CHARACTERIZATION OF FINITE DIMENSIONAL NILPOTENT LIE ALGEBRAS BY THE DIMENSION OF THEIR SCHUR MULTIPLIERS, $s(L) = 5$

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It is known that the dimension of the Schur multiplier of a non-abelian nilpotent Lie algebra L of dimension n is equal to $\frac{1}{2}(n-1)(n-2)+1-s(L)$ for some $s(L) \geq 0$. The structure of all nilpotent Lie algebras has been given for $s(L) \leq 4$ in several papers. Here, we are going to give the structure of all non-abelian nilpotent Lie algebras for $s(L) = 5$.

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1. INTRODUCTION AND MOTIVATION

Let L be a finite dimensional nilpotent Lie algebra such that $L \cong F/R$ for a free Lie algebra F. Then by [4], the Schur multiplier $\mathcal{M}(L)$ of L is isomorphic to $R \cap F^2/[R, F]$. By a result of Moneyhun in [11], there exists a non-negative integer $t(L)$ such that $\dim \mathcal{M}(L) = \frac{1}{2}n(n-1) - t(L)$. It is a classical question to determine the structure of L by looking at the dimension of its Schur multiplier. The answer to this problem was given for $t(L) \leq 8$ in [5, 8, 9] and by putting some conditions on L for $t(L) \leq 16$ in [6] by several authors.

From [13], when L is a non-abelian nilpotent Lie algebra, the dimension of the Shur multiplier of L is equal to $\frac{1}{2}(n-1)(n-2)+1-s(L)$ for some $s(L) \geq 0$. It not only improves the bound of Moneyhun but also let us ask the same natural question about the characterization of Lie algebras in term of size $s(L)$. The answer to this question was given by several papers in [15, 22, 23] for $s(L) \leq 4$ and for $s(L) \leq 15$ when conditions are put on L in [24].

It is not easy to characterize the nilpotent Lie algebras for $s(L) \geq 4$ by using the only methods of previous articles [13, 22, 23]. Thanks to a result of [18] and the classification of indecomposable Lie algebras of Gong [10], here we are able to characterize the structure of all nilpotent Lie algebras L for $s(L) = 5$. It is important to know that the classifications of Gong were originally

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studied in mathematical physics in the papers of Patera, Zassenhaus [19] and Morozov [12], but they are recently reconsidered by Bagarello and Russo in [1, 2, 3]. This emphasizes that the invariant $s(L)$ turns out to be very useful in many contexts of theoretical physics.

Throughout the paper, we may assume that L is a Lie algebra over an algebraically closed field of characteristic not equal to 2 and $A(n)$ and $H(m)$ are used to denote the abelian Lie algebra of dimension n and the Heisenberg Lie algebra of dimension $2m+1$, respectively.

For the sake of convenience for the reader, some notations and terminology from [7, 8, 9, 10] are listed below.

$L_{3,2} \cong H(1)$	with a basis $\{x_1, x_2, x_3\}$ and the multiplication $[x_1, x_2] = x_3$,
$L_{4,3} \cong L(3,4,1,4)$	with a basis $\{x_1, ..., x_4\}$ and the multiplication $[x_1, x_2] = x_3, [x_1, x_3] = x_4$,
$L_{5,5} \cong L(4,5,1,6)$	with a basis $\{x_1, ..., x_5\}$ and the multiplication $[x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5$,
$L_{5,6} \cong L'(7,5,1,7)$	with a basis $\{x_1, ..., x_5\}$ and the multiplication $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = x_5$,
$L_{5,7} \cong L(7,5,1,7)$	with a basis $\{x_1, ..., x_5\}$ and the multiplication $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5$,
$L_{5,8} \cong L(4,5,2,4)$	with a basis $\{x_1, ..., x_5\}$ and the multiplication $[x_1, x_2] = x_4, [x_1, x_3] = x_5$,
$L_{5,9} \cong L(7,5,2,7)$	with a basis $\{x_1, ..., x_5\}$ and the multiplication $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5$,
$L_{6,10}$	with a basis $\{x_1, ..., x_6\}$ and the multiplication $[x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_4, x_5] = x_6$,
$L_{6,22}(\varepsilon)$	with a basis $\{x_1, ..., x_6\}$ and the multiplication $[x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = \varepsilon x_6, [x_3, x_4] = x_5, \varepsilon \in \mathbb{F}$,
$27B$	with a basis $\{x_1, ..., x_7\}$ and the multiplication $[x_1, x_2] = [x_3, x_4] = x_6, [x_1, x_5] = [x_2, x_3] = x_7$,
$27A$	with a basis $\{x_1, ..., x_7\}$ and the multiplication $[x_1, x_2] = x_6, [x_1, x_4] = x_7, [x_3, x_5] = x_7$,
$L_3 = 157$	with a basis $\{x_1, ..., x_7\}$ and the multiplication $[x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = [x_5, x_6] = x_7$
$37B$	with a basis $\{x_1, ..., x_7\}$ and the multiplication $[x_1, x_2] = x_5, [x_2, x_3] = x_6, [x_3, x_4] = x_7$,
$37C$	with a basis $\{x_1, ..., x_7\}$ and the multiplication $[x_1, x_2] = [x_3, x_4] = x_5, [x_2, x_3] = x_6, [x_2, x_4] = x_7$,
$37D$	with a basis $\{x_1, ..., x_7\}$ and the multiplication $[x_1, x_2] = [x_3, x_4] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = x_7$.

