ON AN OVERDETERMINED EIGENVALUE PROBLEM WITH MEMS OPERATOR

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We consider an overdetermined eigenvalue problem related to the MEMS operator given by $L_\tau := \Delta^2 - \tau\Delta$ on a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. We give radial solutions on balls. Moreover, we establish a symmetry result with respect to operator L_τ, that is, under some hypotheses, we show that if a solution does exist to the overdetermined eigenvalue problem, then the domain Ω must be a ball.

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1. INTRODUCTION

In his speech delivered in 1959 and entitled "There's Plenty of Room at the Bottom", Feynman described the technology of the future as a "small-scale" technology, which will concern, among other things, condensed information storage, miniature computers, infinitesimal machines, powerful microscopes, etc. Later on, he described the mechanisms of miniature machines using ideas and techniques ranging from electrostatic actuation to quantum computation at the atomic-electron levels. Feynman was, thus, the precursor of a contemporary technology known as the so-called microelectromechanical systems (MEMS) and the so-called nanoelectromechanical systems (NEMS), and this constitutes a major advance since the resonant gate transistor produced by Nathanson *et al.* in 1967.

The modelling, optimization and design of MEMS and NEMS machines suggest an intensive use of mathematical analysis and numerical simulation (see [9] and the references therein).

In almost every kind of MEMS and NEMS systems, there are electrostatic actuation-based devices in routine operation. The electrostatic actuation is based on an electrostatic-controlled tunable capacitor and is widely used in micromirrors, accelerometers, switches, microresonators, etc.

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Physically speaking, consider two thin parallel plates electrically charged and imagine one of the plates stretches a small vertical distance. By continuum mechanics, the simplest elastic force depends on the Laplace or bi-Laplace of the deformation variable. If we set $u(x)$ ($x \in \Omega$ a bounded domain in $\mathbb{R}^2$) is the plate deformation variable, then in presence of elastic deformation characterized by $u \neq 0$, it is well-known that the elastic energy collects contributions from two sectors, i.e. the stretching energy sector given by

$$P = \int_\Omega \frac{T}{2} |\nabla u|^2 \mathrm{d}x,$$

where $T > 0$ is the tension constant, and the bending energy sector given by

$$Q = \int_\Omega \frac{D}{2} |\Delta u|^2 \mathrm{d}x,$$

where $D = 2h^3 Y/3(1-\nu^2)$, with h is the plate thickness, Y is the Young modulus and ν is the Poisson ratio.

Therefore, the total energy $E = P + Q + W$, where $W = \int_\Omega G(u)\mathrm{d}x$ is the electric potential (see, for example, [9] for more details about W), can be represented as

$$E(u) = \int_\Omega \left\{ \frac{D}{2} |\Delta u|^2 + \frac{T}{2} |\nabla u|^2 + G(u) \right\} \mathrm{d}x,$$

so that its Euler-Lagrange equation is

$$D\Delta^2 u - T\Delta u = -G'(u), \qquad x \in \Omega.$$

We see then that the modelling of the electrostatic actuation for MEMS and NEMS devices involves the fourth order operator of the form

$$L_\tau := \Delta^2 - \tau\Delta$$

(with $\tau = \frac{T}{D} > 0$ when avoiding the zero plate thickness limit $D = 0$).

Now, considering the MEMS operator (or NEMS operator) L_τ, our interest in this paper is to study the overdetermined eigenvalue problem related to L_τ, given by

$$(P) \qquad \begin{cases} L_\tau u = \Delta^2 u - \tau\Delta u = \lambda u + \mu & \text{in } \Omega, \\ u = a, \quad \dfrac{\partial u}{\partial \nu} = b, \quad \Delta u = c & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open connected domain in the Euclidean space $\mathbb{R}^N$ and $\dfrac{\partial}{\partial \nu}$ is the outward normal derivative operator on the boundary $\partial\Omega$.

Let us recall some basic known results concerning overdetermined eigenvalue problems.

For the problem

$$
\begin{cases} \Delta^2 u = -1 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, \quad \Delta u = c & \text{on } \partial\Omega, \end{cases} \tag{1}
$$

Bennett [2] showed that if (1) admits a solution $u \in C^4(\bar{\Omega})$ and Ω is a bounded domain in $\mathbb{R}^N$ with $C^{4+\varepsilon}$ boundary $\partial\Omega$, then Ω must be an open ball of radius $R = [|c|(N^2+2N)]^{1/2}$ and u is radially symmetric given by

$$
u = -\frac{1}{2N}\left\{\frac{N+2}{4}(Nc)^2 + \frac{Nc}{2}r^2 + \frac{1}{4(N+2)}r^4\right\}.
$$

