ON THE KERNEL OF THE PROJECTION MAP $T(V) \to S(V)$

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If V is a vector space over some field F, then we have the well known exact sequence $0 \to \Lambda^2(V) \to T^2(V) \to S^2(V) \to 0$, where the first map is given by $x \wedge y \to x \otimes y - y \otimes x$ and the second by $x \otimes y \mapsto xy$. The obvious generalization, an exact sequence, $0 \to \Lambda^k(V) \to T^k(V) \to S^k(V) \to 0$, does not hold when the degree k is ≥ 3. In this paper, we produce two generalizations, which hold for arbitrary degrees. These results are a pre-requisite for a future paper. In the first section we describe, in terms of generators and relations, the kernel of the projection map $\rho_{T,S} : T(V) \to S(V)$, given by $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n$. Namely, we will define a $\mathbb{Z}_{\geq 2}$-graded $T(V)$-bimodule $M(V)$, which is a quotient of the $T(V)$-bimodule $T(V) \otimes \Lambda^2(V) \otimes T(V)$, and a morphism of $T(V)$-bimodules $\rho_{M,T} : M(V) \to T(V)$, such that the sequence

$$0 \to M(V) \xrightarrow{\rho_{M,T}} T(V) \xrightarrow{\rho_{T,S}} S(V) \to 0$$

is exact. In the second section, we define the algebra $S'(V)$ as $T(V)$ factorised by the bilateral ideal generated by $x \otimes y \otimes z - y \otimes z \otimes x$, with $x, y, z \in V$, and we prove that there is a short exact sequence,

$$0 \to \Lambda^{\geq 2}(V) \xrightarrow{\rho_{\Lambda^{\geq 2},S'}} S'(V) \xrightarrow{\rho_{S',S}} S(V) \to 0.$$

When considering the homogeneous components of degree 2, we have $M^2(V) = \Lambda^2(V)$ and $S'^2(V) = T^2(V)$ so, in both cases, we obtain the exact sequence $0 \to \Lambda^2(V) \to T^2(V) \to S^2(V) \to 0$ mentioned above.

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1. THE BIMODULE $M(V)$

Let V be a vector space over a field F and let $(v_i)_{i \in I}$ be a basis, where $(I, \leq)$ is a totally ordered set.

Let $\rho_{T,S} : T(V) \to S(V)$ be the canonical projection and, for $n \geq 0$, let $\rho_{T^n,S^n} : T^n(V) \to S^n(V)$ be its homogeneous component of degree n.

We denote by $[\cdot, \cdot] : T(V) \times T(V) \to T(V)$ the commutator map,

$$[\xi, \eta] = \xi \otimes \eta - \eta \otimes \xi.$$

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Then $\ker\rho_{T,S}$ is the bilateral ideal generated by $[x,y]$, with $x,y\in V$. We want to describe $\ker\rho_{T,S}$ in terms of generators and relations. On homogeneous components, for $n=0,1$ we have $T^n(V)=S^n(V)$ so $\ker\rho_{T^n,S^n}=0$. The first interesting case is $n=2$. The map $V^2\to T^2(V)$, given by $(x,y)\mapsto[x,y]$, is bilinear and alternating, so it induces a linear map $\rho_{\Lambda^2,T^2}:\Lambda^2(V)\to T^2(V)$, given by $x\wedge y\mapsto[x,y]$. Then, we have the following well known and elementary result.

Proposition 1.1. *We have an exact sequence*

$$0\to\Lambda^2(V)\xrightarrow{\rho_{\Lambda^2,T^2}}T^2(V)\xrightarrow{\rho_{T^2,S^2}}S^2(V)\to 0.$$

We now consider the $T(V)$-bimodule $T(V)\otimes\Lambda^2(V)\otimes T(V)$, generated by $\Lambda^2(V)$. It is $\mathbb{Z}_{\geq 2}$ graded, where for every $n\geq 2$ homogeneous component of degree n is

$$(T(V)\otimes\Lambda^2(V)\otimes T(V))^n=\bigoplus_{i+j=n-2}T^i(V)\otimes\Lambda^2(V)\otimes T^j(V).$$

Note that we can define the commutator $[\cdot,\cdot]$, by the same formula, $[\xi,\eta]=\xi\otimes\eta-\eta\otimes\xi$, also as

$$[\cdot,\cdot]:T(V)\times(T(V)\otimes\Lambda^2(V)\otimes T(V))\to T(V)\otimes\Lambda^2(V)\otimes T(V),$$

or as

$$[\cdot,\cdot]:(T(V)\otimes\Lambda^2(V)\otimes T(V))\times T(V)\to T(V)\otimes\Lambda^2(V)\otimes T(V).$$

We consider the map $1\otimes\rho_{\Lambda^2,T^2}\otimes 1:T(V)\otimes\Lambda^2(V)\otimes T(V)\to T(V)$. Note that $1\otimes\rho_{\Lambda^2,T^2}\otimes 1$ is a morphism of graded $T(V)$-bimodules.

Now $T(V)\otimes\Lambda^2(V)\otimes T(V)$ is spanned by $\xi\otimes x\wedge y\otimes\eta$, with $x,y\in V$ and $\xi,\eta\in T(V)$, so $\mathrm{Im}(1\otimes\rho_{\Lambda^2,T^2}\otimes 1)$ is spanned by

$$(1\otimes\rho_{\Lambda^2,T^2}\otimes 1)(\xi\otimes x\wedge y\otimes\eta)=\xi\otimes[x,y]\otimes\eta,$$

i.e. it is the ideal of $T(V)$ generated by $[x,y]$, with $x,y\in V$. Therefore $\mathrm{Im}(1\otimes\rho_{\Lambda^2,T^2}\otimes 1)=\ker\rho_{T,S}$ and we have the exact sequence

$$T(V)\otimes\Lambda^2(V)\otimes T(V)\xrightarrow{1\otimes\rho_{\Lambda^2,T^2}\otimes 1}T(V)\xrightarrow{\rho_{T,S}}S(V)\to 0.$$

It follows that $\ker\rho_{T,S}\cong\frac{T(V)\otimes\Lambda^2(V)\otimes T(V)}{\ker(1\otimes\rho_{\Lambda^2,T^2}\otimes 1)}$.

Lemma 1.2. *The following elements of $T(V)\otimes\Lambda^2(V)\otimes T(V)$ belong to $\ker(1\otimes\rho_{\Lambda^2,T^2}\otimes 1)$.*

(i) $[x,y]\otimes\xi\otimes z\wedge t-x\wedge y\otimes\xi\otimes[z,t]$, *with $x,y,z,t\in V$, $\xi\in T(V)$.*

(ii) $[x,y\wedge z]+[y,z\wedge x]+[z,x\wedge y]$, *with $x,y,z\in V$.*

Proof. (i) We have

$$(1\otimes\rho_{\Lambda^2,T^2}\otimes 1)([x,y]\otimes\xi\otimes z\wedge t-x\wedge y\otimes\xi\otimes[z,t])$$
$$=[x,y]\otimes\xi\otimes[z,t]-[x,y]\otimes\xi\otimes[z,t]=0.$$

(ii) By the Jacobi identity, we have

$$(1\otimes\rho_{\Lambda^2,T^2}\otimes 1)([x,y\wedge z]+[y,z\wedge x]+[z,x\wedge y])=[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

□

Definition 1. Let $M(V)=(T(V)\otimes\Lambda^2(V)\otimes T(V))/W_M(V)$, where $W_M(V)$ is the subbimodule of $T(V)\otimes\Lambda^2(V)\otimes T(V)$ generated by

$$[x,y]\otimes\xi\otimes z\wedge t-x\wedge y\otimes\xi\otimes[z,t],$$

with $x,y,z,t\in V$, and $\xi\in T(V)$, and $[x,y\wedge z]+[y,z\wedge x]+[z,x\wedge y]$, with $x,y,z\in V$.

If $\eta\in T(V)\otimes\Lambda^2(V)\otimes T(V)$ then we denote by $[\eta]$ its class in $M(V)$.

On $M(V)$ we keep the notation $\otimes$ for the left and right multiplication from $T(V)\otimes\Lambda^2(V)\otimes T(V)$. That is

$$\xi\otimes[\eta]\otimes\xi':=[\xi\otimes\eta\otimes\xi']\quad\forall\xi,\xi'\in T(V),\ [\eta]\in M(V).$$

Note that $W_M(V)$ is generated by homogeneous elements so it is homogeneous. (In the formula $[x,y]\otimes\xi\otimes z\wedge t-x\wedge y\otimes\xi\otimes[z,t]$ we may restrict ourselves to $\xi\in T^k(V)$, with $k\geq 0$, which makes it homogeneous of degree $k+4$.) Therefore, $M(V)$ inherits from $T(V)\otimes\Lambda^2(V)\otimes T(V)$ the property of being a $\mathbb{Z}_{\geq 2}$-graded $T(V)$-bimodule.