We state some results without proof and refer the reader to see [14, 15, 20, 23]. We recall that the Lie algebras $27B$ and $27A$ in [10] are expressed by different notations L_1 and L_2 in the following proposition, respectively.

Proposition 1.1 (See [16, Proposition 2.10]). *The Schur multiplier of Lie algebras* $L_{6,22}(\varepsilon)$, $L_{5,8}$, L_1 *and* L_2 *are abelian Lie algebras of dimension* $8, 6, 9$ *and* 10, *respectively.*

A Lie algebra L is called capable provided that $L \cong H/Z(H)$ for a Lie algebra H. From [21, Definition 1.4], $Z^*(L)$ is used to the denote the epicenter of L. The importance of $Z^*(L)$ is due to the fact that L is capable if and only if $Z^*(L) = 0$. Another notion having relation to the capability is the concept of the exterior center of a Lie algebra $Z^\wedge(L)$ which is introduced in [14]. It is known that from [14, Lemma 3.1]), $Z^*(L) = Z^\wedge(L)$.

Lemma 1.2 (See [23, Corollary 2.3]). *Let L be a non-capable nilpotent Lie algebra of dimension n such that* $\dim L^2 \geq 2$. *Then*

$$n - 3 < s(L).$$

Let $\otimes_{mod}$ be used to denote the operator of usual tensor product of Lie algebras. Then

Theorem 1.3 (See [18, Theorem 2.1]). *Let L be a finite dimensional nilpotent Lie algebra non-abelian Lie algebra of class two. Then*

$$0 \to \ker g \to L^2 \otimes_{mod} L^{ab} \xrightarrow{g} \mathcal{M}(L) \to \mathcal{M}(L^{ab}) \to L^2 \to 0$$

is exact, where

$$g : x \otimes (z + L^2) \in L^2 \otimes_{mod} L^{ab} \mapsto [\overline{x}, \overline{z}] + [R, F] \in \mathcal{M}(L) = R \cap F^2/[R, F],$$

$\pi(\overline{x} + R) = x$ *and* $\pi(\overline{z} + R) = z$. *Moreover, the subalgebra*

$$K = \langle [x, y] \otimes (z + L^2) + [z, x] \otimes (y + L^2) + [y, z] \otimes (x + L^2) \mid x, y, z \in L\rangle$$

is contained in $\ker g$.

2. MAIN RESULTS

We begin with the following lemma that is easily proven.

Lemma 2.1. *There is no n-dimensional nilpotent Lie algebra with $s(L) = 5$, when*

(i) $\dim L^2 \geq 4$;

(ii) $\dim L^2 = 1$.

Proof. (i) Let L be a nilpotent Lie algebra such that $m = \dim L^2 \geq 4$ and $s(L) = 5$. [15, Theorem 3.1] and our assumption implies that

$$\frac{1}{2}(n-1)(n-2)-4 = \dim \mathcal{M}(L) \leq \frac{1}{2}(n+m-2)(n-m-1)+1 \leq \frac{1}{2}(n+2)(n-5)+1.$$

This is a contradiction.

