2.1.

Lemma 3.3 ([6, Lemmas 3.12, 3.13 and 3.15]). *Let G be a group of almost maximal class and order 2^n, $n \geq 5$.*

(i) *If $G/G' \cong \mathbb{Z}_2^3$, then $G = \langle s, s_1, t\rangle$ and $Z_{n-3}(G) = \langle t, G'\rangle$ where $t \in C_G(s) \setminus G'$.*

(ii) *If $\gamma_1(G)/G'$ is cyclic, then $G = \langle s, s_1\rangle$ and $Z_{n-3}(G) = \langle s_1^2, G'\rangle$.*

(iii) *If $G/G' \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, $\gamma_1(G)/G' \cong \mathbb{Z}_2^2$, then $G = \langle s, s_1\rangle$ and $Z_{n-3}(G) = \langle s^2, G'\rangle$.*

Proof of Corollary 2.1. Corollary 2.1 is an immediate consequence of Lemma 3.3 and Theorem 2.1. □

Proof of Corollary 2.2. Indeed, Rhemtulla [15], proved that the wreath product of a c_1-group by a finite cyclic group is again a c_1-group. Repeated application of Rhemtulla's result shows that the group G, satisfies the desired property and the proof is complete. □

Lemma 3.4 ([1, Lemma 3]). *Let A be a normal subgroup of $G = \langle x_1, ..., x_n\rangle$. If A is abelian or A lies in the second center $Z_2(G)$ of G, then every element of $[G, A]$ has the form $\prod_{i=1}^{n}[x_i, a_i]$, where $a_i \in A$.*

Proof of Theorem 2.2. (i) Let G be the wreath product of a nontrivial finite abelian group A and a 2-group of almost maximal class P. By Lemma 3.3, $P = \langle s, s_1\rangle$.

Let $B = Dr_{i=1}^{|P|}A_i$ where $A_i \cong A$, be the base group of G. Now $G = BP$ is the semidirect product of B by P. Hence $[B, P]$ is a normal subgroup of G and $G' = [BP, BP] = [B, P]P'$. Since B is a normal abelian subgroup of G, we see that $[P, B] = [G, B]$. Now by Lemma 3.4, $[B, P] = \{[b_1, s_1][b, s] | b, b_1 \in B\}$. By Theorem 2.1, every element of G' has the form $\gamma = [b_1, s_1][b, s][s, g] = [b_1, s_1][s, gb^{-1}]^b$. Hence $c(\gamma) \leq 2$. Now by [9, Theorem 1] some element of $[P, B]$ is not a commutator. Thus $c(G) = 2$, as required.

The proof of (ii) is similar to the proof of (i). □

4. EXAMPLES

In this section, we collect several examples.

Example 1. To give this example, we will use [14, Corollary 5.6]. First, consider the abelian group $W = \langle u\rangle \times \langle v\rangle \times \langle w\rangle \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ where $|u| = 9$,

$|v| = |w| = 3$. The group G will be constructed by two suitable split extensions, starting with W. Let $A = W \rtimes \langle a \rangle$ in which $|a| = 3$. The action of a on the generators of W is given as follows: $[u, a] = v$, $[v, a] = u^6$ *and* $[w, a] = 1$. Let $G = A \rtimes \langle c \rangle$ in which $|c| = 3$. The action of c on the generators of A is given as follows: $[a, c] = uw^{-1}$, $[u, c] = vw^{-1}$, $[v, c] = u^6$, and $[w, c] = 1$. According to [14, Corollary 5.6], $cl(G) = 4$ and $\Gamma(G) \neq G'$.

It is clear that $|G| = 3^6$, hence $cc(G) = 2$. By Theorem 2.1, $c(G) = 2$ and $d(G) = 2$.

Example 2. To give this example, we will use [14, Corollary 5.4]. Let $p \geq 5$ be a prime. Let $V = \langle u \rangle \times \langle v \rangle \times \langle w \rangle \times \langle z \rangle \cong \mathbb{Z}_p^4$ where $|u| = |v| = |w| = |z| = p$. The group G will be constructed by two suitable split extensions, starting with V. Let $B = V \rtimes \langle b \rangle$ in which $|b| = p$. The action of b on the generators of V is given as follows: $[u, b] = w$, *and* $[v, b] = [w, b] = [z, b] = 1$. Let $G = B \rtimes \langle a \rangle$ in which $|a| = p$. The action of a on the generators of B is given as follows: $[b, a] = u$, $[u, a] = v$, $[v, a] = z$ *and* $[w, a] = [z, a] = 1$. According to [14, Corollary 5.4], $cl(G) = 4$ and $\Gamma(G) \neq G'$.

It is clear that $|G| = p^6$, hence $cc(G) = 2$. By Theorem 2.1, $c(G) = 2$ and $d(G) = 2$.

Next, we give an example of a p-group of almost maximal class of order p^6 for $p > 3$ such that $\Gamma(G) = G'$.

Example 3. Let $G = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma | [\alpha_1, \beta_1] = [\alpha_2, \beta_2] = [\beta, \beta_i] = [\beta_1, \beta_2] = [\alpha_i, \gamma] = [\beta_i, \gamma] = [\beta, \gamma] = 1, [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, [\alpha_1, \beta_2] = [\alpha_2, \beta_1] = \beta^p = \gamma, \alpha_1^p = \beta_1^{-1}\gamma^{-1/2}, \alpha_2^p = \beta_2\gamma^{1/2}, \beta_i^p = \gamma^p = 1, \quad (i = 1, 2) \rangle$ such that $p > 3$. According to [13, p. 619], $|G| = p^6$, $G' \cong (\mathbb{Z}_p)^2 \times \mathbb{Z}_{p^2}$ and $cl(G) = 4$. Hence $cc(G) = 2$ and $d(G') = 3$, therefore by Theorem 3.2, $\Gamma(G) = G'$.

Finally, let p be a prime. We give examples of finite p-groups of almost maximal class in which the commutator length is equal to 1. We will use a special case of a theorem of M. Akhavan-Malayeri [4].

Example 4. Let p be a prime. Set $G = \mathbb{Z}_{p^2} \wr \mathbb{Z}_p$. By [4, Theorem 2] $cl(G) = 2p - 1$, $d(G) = 2$ and $c(G) = 1$. It is clear that $|G| = p^{2p+1}$ therefore, $cc(G) = 2$.

Example 5. Let $G = \langle x, y, t : x^{2^{n-2}} = t^2 = y^2 = 1,\ x^y = x^{-1+2^{n-4}}t,\ x^t = x^{2^{n-3}+1},\ t^y = t \rangle$. According to [7, p. 101], for $n \geq 6$, G is a group of almost maximal class of order 2^n. By Theorem 2.1 (i), $c(G) = 1$.

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