By Lemma 1.2, we have $W_M(V)\subseteq\ker(1\otimes\rho_{\Lambda^2,T^2}\otimes 1)$. It follows that $1\otimes\rho_{\Lambda^2,T^2}\otimes 1:T(V)\otimes\Lambda^2(V)\otimes T(V)\to T(V)$ induces a morphism of graded bimodules $\rho_{M,T}:M(V)\to T(V)$, given by $[\xi\otimes x\wedge y\otimes\eta]\mapsto\xi\otimes[x,y]\otimes\eta$ $\forall x,y\in V$ and $\xi,\eta\in T(V)$. Moreover, we have the exact sequence

$$M(V)\xrightarrow{\rho_{M,T}}T(V)\xrightarrow{\rho_{T,S}}S(V)\to 0.$$

Since $\rho_{M,T}$ is a morphism of graded bimodules, we may consider its homogenous components, $\rho_{M^n,T^n}:M^n(V)\to T^n(V)$.

THEOREM 1.3. *We have* $W_M(V)=\ker(1\otimes\rho_{\Lambda^2,T^2}\otimes 1)$, *i.e.* $\rho_{M,T}$ *is injective and we have the exact sequence*

$$0\to M(V)\xrightarrow{\rho_{M,T}}T(V)\xrightarrow{\rho_{T,S}}S(V)\to 0.$$

Proof. We use induction on n to prove that ρ_{M^n,T^n} is injective. If $n=0,1$ then $(T(V)\otimes\Lambda^2(V)\otimes T(V))^n=0$, so $M^n(V)=0$, so there is nothing to prove.

Before proving the induction step, we need some preliminary results.

We have an action of the symmetric group S_n on $T^n(V)$, given by

$$\tau(x_1\otimes\cdots\otimes x_n)=x_{\tau^{-1}(1)}\otimes\cdots\otimes x_{\tau^{-1}(n)}.$$

For $1\le i\le n-1$, we denote by τ_i the transposition $(i,i+1)\in S_n$ and we denote by $f_i:T^n(V)\to M^n(V)$ the linear map given by

$$x_1\otimes\cdots\otimes x_n\mapsto[x_1\otimes\cdots\otimes x_i\wedge x_{i+1}\otimes\cdots\otimes x_n].$$

(Here we repaced the $\otimes$ sign between x_i and x_{i+1} by $\wedge$.)

LEMMA 1.4. *On $T^n(V)$ we have $\rho_{M^n,T^n}f_i=1-\tau_i$.*

Proof. We verify this relation on generators $\xi=x_1\otimes\cdots\otimes x_n$ of $T^n(V)$. We have

$$\begin{aligned}\rho_{M^n,T^n}f_i(\xi)&=\rho_{M^n,T^n}([x_1\otimes\cdots\otimes x_i\wedge x_{i+1}\otimes\cdots\otimes x_n])\\&=x_1\otimes\cdots\otimes[x_i,x_{i+1}]\otimes\cdots\otimes x_n\\&=x_1\otimes\cdots\otimes(x_i\otimes x_{i+1}-x_{i+1}\otimes x_i)\otimes\cdots\otimes x_n\\&=x_1\otimes\cdots\otimes x_n-x_1\otimes\cdots\otimes x_{i+1}\otimes x_i\cdots\otimes x_n\\&=x_1\otimes\cdots\otimes x_n-x_{\tau_i(1)}\otimes\cdots\otimes x_{\tau_i(n)}=\xi-\tau_i(\xi).\end{aligned}$$

□

COROLLARY 1.5. *For every $1\le i_1,\dots,i_s\le n-1$ in $T(V)$, we have*

$$1-\tau_{i_s}\cdots\tau_{i_1}=\rho_{M^n,T^n}\sum_{k=1}^s f_{i_k}\tau_{i_{k-1}}\cdots\tau_{i_1}.$$

Proof. By Lemma 1.4, for $1\le k\le s$, we have $\rho_{M^n,T^n}f_{i_k}=1-\tau_{i_k}$, so $\rho_{M^n,T^n}f_{i_k}\tau_{i_{k-1}}\cdots\tau_{i_1}=(1-\tau_{i_k})\tau_{i_{k-1}}\cdots\tau_{i_1}$. Hence, $\rho_{M^n,T^n}\sum_{k=1}^s f_{i_k}\tau_{i_{k-1}}\cdots\tau_{i_1}$ is equal to the telescoping sum $\sum_{k=1}^s(1-\tau_{i_k})\tau_{i_{k-1}}\cdots\tau_{i_1}=1-\tau_{i_s}\cdots\tau_{i_1}$. □

LEMMA 1.6. (i) *If $\tau\in S_n$ then there is a map $h_\tau:T(V)\to M(V)$ with $h_\tau=\sum_{k=1}^s f_{i_k}\tau_{i_{k-1}}\cdots\tau_{i_1}$ whenever $\tau=\tau_{i_s}\cdots\tau_{i_1}$. In particular, $h_1=0$ and $h_{\tau_i}=f_i$.*

(ii) *$h_{\sigma\tau}=h_\tau+h_\sigma\tau$ $\forall\sigma,\tau\in S_n$.*

(iii) *On $T^n(V)$ we have $\rho_{M^k,T^k}h_\tau=1-\tau$ $\forall\tau\in S_n$.*

Proof. (i) We use the fact that S_n is generated by $\tau_1,\dots,\tau_{n-1}$, with the relations $\tau_i^2=1$, $\tau_i\tau_j=\tau_j\tau_i$ if $j-i\ge 2$ and $(\tau_i\tau_{i+1})^3=1$. (See, e.g., [1, Theorem 4.1, pag. 152].)

We consider the set of symbols $A=\{\sigma_1,\dots,\sigma_{n-1}\}$. Then S_n is isomorphic to the free monoid $(A\cup A^{-1})^*$ factored by the equivalence relation $\sim$, generated by $\alpha\beta\sim\alpha\gamma\beta$ for every $\alpha,\beta\in(A\cup A^{-1})^*$ and γ of the form $\gamma=\sigma_i\sigma_i^{-1}$

or $\sigma_i^{-1}\sigma_i$, $\gamma = \sigma_i^2$, $\gamma = (\sigma_i\sigma_{i+1})^3$ or $\gamma = \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}$, with $j - i \geq 2$. For any $\sigma \in (A \cup A^{-1})^*$ we denote by $[\sigma]$ its class in $(A \cup A^{-1})^*_{/\sim}$. If $\psi : (A \cup A^{-1})^* \to S_n$ is the morphism of monoids given by $\sigma_i \mapsto \tau_i$ and $\sigma_i^{-1} \mapsto \tau_i^{-1} = \tau_i$, then ψ induces an isomorphism $\tilde{\psi} : (A \cup A^{-1})^*_{/\sim} \to S_n$, given by $\tilde{\psi}([\sigma]) = \psi(\sigma)$ $\forall \sigma \in (A \cup A^{-1})^*$.

For every $\sigma = \sigma_{i_s}^{\pm 1} \cdots \sigma_{i_1}^{\pm 1} \in (A \cup A^{-1})^*$, we define the map

$$g_\sigma = \sum_{k=1}^{s} f_{i_k} \tau_{i_{k-1}} \cdots \tau_{i_1}.$$

(If $\sigma = 1$ then $s = 0$ so $g_1 := 0$.)

Note that if $\alpha = \sigma_{i_s}^{\pm 1} \cdots \sigma_{i_{t+1}}^{\pm 1}$ and $\beta = \sigma_{i_t}^{\pm 1} \cdots \sigma_{i_1}^{\pm 1}$ then $\psi(\beta) = \tau_{i_t} \cdots \tau_{i_1}$ so

$$\begin{aligned} g_{\alpha\beta} &= \sum_{k=1}^{s} f_{i_k} \tau_{i_{k-1}} \cdots \tau_{i_1} = \sum_{k=1}^{t} f_{i_k} \tau_{i_{k-1}} \cdots \tau_{i_1} + \left(\sum_{k=t+1}^{s} f_{i_k} \tau_{i_{k-1}} \cdots \tau_{i_{t+1}} \right) \tau_{i_t} \cdots \tau_{i_1} \\ &= g_\beta + g_\alpha \psi(\beta). \end{aligned}$$

We now prove that if $\sigma \sim \sigma'$ then $g_\sigma = g_{\sigma'}$. It suffices to take the case when $\sigma = \alpha\gamma\beta$ and $\sigma' = \alpha\beta$, where $\alpha, \beta \in (A \cup A^{-1})^*$ and γ is of the form $\sigma_i\sigma_i^{-1}$, $\sigma_i^{-1}\sigma_i$, σ_i^2, $(\sigma_i\sigma_{i+1})^3$ or $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}$, with $j - i \geq 2$. Note that, in all these cases, we have $\psi(\gamma) = 1$. (We have $\tau_i^2 = 1$, $(\tau_i\tau_{i+1})^3 = 1$ and, if $j - i \geq 2$, then $\tau_i\tau_j\tau_i\tau_j = \tau_i^2\tau_j^2 = 1$.)