(ii) By contrary. Let L be a Lie algebra such that $\dim L^2 = 1$ and $s(L) = 5$. Then by using [13, Lemma 3.3], $L \cong H(m) \oplus A(n-2m-1)$ for some $m \geq 1$. Looking at [22, Corollary 2.5] shows that $s(L) = 0$ or $s(L) = 2$, when $m = 1$ or $m \geq 2$, respectively. It is a contradiction. Hence, the result follows. □

By using Lemma 2.1, we may assume that a nilpotent Lie algebras L with $s(L) = 5$ has $2 \leq \dim L^2 \leq 3$. First assume that $\dim L^2 = 2$.

Lemma 2.2. *Let L be an n-dimensional non-capable nilpotent Lie algebra of dimension at most 7 and $\dim L^2 = 2$. Then L is isomorphic to one of the Lie algebras $L_{6,10}$, L_2 or L_3. Moreover, $s(L_{6,10}) = 5$ and $s(L_2) = s(L_3) = 6$.*

Proof. The proof is similar to [23, Theorem 2.6]. □

Theorem 2.3. *Let L be an n-dimensional nilpotent Lie algebra with $s(L) = 5$ and $\dim L^2 = 2$. Then L is isomorphic to one of the Lie algebras $L(4,5,2,4) \oplus A(4)$, $L(3,4,1,4) \oplus A(3)$, $L(4,5,1,6) \oplus A(2)$, $L_{6,22}(\varepsilon) \oplus A(2)$ or $L_{6,10}$.*

Proof. Since $\dim L^2 = 2$, L is nilpotent of class two or three. Let L be a Lie algebra of nilpotency class two. If L is a capable Lie algebra, then it should be isomorphic to one of the Lie algebras $L_{6,22}(\varepsilon) \oplus A$, $L_{5,8} \oplus A$ or $L_1 \oplus A$, for an abelian Lie algebra A by using [16, Corollary 2.13].

Case (i) Let $L \cong L_{6,22}(\varepsilon) \oplus A$. Proposition 1.1 implies $\dim M(L_{6,22}(\varepsilon)) = 8$. Since $5 = s(L) = \frac{1}{2}(n-1)(n-2) + 1 - \dim \mathcal{M}(L)$ and $\dim \mathcal{M}(L) = 8 + \frac{1}{2}(n-6)(n+1)$ by using [5, Theorem 1] and [11, Lemma 23], we have $n = 8$. Hence $L \cong L_{6,22}(\varepsilon) \oplus A(2)$.

Case (ii) Let now $L \cong L_{5,8} \oplus A$. We know from Proposition 1.1 that $\dim \mathcal{M}(L_{5,8}) = 6$. Since $5 = s(L) = \frac{1}{2}(n-1)(n-2) + 1 - \dim \mathcal{M}(L)$ and $\dim \mathcal{M}(L) = 6 + \frac{1}{2}(n-5)n$ by using [5, Theorem 1] and [11, Lemma 23], we have $n = 9$. Therefore $L \cong L_{5,8} \oplus A(4) \cong L(4,5,2,4) \oplus A(4)$.

Case (iii) Let $L \cong L_1 \oplus A$. We know $\dim \mathcal{M}(L_1) = 9$ by using Proposition 1.1. Since $5 = s(L) = \frac{1}{2}(n-1)(n-2) + 1 - \dim \mathcal{M}(L)$ and $\dim \mathcal{M}(L) = 9 + \frac{1}{2}(n-7)(n+2)$ by using [5, Theorem 1] and [11, Lemma 23], we have $n = 5$, which is contradiction. Thus L cannot be isomorphic to $L_1 \oplus A$.

Now let L be a Lie algebra of nilpotency class 3. If L is a capable Lie algebra, then it should be isomorphic to one of the Lie algebras $L_{4,3} \oplus A(n-4)$ or $L_{5,5} \oplus A(n-5)$ by using [17, Theorem 5.5].

Case (i). Let $L \cong L_{4,3} \oplus A(n-4)$. Since $\dim \mathcal{M}(L_{4,3}) = 2$ by using [8, Section 2], we have $\dim \mathcal{M}(L) = 2 + \frac{1}{2}(n-4)(n-1)$ by using [5, Theorem 1] and [11, Lemma 23]. Since $5 = s(L) = \frac{1}{2}(n-1)(n-2) + 1 - \dim \mathcal{M}(L)$ and $\dim \mathcal{M}(L) = 2 + \frac{1}{2}(n-4)(n-1)$, we have $n = 7$. Hence $L \cong L_{4,3} \oplus A(3) \cong L(3,4,1,4) \oplus A(3)$.