Actually, Bennett's result is analogous to that of Serrin [13] and Weinberger [14] for the overdetermined torsion problem, namely,

$$
\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial \nu} = b & \text{on } \partial\Omega. \end{cases} \tag{2}
$$

Serrin [13] proved that for a bounded domain Ω whose boundary is of class C^2, if there exists a function $u \in C^2(\bar{\Omega})$ satisfying problem (2) with $b < 0$, then Ω must be a ball of radius $R = N|b|$ and u is radially symmetric given by

$$
u = \frac{(Nb)^2 - r^2}{2N}.
$$

Serrin used the Alexandrov [1] moving plane technique and the Hopf maximum principles [7, 8], while Weinberger's argument is much more elementary. It also uses the maximum principle but relies on Green's theorem to establish certain identities allowing to solve problem (2).

Bennett [2] used Weinberger's argument, by modifying it, to establish his result announced above for the fourth order problem (1). Precisely, he used a maximum principle for fourth order elliptic equations and several applications of Green's theorem. Unfortunately, Bennet's argument does not extend to more general equations. Using the method of moving planes of Serrin and assuming in addition that $u \geq 0$ in Ω, Dalmasso [4, 5] was able to treat more general biharmonic equations and systems.

Consider now the problem

$$
\begin{cases} \Delta^2 u = \lambda u + \mu & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, \quad \dfrac{\partial^2 u}{\partial \nu^2} = c & \text{on } \partial\Omega. \end{cases} \tag{3}
$$

When $\mu \neq 0$, Dalmasso [6] gives a symmetry result for the problem (3) in dimension $N = 2$: let Ω be a smooth bounded domain in $\mathbb{R}^2$, with $\partial\Omega \in C^{6,\gamma}$ for some $\gamma \in (0, 1]$ and such that Ω is ε close in C^4 sense to unit open ball B in $\mathbb{R}^2$. Suppose $0 < \lambda \leq \varepsilon < \varepsilon_0$. If there exits a solution $u \in C^{4,\gamma}(\bar{\Omega})$ satisfying (3) then Ω is a ball.

When $\mu = 0$, Dalmasso [6] gives again a symmetry result for the problem (3) in dimension $N = 2$: let Ω be a smooth bounded domain in $\mathbb{R}^2$, with $\partial\Omega \in C^{6,\gamma}$ for some $\gamma \in (0, 1]$ and such that Ω is ε close in C^4 sense to unit open ball B in $\mathbb{R}^2$, then there is a positive eigenfunction u for the first Dirichlet eigenvalue $\lambda_1^{(2)}$ of Δ^2 in Ω, which is simple. It is then proved that the domain Ω must be a ball in $\mathbb{R}^2$ and u is radially symmetric, when $\lambda = \lambda_1^{(2)}$.

For more results on this issue, we can also see [3], [12] and the references therein. In [12], we find a survey on the three main methods (Serrin's method, Weinberger's method and the duality method) used in this field.

Coming back to the MEMS operator $L_\tau = \Delta^2 - \tau\Delta$ (with $\tau > 0$) and to the overdetermined problem (P), our aim in this paper is to carry on the previous results.

Our first result deals with radial solutions of the problem (P) when the domain Ω is a ball in $\mathbb{R}^N$. Those radial solutions are expressed by the mean of Bessel functions.

We assume that $\tau > 0$ and $\lambda > 0$, and we set

$$\eta := \frac{\sqrt{\tau^2 + 4\lambda} - \tau}{2} > 0 \qquad \text{and} \qquad \theta := \sqrt{\eta} > 0. \tag{4}$$

THEOREM 1.1. *Let $N \geq 2$ and $\Omega = B(0, 1)$ the unit ball in $\mathbb{R}^N$. Let $\beta := \frac{N-2}{2}$ and consider the Bessel function of the first kind of order β denoted by J_β. Let η and θ given by (4). Then, we have the following.*

(i) *Suppose $\mu = 0$, $a = c = 0$, $b \neq 0$ and θ satisfies $J_\beta(\theta) = 0$. Then, there exists a radial solution to problem (P), given by*

$$u(x) = v(r) = \frac{b}{\theta J'_\beta(\theta)}\, r^{-\beta} J_\beta(\theta r), \qquad \forall r = |x| \in (0, 1].$$

Moreover, we have $\frac{\partial(\Delta u)}{\partial\nu} = -\eta b$ on ∂B.

(ii) *Suppose $\mu = 0$, $a \neq 0$, $b = 0$, $c = -\eta a$ and θ satisfies $\theta J'_\beta(\theta) - \beta J_\beta(\theta) = 0$. Then, there exists a radial solution to problem (P), given by*

$$u(x) = v(r) = \frac{a}{J_\beta(\theta)}\, r^{-\beta} J_\beta(\theta r), \qquad \forall r = |x| \in (0, 1].$$

Moreover, we have $\frac{\partial(\Delta u)}{\partial\nu} = 0$ on ∂B.