The relation $g_{\alpha\beta} = g_{\alpha\gamma\beta}$ writes as $g_\beta + g_\alpha\psi(\beta) = g_\beta + g_{\alpha\gamma}\psi(\beta)$ so it suffices to prove that $g_\alpha = g_{\alpha\gamma}$. But $\psi(\gamma) = 1$ so $g_{\alpha\gamma} = g_\gamma + g_\alpha\psi(\gamma) = g_\gamma + g_\alpha$. Hence, we must prove that $g_\gamma = 0$. We prove that $g_\gamma(\eta) = 0$ for $\eta = x_1 \otimes \cdots \otimes x_n$.

If $\gamma = \sigma_i^{\pm 1}\sigma_i^{\pm 1}$, which includes the cases $\gamma = \sigma_i\sigma_i^{-1}$, $\sigma_i^{-1}\sigma_i$ and σ_i^2, we have

$$\begin{aligned} g_\gamma(\eta) &= f_i(\eta) + f_i(\tau_i(\eta)) \\ &= f_i(x_1 \otimes \cdots \otimes x_n) + f_i(x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n) \\ &= x_1 \otimes \cdots \otimes x_i \wedge x_{i+1} \otimes \cdots \otimes x_n + x_1 \otimes \cdots \otimes x_{i+1} \wedge x_i \otimes \cdots \otimes x_n = 0. \end{aligned}$$

Let now $\gamma = (\sigma_i\sigma_{i+1})^3 = \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}$. We have $\eta = \eta' \otimes x \otimes y \otimes z \otimes \eta''$, where $\eta' = x_1 \otimes \cdots \otimes x_{i-1}$, $\eta'' = x_{i+3} \otimes \cdots \otimes x_n$ and $(x, y, z) = (x_i, x_{i+1}, x_{i+2})$. Note that when we apply successively the transpositions $\tau_i = (i, i+1)$ and $\tau_{i+1} = (i+1, i+2)$ to η only the factors x, y, z of η are permuted, while the factors η' and η'' are unchanged. The factors on the positions i, $i+1$ and $i+2$ in η are x, y, z; in $\tau_{i+1}(\eta)$ they are x, z, y; in $\tau_i\tau_{i+1}(\eta)$ they are z, x, y; in $\tau_{i+1}\tau_i\tau_{i+1}(\eta)$ they are z, y, x; in $\tau_i\tau_{i+1}\tau_i\tau_{i+1}(\eta)$ they are y, z, x; and in $\tau_{i+1}\tau_i\tau_{i+1}\tau_i\tau_{i+1}(\eta)$ they are y, x, z. Therefore,

$$\begin{aligned}g_\gamma(\eta) &= f_{i+1}(\eta) + f_i\tau_{i+1}(\eta) + f_{i+1}\tau_i\tau_{i+1}(\eta) + f_i\tau_{i+1}\tau_i\tau_{i+1}(\eta)\\ &\qquad + f_{i+1}\tau_i\tau_{i+1}\tau_i\tau_{i+1}(\eta) + f_i\tau_{i+1}\tau_i\tau_{i+1}\tau_i\tau_{i+1}(\eta)\\ &= f_{i+1}(\eta'\otimes x\otimes y\otimes z\otimes\eta'') + f_i(\eta'\otimes x\otimes z\otimes y\otimes\eta'')\\ &\qquad + f_{i+1}(\eta'\otimes z\otimes x\otimes y\otimes\eta'') + f_i(\eta'\otimes z\otimes y\otimes x\otimes\eta'')\\ &\qquad + f_{i+1}(\eta'\otimes y\otimes z\otimes x\otimes\eta'') + f_i(\eta'\otimes y\otimes x\otimes z\otimes\eta'')\\ &= [\eta'\otimes x\otimes y\wedge z\otimes\eta''] + [\eta'\otimes x\wedge z\otimes y\otimes\eta''] + [\eta'\otimes z\otimes x\wedge y\otimes\eta'']\\ &\quad + [\eta'\otimes z\wedge y\otimes x\otimes\eta''] + [\eta'\otimes y\otimes z\wedge x\otimes\eta''] + [\eta'\otimes y\wedge x\otimes z\otimes\eta''].\end{aligned}$$

Thus $g_\gamma(\eta) = [\eta'\otimes\xi\otimes\eta'']$, where

$$\begin{aligned}\xi &= x\otimes y\wedge z + x\wedge z\otimes y + z\otimes x\wedge y + z\wedge y\otimes x + y\otimes z\wedge x + y\wedge x\otimes z\\ &= x\otimes y\wedge z - z\wedge x\otimes y + z\otimes x\wedge y - y\wedge z\otimes x + y\otimes z\wedge x - x\wedge y\otimes z\\ &= [x, y\wedge z] + [y, z\wedge x] + [z, x\wedge y].\end{aligned}$$

We have $\xi\in W_M(V)$ so $\eta'\otimes\xi\otimes\eta''\in W_M(V)$ and so $g_\gamma(\eta) = [\eta'\otimes\xi\otimes\eta''] = 0$.

Let now $\gamma = \sigma_i\sigma_j\sigma_i\sigma_j$. We have $\eta = \eta'\otimes x\otimes y\otimes\xi\otimes z\otimes t\otimes\eta''$, where $\eta' = x_1\otimes\cdots\otimes x_{i-1}$, $\xi = x_{i+2}\otimes\cdots\otimes x_{j-1}$, $\eta'' = x_{j+2}\otimes\cdots\otimes x_n$, $(x, y) = (x_i, x_{i+1})$ and $(z, t) = (x_j, x_{j+1})$. Note that τ_i permutes the factors x and y of η and leaves all the other factors unchanged, while τ_j permutes z and t and leaves all the other factors unchanged. We get $\tau_j(\eta) = \eta'\otimes x\otimes y\otimes\xi\otimes t\otimes z\otimes\eta''$, $\tau_i\tau_j(\eta) = \eta'\otimes y\otimes x\otimes\xi\otimes t\otimes z\otimes\eta''$ and $\tau_j\tau_i\tau_j(\eta) = \eta'\otimes y\otimes x\otimes\xi\otimes z\otimes t\otimes\eta''$. Then

$$\begin{aligned}g_\gamma(\eta) &= f_j(\eta) + f_i\tau_j(\eta) + f_j\tau_i\tau_j(\eta) + f_i\tau_j\tau_i\tau_j(\eta)\\ &= f_j(\eta'\otimes x\otimes y\otimes\xi\otimes z\otimes t\otimes\eta'') + f_i(\eta'\otimes x\otimes y\otimes\xi\otimes t\otimes z\otimes\eta'')\\ &\quad + f_j(\eta'\otimes y\otimes x\otimes\xi\otimes t\otimes z\otimes\eta'') + f_i(\eta'\otimes y\otimes x\otimes\xi\otimes z\otimes t\otimes\eta'')\\ &= [\eta'\otimes x\otimes y\otimes\xi\otimes z\wedge t\otimes\eta''] + [\eta'\otimes x\wedge y\otimes\xi\otimes t\otimes z\otimes\eta'']\\ &\quad + [\eta'\otimes y\otimes x\otimes\xi\otimes t\wedge z\otimes\eta''] + [\eta'\otimes y\wedge x\otimes\xi\otimes z\otimes t\otimes\eta''].\end{aligned}$$

Thus $g_\gamma(\eta) = [\eta'\otimes\xi'\otimes\eta'']$, where

$$\begin{aligned}\xi' &= x\otimes y\otimes\xi\otimes z\wedge t + x\wedge y\otimes\xi\otimes t\otimes z + y\otimes x\otimes\xi\otimes t\wedge z\\ &\quad + y\wedge x\otimes\xi\otimes z\otimes t\\ &= x\otimes y\otimes\xi\otimes z\wedge t + x\wedge y\otimes\xi\otimes t\otimes z - y\otimes x\otimes\xi\otimes z\wedge t\\ &\quad - x\wedge y\otimes\xi\otimes z\otimes t\\ &= [x, y]\otimes\xi\otimes z\wedge t - x\wedge y\otimes\xi\otimes[z, t].\end{aligned}$$

We have $\xi'\in W_M(V)$ so $\eta'\otimes\xi'\otimes\eta''\in W_M(V)$ and $g_\gamma(\eta) = [\eta'\otimes\xi'\otimes\eta''] = 0$.