Case (ii). Suppose $L \cong L_{5,5} \oplus A(n-5)$. [8, Section 3] shows that $\dim \mathcal{M}(L_{5,5}) = 4$. Now [5, Theorem 1] and [11, Lemma 23] imply that $\dim \mathcal{M}(L) = 4 + \frac{1}{2}n(n-5)$. Since $5 = s(L) = \frac{1}{2}(n-1)(n-2) + 1 - \dim \mathcal{M}(L)$ and $\dim \mathcal{M}(L) = 4 + \frac{1}{2}n(n-5)$, we have $n = 7$. Therefore $L \cong L_{5,5} \oplus A(2) \cong L(4,5,1,6) \oplus A(2)$.

If L is a non-capable Lie algebra of nilpotency class 2 or 3, then by using Lemma 1.2, we have $n \leq 7$. Therefore, $L \cong L_{6,10}$ by using Lemma 2.2. This completes the proof. $\square$

We now consider the case that $\dim L^2 = 3$. By looking all nilpotent Lie algebras listed in [7], we may choose all n-dimensional nilpotent Lie algebras L such that $\dim L^2 = 3$ for $n = 5$ or 6 in the Table 1.

Table 1

Name	Nonzero multiplication
$L_{5,6}$	$[x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = [x_2,x_3] = x_5$
$L_{5,7}$	$[x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5$
$L_{5,9}$	$[x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_2,x_3] = x_5$
$L_{6,6}$	$[x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = [x_2,x_3] = x_5$
$L_{6,7}$	$[x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5$
$L_{6,9}$	$[x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_2,x_3] = x_5$
$L_{6,11}$	$[x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = [x_2,x_3] = [x_2,x_5] = x_6$
$L_{6,12}$	$[x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = [x_2,x_5] = x_6$
$L_{6,13}$	$[x_1,x_2] = x_3, [x_1,x_3] = [x_2,x_4] = x_5, [x_1,x_5] = [x_3,x_4] = x_6$
$L_{6,19}(\epsilon)$	$[x_1,x_2] = x_4, [x_1,x_3] = x_5, [x_1,x_5] = [x_2,x_4] = x_6, [x_3,x_5] = \epsilon x_6$
$L_{6,20}$	$[x_1,x_2] = x_4, [x_1,x_3] = x_5, [x_1,x_5] = [x_2,x_4] = x_6$
$L_{6,23}$	$[x_1,x_2] = x_3, [x_1,x_3] = [x_2,x_4] = x_5, [x_1,x_4] = x_6$
$L_{6,24}(\epsilon)$	$[x_1,x_2] = x_3, [x_1,x_3] = [x_2,x_4] = x_5, [x_1,x_4] = \varepsilon x_6, [x_2,x_3] = x_6$
$L_{6,25}$	$[x_1,x_2] = x_3, [x_1,x_3] = x_5, [x_1,x_4] = x_6$
$L_{6,26}$	$[x_1,x_2] = x_4, [x_1,x_3] = x_5, [x_2,x_3] = x_6$

Assume L is nilpotent Lie algebra of dimension 7 such that $\dim L^2 = 3$. By looking at the classification of all nilpotent Lie algebras in [10], L must be isomorphic to one of the Lie algebras listed in Tables 2 and 3.