(iii) *Suppose* $\mu \neq 0$, $a = b = 0$, $c = -\frac{\eta\mu}{\lambda}$ *and* θ *satisfies* $\theta J'_\beta(\theta) - \beta J_\beta(\theta) = 0$. *Then, there exists a radial solution to problem* (P), *given by*

$$u(x) = v(r) = \frac{\mu}{\lambda}\left[\frac{1}{J_\beta(\theta)}\, r^{-\beta} J_\beta(\theta r) - 1\right], \qquad \forall r = |x| \in (0,1].$$

Moreover, we have $\frac{\partial(\Delta u)}{\partial\nu} = 0$ *on* ∂B.

Our second result is a symmetry result for problem (P), that is, existence of solution to problem (P) implies that necessarily the domain Ω must be a ball in $\mathbb{R}^N$. We establish such a result under conditions required for the boundary parameters a, b and c, in both cases $\mu = 0$ and $\mu \neq 0$.

We set

$$\lambda_\tau := \eta_1(\eta_1 + \tau) > 0, \tag{5}$$

where η_1 is the first eigenvalue of the Laplacian under Dirichlet boundary condition. We have the following theorem.

THEOREM 1.2. *Let* $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ *be a smooth bounded domain with* $\partial\Omega \in C^{4,\gamma}$ *for some* $\gamma \in (0,1]$. *Suppose that there exists a solution* u *in* $C^{4,\gamma}(\bar{\Omega})$ *to problem* (P). *Then, we have the following.*

(i) *If* $\mu = 0$, $c = -\eta a$, $\frac{\partial(\Delta u)}{\partial\nu} = -\eta b$ *on* $\partial\Omega$ *and either* $\lambda = \lambda_\tau$ *or* $ab > 0$, *then* Ω *must be a ball.*

(ii) *If* $\mu \neq 0$, $a = b = 0$, $c = -\frac{\eta\mu}{\lambda}$, $\frac{\partial(\Delta u)}{\partial\nu} = 0$ *on* $\partial\Omega$ *and* $\lambda = \lambda_\tau$, *then* Ω *must be a ball.*

In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2 using results due to Dalmasso [6] and to Chamberland, Gladwell and Willms [3], which we recall in the Appendix at the end of the paper.

2. RADIAL SOLUTIONS OF (P) ON BALLS

In this section, we prove Theorem 1.1. We assume that the domain Ω is a ball in $\mathbb{R}^N$ with $N \geq 2$ and for sake of simplicity, we consider the unit ball $B(0,1)$. As mentioned in the Introduction, radial solutions for problem (P) are constructed with the use of Bessel functions.

We recall that the Bessel functions are canonical solutions $y(r)$ of the differential Bessel equation

$$r^2 y''(r) + r y'(r) + (r^2 - \alpha^2) y(r) = 0, \tag{6}$$

for any real or complex number α.

For $\beta = \frac{N-2}{2}$ $(N \geq 2)$ (is an integer or half-integer), we consider the Bessel functions of the first kind J_β.

Now, writing $y(r) = J_\beta(\theta r)$ and taking $\alpha = \beta$ in the equation (6), we have that $J_\beta(\theta r)$ satisfies the equation

$$J''_\beta(\theta r) + \frac{1}{\theta r} J'_\beta(\theta r) + \Big(1 - \frac{\beta^2}{\theta^2 r^2}\Big) J_\beta(\theta r) = 0. \tag{7}$$

Proof of Theorem 1.1. Let $\theta > 0$ given by (4) and consider the Bessel function $J_\beta(\theta r)$ satisfying equation (7).

Proof of (i) : Let $b \neq 0$ and suppose that θ satisfies $J_\beta(\theta) = 0$. Consider the radial function

$$u(x) = v(r) := \frac{b}{\theta J'_\beta(\theta)} r^{-\beta} J_\beta(\theta r), \quad \forall r = |x| \in (0, 1]. \tag{8}$$

We have

$$v'(r) = -\frac{b\beta}{\theta J'_\beta(\theta)}\, r^{-\beta-1} J_\beta(\theta r) + \frac{b}{J'_\beta(\theta)}\, r^{-\beta} J'_\beta(\theta r).$$

Using (7), we have

$$v''(r) = (2\beta+1) r^{-1} \left[\frac{b\beta}{\theta J'_\beta(\theta)}\, r^{-\beta-1} J_\beta(\theta r) - \frac{b}{J'_\beta(\theta)}\, r^{-\beta} J'_\beta(\theta r) \right] - \frac{\theta b\, r^{-\beta}}{J'_\beta(\theta)} J_\beta(\theta r)$$

$$= -(2\beta+1) r^{-1} v'(r) - \theta^2 v(r) = -(N-1) r^{-1} v'(r) - \theta^2 v(r).$$