Since the map $\sigma\mapsto g_\sigma$, defined on $(A\cup A^{-1})^*$, is invariant to the equivalence relation $\sim$, it induces a map defined on $(A\cup A^{-1})^*_{/\sim}$, given by $[\sigma]\mapsto g_\sigma$. Since $\bar\psi : (A\cup A^{-1})^*_{/\sim}\to S_n$ is an isomorphism, we get a map $\tau\mapsto h_\tau$, where

$h_\tau = g_\sigma$ for any $\sigma \in (A \cup A^{-1})^*$ such that $\tau = \bar{\psi}([\sigma]) = \psi(\sigma)$. If $\tau = \tau_{i_s} \cdots \tau_{i_1}$ then $\tau = \psi(\sigma)$, with $\sigma = \sigma_{i_s} \cdots \sigma_{i_1}$. Hence, $h_\tau = g_\sigma = \sum_{k=1}^{s} f_k \tau_{k-1} \cdots \tau_1$, as claimed.

(ii) Let $\alpha, \beta \in (A \cup A^{-1})^*$ with $\sigma = \psi(\alpha)$ and $\tau = \psi(\beta)$ so that $\sigma\tau = \psi(\alpha\beta)$. Then $h_\sigma = g_\alpha$, $h_\tau = g_\beta$ and $h_{\sigma\tau} = g_{\alpha\beta} = g_\beta + g_\alpha \psi(\beta) = h_\tau + h_\sigma \tau$.

(iii) If we write $\tau = \tau_{i_s} \cdots \tau_{i_1}$ then our result is just Corollary 1.5. □

Proof of the induction step. We must prove that if $[\eta] \in \ker \rho_{M^n,T^n}$ then $[\eta] = 0$. Note that η is a finite linear combination of products of the form $v_{i_1} \otimes \cdots \otimes v_{i_k} \wedge v_{i_{k+1}} \otimes \cdots \otimes v_{i_n}$, with $i_1, \ldots, i_n \in I$ and $i_k < i_{k+1}$. We denote by $J = \{j_1, \ldots, j_m\}$ with $j_1, \ldots, j_m \in I$, $j_1 < \cdots < j_m$, the set of all indices $i \in I$ such that v_i is one of the factors v_{i_h} from one of the products in the linear combination that gives η. We will prove our result by induction on m. If $m = 0$ then $\eta = 0$ so there is nothing to be proved. Suppose that $m \geq 1$. Let $J' = \{j_1, \ldots, j_{m-1}\}$.

We have $\eta \in W$, where $W \subseteq (T(V) \otimes \Lambda^2(V) \otimes T(V))^n$ is spanned by $v_{i_1} \otimes \cdots \otimes v_{i_k} \wedge v_{i_{k+1}} \otimes \cdots \otimes v_{i_n}$, with $(i_1, \ldots, i_n; k) \in A$, where A is the set of all $(i_1, \ldots, i_n; k)$ with $i_1, \ldots, i_n \in J$, $1 \leq k \leq n-1$ and $i_k < i_{k+1}$. We also denote by $U \subseteq T^n(V)$ the space generated by $v_{i_1} \otimes \cdots \otimes v_{i_n}$, with $(i_1, \ldots, i_n) \in B := J^n$.

We have $A = A_1 \sqcup A_2$ where A_1 is the set of all $(i_1, \ldots, i_n; k) \in A$ with $i_1, \ldots, i_n \in J'$ and A_2 is the set of those where at least one of $i_1, \ldots, i_n$ is j_m. Similarly, $B = B_1 \sqcup B_2$ where $B_1 = J'^n$ and $B_2 = J^n \setminus J'^n$, i.e. B_2 is the set of all $(i_1, \ldots, i_n) \in J^n$ such that at least one of $i_1, \ldots, i_n$ is j_m. For $\alpha = 1, 2$ we denote by W_α the subspace of W spanned by $v_{i_1} \otimes \cdots \otimes v_{i_k} \wedge v_{i_{k+1}} \otimes \cdots \otimes v_{i_n}$ with $(i_1, \ldots, i_n; k) \in A_\alpha$ and by U_α the subspace of U spanned by $v_{i_1} \otimes \cdots \otimes v_{i_n}$ with $(i_1, \ldots, i_n) \in B_\alpha$. Then from $A = A_1 \sqcup A_2$ and $B = B_1 \sqcup B_2$ we deduce that $W = W_1 \oplus W_2$ and $U = U_1 \oplus U_2$.

We have $(1 \otimes \rho_{\Lambda^2,T^2} \otimes 1)(v_{i_1} \otimes \cdots \otimes v_{i_k} \wedge v_{i_{k+1}} \otimes \cdots \otimes v_{i_n}) = v_{i_1} \otimes \cdots \otimes v_{i_n} - v_{i_1} \otimes \cdots \otimes v_{i_{k+1}} \otimes v_{i_k} \otimes \cdots \otimes v_{i_n}$. If $(i_1, \ldots, i_n; k) \in A$, A_1 or A_2 then both $(i_1, \ldots, i_n)$ and $(i_1, \ldots, i_{k+1}, i_k, \ldots, i_n)$ belong to B, B_1 or B_2 and so $(1 \otimes \rho_{\Lambda^2,T^2} \otimes 1)(v_{i_1} \otimes \cdots \otimes v_{i_k} \wedge v_{i_{k+1}} \otimes \cdots \otimes v_{i_n}) \in U$, U_1 or U_2, respectively. It follows that $(1 \otimes \rho_{\Lambda^2,T^2} \otimes 1)(W) \subseteq U$ and $(1 \otimes \rho_{\Lambda^2,T^2} \otimes 1)(W_\alpha) \subseteq U_\alpha$ for $\alpha = 1, 2$. Equivalently, if $\xi \in W$, W_1 or W_2 then $\rho_{M^n,T^n}([\xi]) = (1 \otimes \rho_{\Lambda^2,T^2} \otimes 1)(\xi) \in U$, U_1 or U_2, respectively.

We define $\psi : U_2 \to M^n(V)$ on elements on the basis as follows. If $\xi = v_{i_1} \otimes \cdots \otimes v_{i_n}$ with $(i_1, \ldots, i_n) \in B_2$ and l is the smallest index vith $i_l = j_m$, then $\psi(\xi) := h_{\sigma_l}(\xi)$, where $\sigma_l \in S_n$ is the cyclic permutation $(1, 2, \ldots, l)$.

After these preliminaries, we start our proof of the induction step.

Since $W = W_1 \oplus W_2$ we have $\eta = \eta_1 + \eta_2$, with $\eta_\alpha \in W_\alpha$. Then $\rho_{M^n,T^n}([\eta_\alpha]) \in U_\alpha$. Since $\rho_{M^n,T^n}([\eta_1]) + \rho_{M^n,T^n}([\eta_2]) = \rho_{M^n,T^n}([\eta]) = 0$ and the sum $U_1 + U_2$ is direct, this implies that $\rho_{M^n,T^n}([\eta_1]) = \rho_{M^n,T^n}([\eta_2]) = 0$. Since $\eta_1 \in W_1$, it can be written in terms of only v_i with $i \in J'$. Since $|J'| = m-1$, by the induction hypothesis, $\rho_{M^n,T^n}([\eta_1]) = 0$ implies $[\eta_1] = 0$ so $[\eta] = [\eta_2]$. So we have reduced to the case when $\eta \in W_2$. Then η writes as

$$\eta = \sum a_{i_1,\dots,i_n;k} v_{i_1} \otimes \cdots \otimes v_{i_k} \wedge v_{i_{k+1}} \otimes \cdots \otimes v_{i_n},$$

where the sum is take over $(i_1,\dots,i_n;k) \in A_2$ and $a_{i_1,\dots,i_n;k} \in F$. Since $[v_{i_1} \otimes \cdots \otimes v_{i_k} \wedge v_{i_{k+1}} \otimes \cdots \otimes v_{i_n}] = f_k(v_{i_1} \otimes \cdots \otimes v_{i_n})$ we have

$$[\eta] = \sum a_{i_1,\dots,i_n;k} f_k(v_{i_1} \otimes \cdots \otimes v_{i_n}).$$

By Lemma 1.4, on $T^n(V)$ we have $\rho_{M^n,T^n} f_k = 1 - \tau_k$. It follows that

$$0 = \rho_{M^n,T^n}[\eta] = \sum a_{i_1,\dots,i_n;k}(v_{i_1} \otimes \cdots \otimes v_{i_n} - \tau_k(v_{i_1} \otimes \cdots \otimes v_{i_n})).$$