Table 2 – 7-dimensional indecomposable nilpotent Lie algebras

Name	Nonzero multiplication
$37A$	$[x_1,x_2]=x_5,[x_2,x_3]=x_6,[x_2,x_4]=x_7$
$37B$	$[x_1,x_2]=x_5,[x_2,x_3]=x_6,[x_3,x_4]=x_7$
$37C$	$[x_1,x_2]=[x_3,x_4]=x_5,[x_2,x_3]=x_6,[x_2,x_4]=x_7$
$37D$	$[x_1,x_2]=[x_3,x_4]=x_5,[x_1,x_3]=x_6,[x_2,x_4]=x_7$
$257A$	$[x_1,x_2]=x_3,[x_1,x_3]=[x_2,x_4]=x_6,[x_1,x_5]=x_7$
$257B$	$[x_1,x_2]=x_3,[x_1,x_3]=x_6,[x_1,x_4]=[x_2,x_5]=x_7$
$257C$	$[x_1,x_2]=x_3,[x_1,x_3]=[x_2,x_4]=x_6,[x_2,x_5]=x_7$
$257D$	$[x_1,x_2]=x_3,[x_1,x_3]=[x_2,x_4]=x_6,\ [x_1,x_4]=[x_2,x_5]=x_7$
$257E$	$[x_1,x_2]=x_3,[x_1,x_3]=[x_4,x_5]=x_6,[x_2,x_4]=x_7$
$257F$	$[x_1,x_2]=x_3,[x_2,x_3]=[x_4,x_5]=x_6,[x_2,x_4]=x_7$
$257G$	$[x_1,x_2]=x_3,[x_1,x_3]=[x_4,x_5]=x_6,\ [x_1,x_5]=[x_2,x_4]=x_7$
$257H$	$[x_1,x_2]=x_3,[x_1,x_3]=[x_2,x_4]=x_6,[x_4,x_5]=x_7$
$257I$	$[x_1,x_2]=x_3,[x_1,x_3]=[x_1,x_4]=x_6,\ [x_1,x_5]=[x_2,x_3]=x_7$
$257J$	$[x_1,x_2]=x_3,[x_1,x_3]=[x_2,x_4]=x_6,\ [x_1,x_5]=[x_2,x_3]=x_7$
$257K$	$[x_1,x_2]=x_3,[x_1,x_3]=x_6,[x_2,x_3]=[x_4,x_5]=x_7$
$257L$	$[x_1,x_2]=x_3,[x_1,x_3]=[x_2,x_4]=x_6,\ [x_2,x_3]=[x_4,x_5]=x_7$
$147A$	$[x_1,x_2]=x_4,[x_1,x_3]=x_5,\ [x_1,x_6]=[x_2,x_5]=[x_3,x_4]=x_7$
$147B$	$[x_1,x_2]=x_4,[x_1,x_3]=x_5,\ [x_1,x_4]=[x_2,x_6]=[x_3,x_5]=x_7$
$1457A$	$[x_1,x_i]=x_{i+1}\quad i=2,3,\quad [x_1,x_4]=[x_5,x_6]=x_7$
$1457B$	$[x_1,x_i]=x_{i+1}\quad i=2,3,\ [x_1,x_4]=[x_2,x_3]=[x_5,x_6]=x_7$
$137A$	$[x_1,x_2]=x_5,[x_1,x_5]=[x_3,x_6]=x_7,[x_3,x_4]=x_6$
$137B$	$[x_1,x_2]=x_5,[x_3,x_4]=x_6,\ [x_1,x_5]=[x_2,x_4]=[x_3,x_6]=x_7$
$137C$	$[x_1,x_2]=x_5,[x_1,x_4]=[x_2,x_3]=x_6,\ [x_1,x_6]=x_7,[x_3,x_5]=-x_7$
$137D$	$[x_1,x_2]=x_5,[x_1,x_4]=[x_2,x_3]=x_6,\ [x_1,x_6]=[x_2,x_4]=x_7,[x_3,x_5]=-x_7$
$1357A$	$[x_1,x_2]=x_4,[x_1,x_4]=[x_2,x_3]=x_5,\ [x_1,x_5]=[x_2,x_6]=x_7,[x_3,x_4]=-x_7$
$1357B$	$[x_1,x_2]=x_4,[x_1,x_4]=[x_2,x_3]=x_5,\ [x_1,x_5]=[x_3,x_6]=x_7,[x_3,x_4]=-x_7$
$1357C$	$[x_1,x_2]=x_4,[x_1,x_4]=[x_2,x_3]=x_5,\ [x_1,x_5]=[x_2,x_4]=x_7,[x_3,x_4]=-x_7$

Table 3 – 7-dimensional decomposable nilpotent Lie algebras

Name	Name
$L_{4,3} \oplus H(1)$	$L_{6,19}(\epsilon) \oplus A(1)$
$L_{5,6} \oplus A(2)$	$L_{6,20} \oplus A(1)$
$L_{5,7} \oplus A(2)$	$L_{6,23} \oplus A(1)$
$L_{5,9} \oplus A(2)$	$L_{6,24}(\epsilon) \oplus A(1)$
$L_{6,11} \oplus A(1)$	$L_{6,25} \oplus A(1)$
$L_{6,12} \oplus A(1)$	$L_{6,26} \oplus A(1)$
$L_{6,13} \oplus A(1)$	

We need the following lemma from [23, Lemma 2.7] for the proof of the Main Theorem.

LEMMA 2.4. *Let L be an n-dimensional nilpotent Lie algebra such that $n = 5, 6$ or 7, $\dim L^2 = \dim Z(L) = 3$ and $Z(L) = L^2$. Then the structure and the Schur multiplier of L are given in the following table.*

Table 4

Name	$\dim \mathcal{M}(L)$	$s(L)$	*Name*	$\dim \mathcal{M}(L)$	$s(L)$
$L_{6,26}$	8	3	$37C$	11	5
$37A$	12	4	$37D$	11	5
$37B$	11	5			

LEMMA 2.5. *Let L be a nilpotent Lie algebra of dimension at most 7 such that $\dim L^2 = 3$, $\dim Z(L) = 2$ and $Z(L) \subset L^2$. Then the structure and the Schur multiplier of L are given in the following table.*

Table 5

Name	$\dim \mathcal{M}(L)$	$s(L)$	*Name*	$\dim \mathcal{M}(L)$	$s(L)$
$L_{5,9}$	3	4	$257E$	8	8
$L_{6,23}$	6	5	$257F$	9	7
$L_{6,24}(\epsilon)$	5	6	$257G$	8	8
$L_{6,25}$	6	5	$257H$	8	8
$257A$	9	7	$257I$	8	8
$257B$	8	8	$257J$	8	8
$257C$	9	7	$257K$	6	10
$257D$	8	8	$257L$	6	10

Proof. The proof is similar to [23, Lemma 2.5]. □

LEMMA 2.6. *Let L be an n-dimensional nilpotent Lie algebra such that $n = 7$, $\dim L^2 = 3$ and $\dim Z(L) = 4$. Then the structure and the Schur multiplier of L are given in the following table.*

Table 6

Name	$\dim \mathcal{M}(L)$	$s(L)$
$L_{5,9} \oplus A(2)$	8	8
$L_{6,26} \oplus A(1)$	11	5

Proof. Since $\dim Z(L) = 4$, L is isomorphic to $L_{5,9} \oplus A(2)$ or $L_{6,26} \oplus A(1)$ by searching in Tables 2 and 3. Let $L \cong L_{6,26} \oplus A(1)$. Since $\dim \mathcal{M}(L_{6,26}) = 8$ by using Table 4, we have $\dim \mathcal{M}(L) = 11$ by using [5, Theorem 1] and [11, Lemma 23]. Hence $s(L) = 5$. Also by using similar method, we can see $\dim \mathcal{M}(L_{5,9} \oplus A(2)) = 8$ and $s(L) = 8$. □

LEMMA 2.7 ([23, Lemma 2.9]). *Let L be an n-dimensional nilpotent Lie algebra such that $n = 5, 6$ or 7, $\dim L^2 = 3$ and $\dim Z(L) = 1$. Then the structure and the Schur multiplier of L are given in the following table.*

Table 7

Name	$\dim \mathcal{M}(L)$	$s(L)$	*Name*	$\dim \mathcal{M}(L)$	$s(L)$
$L_{5,6}$	3	4	$1457A$	6	10
$L_{5,7}$	3	4	$1457B$	6	10
$L_{6,11}$	5	6	$137A$	7	9
$L_{6,12}$	5	6	$137B$	7	9
$L_{6,13}$	4	7	$137C$	7	9
$L_{6,19}(\epsilon)$	5	6	$137D$	7	9
$L_{6,20}$	5	6	$1357A$	7	9
$147A$	8	8	$1357B$	6	10
$147B$	8	8	$1357C$	6	10

Recall that a Lie algebra L is called generalized Heisenberg of rank n if $L^2 = Z(L)$ and $\dim L^2 = n$.

LEMMA 2.8. *Let L be an n-dimensional generalized Heisenberg of rank* 3 *with $s(L) = 5$, then $n \leq 7$.*

Proof. By Theorem 1.3, we have $\dim \ker g = \dim \mathcal{M}(L^{ab}) - \dim L^2 + \dim L^{ab} \otimes_{mod} L^2 - \dim \mathcal{M}(L)$. Since $\dim \mathcal{M}(L) = \frac{1}{2}(n-1)(n-2) - 4$ and $\dim L^{ab} = n - 3$, we have $\dim \ker g = n - 3$.