But for every $(i_1,\dots,i_n;k) \in A_2$, we have $(i_1,\dots,i_n) \in B_2$, which implies that also $(i_{\tau_k(1)},\dots,i_{\tau_k(n)}) \in B_2$. Thus, both $v_{i_1} \otimes \cdots \otimes v_{i_n}$ and $\tau_k(v_{i_1} \otimes \cdots \otimes v_{i_n})$ belong to U_2 so we can apply ψ to the formula above. We get

$$0 = \sum a_{i_1,\dots,i_n;k}(\psi(v_{i_1} \otimes \cdots \otimes v_{i_n}) - \psi\tau_k(v_{i_1} \otimes \cdots \otimes v_{i_n})).$$

Let $(i_1,\dots,i_n;k) \in A_2$. We have $\psi(v_{i_1} \otimes \cdots \otimes v_{i_n}) = h_{\sigma_l}(v_{i_1} \otimes \cdots \otimes v_{i_n})$ and $\psi\tau_k(v_{i_1} \otimes \cdots \otimes v_{i_n}) = h_{\sigma_{l'}}\tau_k(v_{i_1} \otimes \cdots \otimes v_{i_n})$, where l is the smallest index with $i_l = j_m$ and l' is the smallest index with $i_{\tau_k(l')} = j_m$. Since $i_k < i_{k+1} \le j_m$, we cannot have $l = k$. Since $(i_{\tau_k(1)},\dots,i_{\tau_k(n)}) = (i_1,\dots,i_{k-1},i_{k+1},i_k,i_{k+2},\dots,i_n)$, if $l \le k-1$ or $l \ge k+2$ then $l' = l$. If $l = k+1$ then $l' = k$.

Let $\tau = \sigma_{l'}\tau_k\sigma_l^{-1}$ so that $\tau\sigma_l = \sigma_{l'}\tau_k$. Then, by Lemma 1.6(ii), the relation $h_{\tau\sigma_l} = h_{\sigma_{l'}\tau_k}$ writes as

$$h_{\sigma_l} + h_\tau\sigma_l = h_{\tau_k} + h_{\sigma_{l'}}\tau_k = f_k + h_{\sigma_{l'}}\tau_k$$

so $f_k = h_\tau\sigma_l + (h_{\sigma_l} - h_{\sigma_{l'}}\tau_k)$. Since $h_{\sigma_l}(v_{i_1} \otimes \cdots \otimes v_{i_n}) = \psi(v_{i_1} \otimes \cdots \otimes v_{i_n})$ and $h_{\sigma_{l'}}\tau_k(v_{i_1} \otimes \cdots \otimes v_{i_n}) = \psi\tau_k(v_{i_1} \otimes \cdots \otimes v_{i_n})$, this implies that

$$f_k(v_{i_1} \otimes \cdots \otimes v_{i_n}) = [\zeta_{i_1,\dots,i_n;k}] + (\psi(v_{i_1} \otimes \cdots \otimes v_{i_n}) - \psi\tau_k(v_{i_1} \otimes \cdots \otimes v_{i_n})),$$

where $[\zeta_{i_1,\dots,i_n;k}] = h_\tau\sigma_l(v_{i_1} \otimes \cdots \otimes v_{i_n})$.

It follows that

$$\begin{aligned}[\eta] &= \sum a_{i_1,\dots,i_n;k} f_k(v_{i_1} \otimes \cdots \otimes v_{i_n}) \\ &= \sum a_{i_1,\dots,i_n;k}[\zeta_{i_1,\dots,i_n;k}] \\ &\qquad + \sum a_{i_1,\dots,i_n;k}(\psi(v_{i_1} \otimes \cdots \otimes v_{i_n}) - \psi\tau_k(v_{i_1} \otimes \cdots \otimes v_{i_n})) \\ &= \sum a_{i_1,\dots,i_n;k}[\zeta_{i_1,\dots,i_n;k}].\end{aligned}$$

We now claim that if $(i_1, \dots, i_n; k) \in A_2$, then $[\zeta_{i_1,\dots,i_n;k}] = v_{j_m} \otimes [\zeta'_{i_1,\dots,i_n;k}]$ for some $\zeta'_{i_1,\dots,i_n;k} \in (T(V) \otimes \Lambda^2(V) \otimes T(V))^{n-1}$.

If $l = k+1$ then $l' = k$ so $\tau = \sigma_k \tau_k \sigma_{k+1}^{-1}$. But in S_n we have

$$(1, 2, \dots, k)(k, k+1) = (1, 2, \dots, k+1),$$

i.e. $\sigma_k \tau_k = \sigma_{k+1}$, so $\tau = 1$, which implies $h_\tau = 0$, so $[\zeta_{i_1,\dots,i_n;k}] = h_\tau \sigma_l (v_{i_1} \otimes \cdots \otimes v_{i_n}) = 0$ and we may take $\zeta'_{i_1,\dots,i_n;k} = 0$.

Suppose now that $l \leq k-1$ or $l \geq k+2$, so that $l' = l$. We have $\sigma_l(v_{i_1} \otimes \cdots \otimes v_{i_n}) = v_{i'_1} \otimes \cdots \otimes v_{i'_n}$, with $i'_h = i_{\sigma_l^{-1}(h)}$. But $\sigma_l(l) = 1$ so $i'_1 = i_{\sigma_l^{-1}(1)} = i_l = j_m$. So $\sigma_l(v_{i_1} \otimes \cdots \otimes v_{i_n}) = v_{j_m} \otimes v_{i'_2} \otimes \cdots \otimes v_{i'_n}$.

Since $l' = l$ we have $\tau = \sigma_l \tau_k \sigma_l^{-1} = \sigma_l (k, k+1) \sigma_l^{-1} = (\sigma_l(k), \sigma_l(k+1))$.

If $l \leq k-1$ then $\sigma_l(k) = k$ and $\sigma_l(k+1) = k+1$ so $\tau = (k, k+1) = \tau_k$, so $h_\tau = f_k$. Thus $[\zeta_{i_1,\dots,i_n;k}] = h_\tau \sigma_l(v_{i_1} \otimes \cdots \otimes v_{i_n}) = f_k(v_{j_m} \otimes v_{i'_1} \otimes \cdots \otimes v_{i'_n}) = [v_{j_m} \otimes \zeta'_{i_1,\dots,i_n;k}]$, where $\zeta'_{i_1,\dots,i_n;k} = v_{i'_2} \otimes \cdots \otimes v_{i'_k} \wedge v_{i'_{k+1}} \otimes \cdots \otimes v_{i'_n}$. (Note that $l \leq k-1$ implies $k \geq 2$.)

If $l \geq k+2$ then $\sigma_l(k) = k+1$ and $\sigma_l(k+1) = k+2$ so $\tau = (k+1, k+2) = \tau_{k+1}$, so $h_\tau = f_{k+1}$. Then, by the same reasoning from the case $l \leq k-1$, we get $[\zeta_{i_1,\dots,i_n;k}] = [v_{j_m} \otimes \zeta'_{i_1,\dots,i_n;k}]$, where $\zeta'_{i_1,\dots,i_n;k} = v_{i'_2} \otimes \cdots \otimes v_{i'_{k+1}} \wedge v_{i'_{k+2}} \otimes \cdots \otimes v_{i'_n}$.

Since $[\zeta_{i_1,\dots,i_n;k}] = v_{j_m} \otimes [\zeta'_{i_1,\dots,i_n;k}]$ we have

$$[\eta] = \sum a_{i_1,\dots,i_n;k} [\zeta_{i_1,\dots,i_n;k}] = v_{j_m} \otimes [\eta'],$$

where $\eta' = \sum a_{i_1,\dots,i_n;k} \zeta'_{i_1,\dots,i_n;k}$. Then $0 = \rho_{M^n,T^n}([\eta]) = \rho_{M^n,T^n}(v_{j_m} \otimes [\eta']) = v_{j_m} \otimes \rho_{M^{n-1},T^{n-1}}([\eta'])$. It follows that $\rho_{M^{n-1},T^{n-1}}([\eta']) = 0$. But, by the induction hypothesis, $\rho_{M^{n-1},T^{n-1}}$ is injective, so we have $[\eta'] = 0$. It follows that $[\eta] = v_{j_m} \otimes [\eta'] = 0$. □

Remark. When $n = 2$, we have $(T(V) \otimes \Lambda^2(V) \otimes T(V))^2 = \Lambda^2(V)$ and $W_M^2(V) = 0$. (All generators of $W_M(V)$ have degrees ≥ 3.) Thus, $M^2(V) = \Lambda^2(V)$ and ρ_{M^2,T^2} coincides with ρ_{Λ^2,T^2}. Hence,

$$0 \to M^2(V) \xrightarrow{\rho_{M^2,T^2}} T^2(V) \xrightarrow{\rho_{T^2,S^2}} S^2(V) \to 0$$

coincides with the short exact sequence from Proposition 1.1.