By contrary, let $n \geq 8$. Then $d = \dim L^{ab} = n-3 \geq 5$. Since $\dim L^2 = 3$, we can choose a basis $\{x_1 + L^2, ..., x_d + L^2\}$ for L^{ab} such that $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_4]$ are non-trivial in L^2. Thus

$$L^{ab} \otimes_{mod} L^2 \cong \bigoplus_{i=1}^{d} (\langle x_i + L^2 \rangle \rangle \otimes_{mod} L^2).$$

Hence, all elements of

$\{[x_1, x_2] \otimes x_i + L^2 \oplus [x_i, x_1] \otimes x_2 + L^2 \oplus [x_2, x_i] \otimes x_1 + L^2, \mid 3 \leq i \leq d, i \neq 1, 2\}$

and

$\{[x_2, x_3] \otimes x_i + L^2 \oplus [x_i, x_3] \otimes x_2 + L^2 \oplus [x_3, x_i] \otimes x_2 + L^2, \mid 3 \leq i \leq d, i \neq 1, 2, 3\}$
$\{[x_3, x_4] \otimes x_i + L^2 \oplus [x_i, x_3] \otimes x_4 + L^2 \oplus [x_4, x_i] \otimes x_3 + L^2, \mid 3 \leq i \leq d, i \neq 2, 3, 4\}$
are linearly independent and so $2(n-6)+n-5 \leq \ker g$. That is a contradiction for $n \geq 8$. Therefore, the assumption is false and the result follows. □

Let $c(L)$ be used to show the nilpotency class of L. Then

LEMMA 2.9. *There is no nilpotent Lie algebra L with* $\dim L^2 = 3$, $\dim Z(L) = 1$ *and* $s(L) = 5$ *such that* $L/Z(L) \cong L_{5,8} \oplus A(2)$.

Proof. By contrary, let L be a nilpotent Lie algebra L with $\dim L^2 = 3$, $\dim Z(L) = 1$ and $s(L) = 5$ such that $L/Z(L) \cong L_{5,8} \oplus A(2)$. Then $\dim L = 8$ and $cl(L) = 3$. Since $cl(L) = 3$ and $\dim Z(L) = 1$, we have $L^3 = Z(L)$. On the other hand, $\dim \mathcal{M}(L) = \dim \mathcal{M}(L/Z(L)) + (\dim L/L^2 - 1) \dim Z(L) - \dim \ker \lambda_3$ and $\dim \ker \lambda_3 \geq 2$ by using proof [20, Theorem 1.1]. Thus
$\dim \mathcal{M}(L) \leq \dim \mathcal{M}(L/Z(L)) + (\dim L/L^2 - 1) \dim Z(L) - 2$.
It is a contradiction. □

THEOREM 2.10. *Let L be an n-dimensional nilpotent Lie algebra with* $s(L) = 5$ *and* $\dim L^2 = 3$. *Then L is isomorphic to one of the Lie algebras* $L_{6,23}$, $L_{6,25}$, $37B$, $37C$ *or* $37D$.

Proof. First assume that $\dim Z(L) \geq 5$, or $\dim Z(L) = 3$ and $Z(L) \neq L^2$, or $\dim Z(L) = 2$ and $Z(L) \not\subset L^2$. We show that in these cases, there is no such Lie algebra L of dimension n with $s(L) = 5$.

Let I be a central ideal of L of dimension one such that $L^2 \cap I = 0$. Since $\dim(L/I)^2 = 3$, by using [15, Theorem 3.1], we have

$$\dim \mathcal{M}(L/I) \leq \frac{1}{2} n(n-5) + 1.$$

If the equality holds, then

$$\frac{1}{2}(n-2)(n-3) + 1 - s(L/I) = \dim \mathcal{M}(L/I) = \frac{1}{2} n(n-5) + 1.$$

Therefore, $s(L/I)=3$ and by using [22, Theorem 3.2], there is no Lie algebra satisfying in $\dim(L/I)^2 = 3$. Thus [15, Corollary 2.3] and our assumption implies

$$\dim \mathcal{M}(L) = \frac{1}{2}(n-1)(n-2) - 4 \leq \frac{1}{2}n(n-5) + (n-4),$$

which is a contradiction. Therefore we may assume that $\dim Z(L) = 4$, or $\dim Z(L)= 3$ and $L^2= Z(L)$, or $\dim Z(L)= 2$ and $Z(L)\subset L^2$, or $\dim Z(L) = 1$.