2. THE ALGEBRA $S'(V)$

Definition 2. We define the algebra $S'(V)$ as $S'(V) = T(V)/W_{S'}(V)$, where $W_{S'}(V)$ is the bilateral ideal of $T(V)$ generated by $x \otimes y \otimes z - y \otimes z \otimes x$, with $x, y, z \in V$.

If $x_1 \dots, x_n \in V$, then we denote by $x_1 \odot \cdots \odot x_n$ the class of $x_1 \otimes \cdots \otimes x_n$ in $S'(V)$. So $(S'(V), +, \odot)$ is an algebra.

Since $W_{S'}(V)$ is a homogeneous ideal, $S'(V)$ is a graded algebra. For $n \geq 0$ we denote by $S'^n(V)$ the homogeneous component of degree n of $S'(V)$. We have $S'^n(V) = T^n(V)/W^n_{S'}(V)$, where $W^n_{S'}(V) = W_{S'}(V) \cap T^n(V)$.

Since the generators of $W_{S'}(V)$ have degree 3, for $n \leq 2$ we have $W^n_{S'}(V) = 0$ so $S'^n(V) = T^n(V)$.

Note that $\rho_{T,S}(x\otimes y\otimes z - y\otimes z\otimes x) = xyz - yzx = 0$ so $x\otimes y\otimes z - y\otimes z\otimes x \in \ker \rho_{T,S}$ $\forall x, y, z \in V$. It follows that $W_{S'}(V) \subseteq \ker \rho_{T,S}$. Therefore, $\rho_{T,S}$ induces a surjective morphism of algebras defined on $T(V)/W_{S'}(V) = S'(V)$. Namely, we have:

Proposition 2.1. *There is a surjective morphism of algebras* $\rho_{S',S} : S'(V) \to S(V)$ *given by* $x_1 \odot \cdots \odot x_n \mapsto x_1 \cdots x_n$.

Proposition 2.2. *The subspace* $W^n_{S'}(V)$ *of* $T(V)$ *is spanned by* $x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$, *with* $x_1, \dots, x_n \in V$ *and* $\sigma \in A_n$, *where* A_n *is the alternating group of degree* n.

Proof. The bilateral ideal $W_{S'}(V)$ of $T(V)$ is generated by $f(x, y, z)$, where $f : V^3 \to T^3(V)$ is given by $(x, y, z) \mapsto x \otimes y \otimes z - y \otimes z \otimes x$. Therefore, $W^n_{S'}(V)$ is spanned by

$$\begin{aligned} &x_1 \otimes \cdots \otimes x_{i-1} \otimes f(x_i, x_{i+1}, x_{i+2}) \otimes x_{i+3} \otimes \cdots \otimes x_n \\ &\quad = x_1 \otimes \cdots \otimes x_n - x_1 \otimes \cdots \otimes x_{i+1} \otimes x_{i+2} \otimes x_i \otimes \cdots \otimes x_n \\ &\quad = x_1 \otimes \cdots \otimes x_n - x_{\sigma_i(1)} \otimes \cdots \otimes x_{\sigma_i(n)}, \end{aligned}$$

with $x_1, \dots x_n \in V$ and $1 \leq i \leq n-2$, where $\sigma_i \in S_n$ is the cycle $(i, i+1, i+2)$.

Hence, in $S'^n(V)$ we have $x_1 \odot \cdots \odot x_n = x_{\sigma_i(1)} \odot \cdots \odot x_{\sigma_i(n)}$ $\forall x_1, \dots, x_n \in V$ and $1 \leq i \leq n-2$. But the cycles σ_i generate the the alternating group A_n so in $S'^n(V)$ we have $x_1 \odot \cdots \odot x_n = x_{\sigma(1)} \odot \cdots \odot x_{\sigma(n)}$ $\forall \sigma \in A_n$. Hence, $W^n_{S'}(V)$ contains $x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ $\forall x_1, \dots, x_n \in V$ and $\sigma \in A_n$, which are more general than the original generators, where $\sigma = \sigma_i$ for some $1 \leq i \leq n-2$. □

Proposition 2.3. (i) *If* $n \geq 2$ *then we have a linear map* $c : S'^n(V) \to S'^n(V)$ *given by* $x_1 \odot \cdots \odot x_n \mapsto x_2 \odot x_1 \odot x_3 \odot \cdots \odot x_n$, $\forall x_1, \dots, x_n \in V$.

(ii) *We have* $c^2 = 1$ *and* $\rho_{S',S} c = \rho_{S',S}$

(iii) *If* $x_1, \dots, x_n \in V$ *and* $\sigma \in S_n$ *then*

$$x_{\sigma(1)} \odot \cdots \odot x_{\sigma(n)} = \begin{cases} x_1 \odot \cdots \odot x_n & \text{if } \sigma \in A_n \\ c(x_1 \odot \cdots \odot x_n) & \text{if } \sigma \in S_n \setminus A_n \end{cases}.$$

(iv) *If there are* $i < j$ *with* $x_i = x_j$ *then* $c(x_1 \odot \cdots \odot x_n) = x_1 \odot \cdots \odot x_n$. *Consequently,* $x_{\sigma(1)} \odot \cdots \odot x_{\sigma(n)} = x_1 \odot \cdots \odot x_n$ *holds regardless of the parity of* σ.

Proof. (i) We define $\bar{c} : T^n(V) \to T^n(V)$ by $x_1 \otimes \cdots \otimes x_n \mapsto x_2 \otimes x_1 \otimes x_3 \otimes \cdots \otimes x_n = x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)}$, where $\tau \in S_n$ is the transposition $(1,2)$. To prove that $\bar{c}$ induces the morphism $c : S'^n(V) \to S'^n(V)$ given by $x_1 \odot \cdots \odot x_n \mapsto x_2 \odot x_1 \odot x_3 \odot \cdots \odot x_n$, one must prove that $\bar{c}(W^n_{S'}(V)) \subseteq W^n_{S'}(V)$.

Let $\xi = x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ be a generator of $W^n_{S'}$, with $x_1, \ldots, x_n \in V$ and $\sigma \in A_n$. Then $\bar{c}(\xi) = x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)} - x_{\sigma\tau(1)} \otimes \cdots \otimes x_{\sigma\tau(n)}$. If we denote $y_i = x_{\tau(i)}$, so that $x_i = y_{\tau^{-1}(i)} = y_{\tau(i)}$, then $x_{\sigma\tau(i)} = y_{\tau\sigma\tau(i)}$. Hence, $\bar{c}(\xi) = y_1 \otimes \cdots \otimes y_n - y_{\sigma'(1)} \otimes \cdots \otimes y_{\sigma'(n)}$, where $\sigma' = \tau\sigma\tau$. But $\sigma \in A_n$, which implies that $\sigma' \in A_n$ and so $\bar{c}(\xi) \in W^n_{S'}(V)$.

(ii) Let $\xi = x_1 \odot \cdots \odot x_n$. Applying c to ξ twice permutes the first two factors of ξ twice, so we have $c^2(\xi) = \xi$. We also have $\rho_{S',S} c(\xi) = x_2 x_1 x_3 \cdots x_n = x_1 \cdots x_n = \rho_{S',S}(\xi)$.

(iii) We have $c(x_1 \odot \cdots \odot x_n) = y_1 \odot \cdots \odot y_n$, where $y_i = x_{\tau(i)}$, with $\tau = (1,2) \in S_n$. Then $x_i = y_{\tau(i)}$.

If $\sigma \in A_n$ then $x_{\sigma(1)} \odot \cdots \odot x_{\sigma(n)} = x_1 \odot \cdots \odot x_n$ follows from Proposition 2.2. If $\sigma \notin A_n$ then note that $x_{\sigma(i)} = y_{\tau\sigma(i)}$ and, since $\tau, \sigma \notin A_n$, we have $\tau\sigma \in A_n$. Therefore $x_{\sigma(1)} \odot \cdots \odot x_{\sigma(n)} = y_{\tau\sigma(1)} \odot \cdots \odot y_{\tau\sigma(n)} = y_1 \odot \cdots \odot y_n = c(x_1 \odot \cdots \odot x_n)$.

(iv) Let $\tau \in S_n$, $\tau = (i,j)$. Since $x_i = x_j$, permuting the factors x_i and x_j has no effect on te product $x_1 \odot \cdots \odot x_n$. Hence $x_{\tau(1)} \odot \cdots \odot x_{\tau(n)} = x_1 \odot \cdots \odot x_n$. But $\tau \in S_n \setminus A_n$ so, by (iii), $x_{\tau(1)} \odot \cdots \odot x_{\tau(n)} = c(x_1 \odot \cdots \odot x_n)$. Hence the conclusion. □

We now produce a basis for $S'^n(V)$. For this purpose, we need the following elementary result.

LEMMA 2.4. *Let U be a vector space with the basis $(u_\alpha)_{\alpha \in A}$. Let $\sim$ be an equivalence relation on A and let B be a set of representatives for $A_{/\sim}$.*

Let $W \subseteq U$ be the subspace generated by all $u_\alpha - u_\beta$, with $\alpha, \beta \in A$ such that $\alpha \sim \beta$. For every $u \in U$ we denote by $\bar{u}$ its class in U/W.

Then $(\bar{u}_\alpha)_{\alpha \in B}$ is a basis for U/W.

Proof. Let $U' \subseteq U$ be the subspace generated by u_α, with $\alpha \in B$. Let $f : U \to U'$ be the linear function given by $u_\alpha \mapsto u_\beta$, where β is the unique element of B such that $\alpha \sim \beta$. If $u_\alpha - u_\beta$, with $\alpha \sim \beta$, is a generator of W and $\gamma \in B$ such that $\alpha \sim \gamma$ then, we also have $\beta \sim \gamma$ and so $f(u_\alpha) = f(u_\beta) = u_\gamma$. It follows that $u_\alpha - u_\beta \in \ker f$ and so $W \subseteq \ker f$. Therefore, f induces a linear map $\bar{f} : U/W \to U'$, given by $\bar{u}_\alpha \mapsto u_\beta$, where $\beta \in B$ such that $\alpha \sim \beta$.

We now define $g : U' \to U/W$, given by $u_\alpha \mapsto \bar{u}_\alpha$ $\forall \alpha \in B$. For every $\alpha \in B$, we have $\bar{f}g(u_\alpha) = \bar{f}(\bar{u}_\alpha) = u_\alpha$. (We have $\alpha \in B$ and $\alpha \sim \alpha$.) Thus

$\bar{f}g = 1_{U'}$. If $\alpha \in A$ and $\beta \in B$ such that $\alpha \sim \beta$ then $g\bar{f}(\bar{u}_\alpha) = g(u_\beta) = \bar{u}_\beta = \bar{u}_\alpha$. (We have $\alpha \sim \beta$ so $u_\alpha - u_\beta \in W$ so $\bar{u}_\alpha = \bar{u}_\beta$.) Thus $g\bar{f} = 1_{U/W}$.

Thus, $g : U' \to U/W$ is an isomorphism and $\bar{f}$ its inverse. Since $(\bar{u}_\alpha)_{\alpha\in B}$ is the image with respect to g of the basis $(u_\alpha)_{\alpha\in B}$ of U', it will be a basis for U/W. □

Proposition 2.5. *Let* $J = \{(i_1, \ldots, i_n) \in I^n \mid i_1 \leq \cdots \leq i_n\}$, $J_1 = \{(i_1, \ldots, i_n) \in I^n \mid i_1 < \cdots < i_n\}$ *and* $J_2 = J \setminus J_1$. *Then*

$$\{v_{i_1} \odot \ldots \odot v_{i_n} \mid (i_1, \ldots, i_n) \in J\} \cup \{c(v_{i_1} \odot \ldots \odot v_{i_n}) \mid (i_1, \ldots, i_n) \in J_1\}$$

is a basis of $S'^n(V)$.

Proof. We use Lemma 2.4 for $U = T^n(V)$, with the basis $(u_\alpha)_{\alpha\in A}$, where $A = I^n$ and $u_{i_1,\ldots,i_n} = v_{i_1} \otimes \cdots \otimes v_{i_n}$ $\forall (i_1, \ldots, i_n) \in A$. The equivalence relation $\sim$ on A is given by $(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n)$ if $(j_1, \ldots, j_n) = (i_{\sigma(1)}, \ldots, i_{\sigma(n)})$ for some $\sigma \in A_n$ and $W \subseteq U$ is generated by $u_\alpha - u_\beta$, with $\alpha, \beta \in A$, $\alpha \sim \beta$.

We claim that $B = J \cup \{(i_2, i_1, i_3 \ldots, i_n) \mid (i_1, \ldots, i_n) \in J_1\}$ is a set of representatives for $A_{/\sim}$. First, we show that if $\alpha, \beta \in B$, $\alpha \neq \beta$, then $\alpha \not\sim \beta$. We note that if $(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n)$ then then the sequences $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ written in increasing order are the same. Therefore, if $(i_1, \ldots, i_n)$, $(j_1, \ldots, j_n) \in J$ with $(i_1, \ldots, i_n) \neq (j_1, \ldots, j_n)$ then $(i_1, \ldots, i_n)$ or $(i_2, i_1, i_3, \ldots, i_n)$ cannot be in the $\sim$ relation with $(j_1, \ldots, j_n)$ or $(j_2, j_1, j_3, \ldots, j_n)$. This proves that if $\alpha, \beta \in B$, with $\alpha \neq \beta$ then $\alpha \not\sim \beta$ unless $\alpha = (i_1, \ldots, i_n)$ and $\beta = (i_2, i_1, i_3, \ldots, i_n)$ (or vice versa) for some $(i_1, \ldots, i_n) \in J_1$, i.e. with $i_1 < \cdots < i_n$. But in this case the only $\sigma \in S_n$ such that $(i_2, i_1, i_3, \ldots, i_n) = (i_{\sigma(1)}, \ldots, i_{\sigma(n)})$ is $\sigma = (1, 2)$, which is odd. Hence, again, $\alpha \not\sim \beta$.

Next, we prove that if $\alpha = (i_1, \ldots, i_n) \in A$ then there is $\beta \in B$ with $\alpha \sim \beta$. We write the sequence $i_1, \ldots, i_n$ in increasing order as $j_1, \ldots, j_n$. Then $(j_1, \ldots, j_n) \in J \subseteq B$ and we have $(j_1, \ldots, j_n) = (i_{\sigma(1)}, \ldots, i_{\sigma(n)})$ for some $\sigma \in S_n$. If $\sigma \in A_n$ then $(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n)$ so we may take $\beta = (j_1, \ldots, j_n)$. Suppose now that $\sigma \in S_n \setminus A_n$. If $(j_1, \ldots, j_n) \in J_2$ then $j_k = j_{k+1}$ for some $1 \leq k \leq n-1$. Hence, if $\tau = (k, k+1) \in S_n$ then $(j_1, \ldots, j_n) = (j_{\tau(1)}, \ldots, j_{\tau(n)}) = (i_{\sigma\tau(1)}, \ldots, i_{\sigma\tau(n)})$. Since $\sigma, \tau \in S_n \setminus A_n$ we have $\sigma\tau \in A_n$ so $(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n)$. So, again, we may take $\beta = (j_1, \ldots, j_n)$. If $(j_1, \ldots, j_n) \in J_1$ then we also have $(j_2, j_1, j_3 \ldots, j_n) \in B$. If $\tau = (1, 2) \in S_n$ then $(j_2, j_1, j_3, \ldots, j_n) = (j_{\tau(1)}, \ldots, j_{\tau(n)}) = (i_{\sigma\tau(1)}, \ldots, i_{\sigma\tau(n)})$. Since $\sigma, \tau \in S_n \setminus A_n$ we have $\sigma\tau \in A_n$ so $(i_1, \ldots, i_n) \sim (j_2, j_1, j_3, \ldots, j_n)$. So this time we may take $\beta = (j_2, j_1, j_3, \ldots, j_n)$.

The subspace $W^n_{S'}(V)$ of $T^n(V)$ is generated by $f_\sigma(x_1, \ldots, x_n)$ with $x_1, \ldots, x_n$ and $\sigma \in A_n$, where $f_\sigma : V^n \to T^n(V)$ is given by $(x_1, \ldots, x_n) \mapsto$

$x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$. But f_σ is multilinear so we may restrict ourselves to the case when $x_1, \ldots, x_n$ belong to the basis v_i, with $i \in I$, of V. Then $W^n_{S'}(V)$ is generated by $f_\sigma(v_{i_1}, \ldots, v_{i_n}) = v_{i_1} \otimes \cdots \otimes v_{i_n} - v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(n)}} = u_{i_1,\ldots,i_n} - u_{i_{\sigma(1)},\ldots,i_{\sigma(n)}}$, with $i_1, \ldots, i_n \in I$ and $\sigma \in A_n$. Equivalently, $W^n_{S'}(V)$ is generated by $u_\alpha - u_\beta$, with $\alpha, \beta \in A$, $\alpha \sim \beta$, i.e. $W^n_{S'}(V) = W$. It follows that $U/W = T^n(V)/W^n_{S'}(V) = S'^n(V)$. Also if $u = x_1 \otimes \cdots \otimes x_n \in U = T^n(V)$ then its class in $U/W = S'^n(V)$ is $\bar{u} = x_1 \odot \cdots \odot x_n$. In particular, $\bar{u}_{i_1,\ldots,i_n} = v_{i_1} \odot \cdots \odot v_{i_n}$.