If $\dim Z(L) = 4$, then there is a central ideal of L of dimension one such that $L^2 \cap I = 0$. Since $\dim(L/I)^2 = 3$, by using [15, Theorem 3.1], we have

$$\dim \mathcal{M}(L/I) \leq \frac{1}{2}n(n-5) + 1.$$

If the equality holds, then

$$\frac{1}{2}(n-2)(n-3) + 1 - s(L/I) = \dim \mathcal{M}(L/I) = \frac{1}{2}n(n-5) + 1.$$

Therefore $s(L/I) = 3$ and by using Table 4, $L/I \cong L_{6,26}$. Since $\dim Z(L) = 4$ and $\dim L = 7$, we have $L \cong L_{6,26} \oplus A(1)$ by using Lemma 2.6. Now let $\dim \mathcal{M}(L) \leq \frac{1}{2}n(n-5)$. Thus [15, corollary 2.3] and our assumption imply

$$\dim \mathcal{M}(L) = \frac{1}{2}(n-1)(n-2) - 4 \leq \frac{1}{2}n(n-5) + (n-4),$$

which is a contradiction.

If $\dim Z(L) = 3$ and $L^2 = Z(L)$, then L is isomorphic to one of the Lie algebras $37B$, $37C$ or $37D$ by using Lemmas 2.4 and 2.8.

Assume that $\dim Z(L) = 2$ and $Z(L) \subset L^2$. Then $\dim(L/Z(L))^2 = 1$. Since $L/Z(L)$ capable, by using [14, Theorem 3.5] and [13, Lemma 3.3], we have $L/Z(L) \cong H(1) \oplus A(n-5)$. Hence L is nilpotent of class 3. Therefore, by using [22, Theorem 2.6] for $c = 3$, we have

$$\dim L^3 + \frac{1}{2}(n-1)(n-2) - 3 \leq \dim \mathcal{M}(L/L^3) + \dim(L/Z_2(L) \otimes L^3).$$

Now since $1 \leq \dim L^3 \leq 2$, we can obtain that $n \leq 7$. Hence Lemma 2.5 implies that $L \cong L_{6,23}$ or $L \cong L_{6,25}$.

Finally, assume that $\dim Z(L) = 1$. Then $\dim(L/Z(L))^2 = 2$. By using [15, Corollary 2.3], we have

$$\frac{1}{2}(n-1)(n-2) - 3 \leq \frac{1}{2}(n-2)(n-3) + 1 - s(L/Z(L)) + n - 3.$$

Thus $s(L/Z(L)) \leq 3$.

If $s(L/Z(L)) = 0$, then $L \cong H(1) \oplus A(n-4)$ by [15, Theorem 3.1]. This case cannot occur, since $\dim(L/Z(L))^2 = 2$.

If $s(L/Z(L)) = 1$, then [13, Theorem 3.9] implies that $L \cong L(4,5,2,4)$. Therefore, $n = 6$.

If $s(L/Z(L)) = 2$, then $L/Z(L)$ is isomorphic to one of the Lie algebras $L(3,4,1,4)$, $L(4,5,2,4) \oplus A(1)$ or $H(m) \oplus A(n-2m-1)(m \geq 2)$ by using [13, Theorem 4.5]. In the case $L(3,4,1,4)$ or $L(4,5,2,4) \oplus A(1)$, we have $n=5$ or 7.

In the case $L/Z(L) \cong H(m) \oplus A(n-2m-1)(m \geq 2)$, then we have a contradiction, since $\dim(L/Z(L))^2 = 2$.

If $s(L/Z(L)) = 3$, $L/Z(L)$ is isomorphic to one of the Lie algebras $L(4,5,1,6)$, $L(5,6,2,7)$, $L'(5,6,2,7)$, $L(7,6,2,7)$, $L'(7,6,2,7)$ or $L(3,4,1,4) \oplus A(1)$ by [22, Main Theorem] and Lemma 2.9.

Hence $n = 5, 6$ or 7 when $\dim Z(L) = 1$. But there is no such Lie algebra by Lemma 2.7. This completes proof. □

THEOREM 2.11. *Let L be a non-abelian n-dimensional nilpotent Lie algebra. Then $s(L) = 5$ if and only if L is isomorphic to one of the Lie algebras $L(4,5,2,4) \oplus A(4)$, $L(3,4,1,4) \oplus A(3)$, $L(4,5,1,6) \oplus A(2)$, $L_{6,22}(\varepsilon) \oplus A(2)$, $L_{6,26} \oplus A(1)$, $L_{6,10}$, $L_{6,23}$, $L_{6,25}$, $37B$, $37C$ or $37D$.*

Proof. By using Lemma 2.1, Theorems 2.3 and 2.10, we can obtain the result. □

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