By Lemma 2.4, $\bar{u}_\alpha$ with $\alpha \in B$ are a basis of $U/W = S'^n(V)$. If $\alpha = (i_1, \ldots, i_n) \in J$ then $\bar{u}_\alpha = v_{i_1} \odot \cdots \odot v_{i_n}$. If $\alpha = (i_2, i_1, i_3, \ldots, i_n)$ for some $(i_1, \ldots, i_n) \in J_1$ then $\bar{u}_\alpha = v_{i_2} \odot v_{i_1} \odot v_{i_3} \odot \cdots \odot v_{i_n} = c(v_{i_1} \odot \cdots \odot v_{i_n})$. Hence, the conclusion. □

Theorem 2.6. *For $n \geq 2$, we have the exact sequence*

$$0 \to \Lambda^n(V) \xrightarrow{\rho_{\Lambda^n,S'^n}} S'^n(V) \xrightarrow{\rho_{S'^n,S^n}} S^n(V) \to 0,$$

where ρ_{Λ^n,S'^n} is given by $x_1 \wedge \cdots \wedge x_n \mapsto x_1 \odot \cdots \odot x_n - c(x_1 \odot \cdots \odot x_n)$.

Proof. We already know that $\rho_{S'^n,S^n}$ is surjective.

The map $(x_1, \ldots, x_n) \mapsto x_1 \odot \cdots \odot x_n - c(x_1 \odot \cdots \odot x_n)$ is linear in each variable and anti-symmetric. (If $x_i = x_j$ for some $i \neq j$ then, by Proposition 2.3(iv), we have $c(x_1 \odot \cdots \odot x_n) = x_1 \odot \cdots \odot x_n$.) Hence the map ρ_{Λ^n,S'^n}, given by $x_1 \wedge \cdots \wedge x_n \mapsto x_1 \odot \cdots \odot x_n - c(x_1 \odot \cdots \odot x_n)$, is well defined.

We prove the injectivity of ρ_{Λ^n,S'^n}. We use the notations of Proposition 2.5. The set $\{v_{i_1} \wedge \cdots \wedge v_{i_n} \mid (i_1, \ldots, i_n) \in J_1\}$ is a basis of $\Lambda^n(V)$. Let $\alpha \in \ker \rho_{\Lambda^n,S'^n}$. We write $\alpha = \sum_{(i_1,\ldots,i_n)\in J_1} a_{i_1,\ldots,i_n} v_{i_1} \wedge \cdots \wedge v_{i_n}$, with $a_{i_1,\ldots,i_n} \in F$. Then

$$0 = \rho_{\Lambda^n,S'^n}(\alpha) = \sum_{(i_1,\ldots,i_n)\in J_1} a_{i_1,\ldots,i_n}(v_{i_1} \odot \cdots \odot v_{i_n} - c(v_{i_1} \odot \cdots \odot v_{i_n})).$$

Since $v_{i_1} \odot \cdots \odot v_{i_n}$ and $c(v_{i_1} \odot \cdots \odot v_{i_n})$, with $(i_1, \ldots, i_n) \in J_1$, are part of the basis of $S'^n(V)$ form Proposition 2.5, this implies that $a_{i_1,\ldots,i_n} = 0$ $\forall (i_1, \ldots, i_n) \in J_1$ so $\alpha = 0$. Hence ρ_{Λ^n,S'^n} is injective.

For the exactness in the second term, we use the formulas $\rho_{S'^n,S^n}(x_1 \odot \cdots \odot x_n) = \rho_{S'^n,S^n} c(x_1 \odot \cdots \odot x_n) = x_1 \cdots x_n$. (See Proposition 2.3 (ii).)

The map $\rho_{S'^n,S^n}\rho_{\Lambda^n,S'^n}$ is given by $x_1 \wedge \cdots \wedge x_k \mapsto \rho_{S'^n,S^n}(x_1 \odot \cdots \odot x_n - c(x_1 \odot \cdots \odot x_n)) = x_1 \cdots x_n - x_1 \cdots x_n = 0$. Hence, $\rho_{S'^n,S^n}\rho_{\Lambda^n,S'^n} = 0$ so $\ker \rho_{S'^n,S^n} \supseteq \operatorname{Im} \rho_{\Lambda^n,S'^n}$.

For the reverse inclusion let $\alpha \in \ker \rho_{S'^n,S^n}$. By Proposition 2.5, α writes as

$$\alpha = \sum_{(i_1,\ldots,i_n)\in J} a_{i_1,\ldots,i_n} v_{i_1} \odot \cdots \odot v_{i_n} + \sum_{(i_1,\ldots,i_n)\in J_1} b_{i_1,\ldots,i_n} c(v_{i_1} \odot \cdots \odot v_{i_n}),$$

where $a_{i_1,\dots,i_n}, b_{i_1,\dots,i_n} \in F$. Then, we have

$$0 = \rho_{S'^n,S^n}(\alpha) = \sum_{(i_1,\dots,i_n)\in J} a_{i_1,\dots,i_n} v_{i_1}\cdots v_{i_n} + \sum_{(i_1,\dots,i_n)\in J_1} b_{i_1,\dots,i_n} v_{i_1}\cdots v_{i_n}$$
$$= \sum_{(i_1,\dots,i_n)\in J_1} (a_{i_1,\dots,i_n} + b_{i_1,\dots,i_n}) v_{i_1}\cdots v_{i_n} + \sum_{(i_1,\dots,i_n)\in J_2} a_{i_1,\dots,i_n} v_{i_1}\cdots v_{i_n}.$$

Since $v_{i_1}\cdots v_{i_n}$ with $(i_1,\dots,i_n) \in J = J_1 \sqcup J_2$ are a basis of $S^n(V)$, we get $a_{i_1,\dots,i_n} = 0\ \forall (i_1,\dots,i_n) \in J_2$ and $a_{i_1,\dots,i_n} + b_{i_1,\dots,i_n} = 0$, so $b_{i_1,\dots,i_n} = -a_{i_1,\dots,i_n}$, $\forall (i_1,\dots,i_n) \in J_1$. It follows that

$$\alpha = \sum_{(i_1,\dots,i_n)\in J_1} a_{i_1,\dots,i_n} v_{i_1}\odot\cdots\odot v_{i_n} + \sum_{(i_1,\dots,i_n)\in J_1} -a_{i_1,\dots,i_n} c(v_{i_1}\odot\cdots\odot v_{i_n})$$
$$= \sum_{(i_1,\dots,i_n)\in J_1} a_{i_1,\dots,i_n} (v_{i_1}\odot\cdots\odot v_{i_n} - c(v_{i_1}\odot\cdots\odot v_{i_n})) = \rho_{\Lambda^n,S'^n}(\beta),$$

where $\beta = \sum_{(i_1,\dots,i_n)\in J_1} a_{i_1,\dots,i_n} v_{i_1}\wedge\cdots\wedge v_{i_n}$. Thus $\alpha \in \operatorname{Im}\rho_{\Lambda^n,S'^n}$. □

Recall that if $n \le 2$ then $S'^n(V) = T^n(V)$.

If $n = 0, 1$ then $S^n(V) = S'^n(V) = T^n(V)$ and $\rho_{S'^n,S^n}$ is the identity map so we have the short exact sequence $0 \to 0 \to S'^n(V) \xrightarrow{\rho_{S'^n,S^n}} S^n(V) \to 0$. By putting together these two sequences with those for $n \ge 2$ from Theorem 2.6, we get:

Corollary 2.7. *We have a short exact sequence,*

$$0 \to \Lambda^{\ge 2}(V) \xrightarrow{\rho_{\Lambda^{\ge 2},S'}} S'(V) \xrightarrow{\rho_{S',S}} S(V) \to 0.$$

Remark. The maps ρ_{Λ^2,S'^2} and $\rho_{S'^2,S^2}$ are given by $x\wedge y \mapsto x\odot y - y\odot x$ and $x\odot y \mapsto xy$, respectively. But when we identify $S'^2(V)$ with $T^2(V)$ they write as $x\wedge y \mapsto x\otimes y - y\otimes x = [x,y]$ and $x\otimes y \mapsto xy$ so they coincide with ρ_{Λ^2,T^2} and ρ_{T^2,S^2}. Therefore, the short exact sequences form Theorem 2.6, in the case $n = 2$, and from Proposition 1.1 are the same.

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