Then

$$\Delta u(x) = \frac{N-1}{r}\, v'(r) + v''(r) = -\theta^2 v(r) = -\theta^2 u(x),$$

and so

$$\Delta^2 u(x) = \Delta(\Delta u) = -\theta^2 \Delta u(x) = \theta^4 u(x).$$

Therefore, we obtain

$$L_\tau u(x) = \Delta^2 u(x) - \tau \Delta u(x) = (\theta^4 + \tau\theta^2) u(x).$$

As by (4), we have $\lambda = \theta^4 + \tau\theta^2$, then $u(x) = v(r)$ given by (8) is a solution of the equation

$$L_\tau u = \lambda u \quad \text{ on } \; B(0, 1).$$

Moreover, on ∂B, we have

$$u(x) = v(r)\Big|_{r=1} = \frac{b}{\theta J'_\beta(\theta)} J_\beta(\theta) = 0,$$

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x) \cdot \vec{\nu}(x) = \frac{1}{r}\, v'(r)\, \vec{x} \cdot \vec{\nu}(x)\Big|_{r=1} = -\frac{b\beta}{\theta J'_\beta(\theta)} J_\beta(\theta) + \frac{b}{J'_\beta(\theta)} J'_\beta(\theta) = b,$$

$$\Delta u(x) = -\theta^2 u(x) = -\theta^2 \, \frac{b}{\theta J_\beta'(\theta)} r^{-\beta} J_\beta(\theta r)\Big|_{r=1} = -\theta \frac{b}{J_\beta'(\theta)} J_\beta(\theta) = 0,$$

$$\frac{\partial}{\partial \nu}\big(\Delta u(x)\big) = -\theta^2 \, \frac{\partial u(x)}{\partial \nu} = -\,\theta^2 b = -\eta b.$$

Proof of (ii) : Let $a \neq 0$ and suppose that θ satisfies $\theta J_\beta'(\theta) - \beta J_\beta(\theta) = 0$. Consider the radial function

(9) $$u(x) = v(r) := \frac{a}{J_\beta(\theta)} \, r^{-\beta} J_\beta(\theta r), \quad \forall r = |x| \in (0, 1].$$

We have

$$v'(r) = -\frac{\beta a}{J_\beta(\theta)} \, r^{-\beta-1} J_\beta(\theta r) + \frac{\theta a}{J_\beta(\theta)} \, r^{-\beta} J_\beta'(\theta r).$$

Using (7), we obtain

$$v''(r) = (2\beta+1) r^{-1} \left[\frac{\beta a}{J_\beta(\theta)} \, r^{-\beta-1} J_\beta(\theta r) - \frac{\theta a}{J_\beta(\theta)} \, r^{-\beta} J_\beta'(\theta r) \right] - \frac{\theta^2 a r^{-\beta}}{J_\beta(\theta)} J_\beta(\theta r)$$

$$= -(2\beta+1) r^{-1} v'(r) - \theta^2 v(r) = -(N-1) r^{-1} v'(r) - \theta^2 v(r).$$

Therefore,

$$\Delta u(x) = \frac{N-1}{r} \, v'(r) + v''(r) = -\theta^2 v(r) = -\theta^2 u(x),$$

and we have

$$\Delta^2 u(x) = -\theta^2 \Delta u(x) = \theta^4 u(x),$$

then

$$L_\tau u(x) = \Delta^2 u(x) - \tau \Delta u(x) = (\theta^4 + \tau\theta^2) u(x).$$

As by (4), we have $\lambda = \theta^4 + \tau\theta^2$, then $u(x) = v(r)$ given by (9) is a solution of the equation

$$L_\tau u = \lambda u \quad \text{on } B(0,1).$$

Moreover, on ∂B, we have

$$u(x) = v(r)\Big|_{r=1} = \frac{a}{J_\beta(\theta)} J_\beta(\theta) = a,$$

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x) \cdot \vec{\nu}(x) = \frac{1}{r} \, v'(r) \; \vec{x} \cdot \vec{\nu}(x)\Big|_{r=1} = \Big(\frac{a}{J_\beta(\theta)} \Big) \Big(\theta \, J_\beta'(\theta) - \beta \, J_\beta(\theta) \Big) = 0,$$

$$\Delta u(x) = -\theta^2 u(x) = -\theta^2 \, \frac{a}{J_\beta(\theta)} r^{-\beta} J_\beta(\theta r)\Big|_{r=1} = -\frac{a\,\theta^2}{J_\beta(\theta)} J_\beta(\theta) = -a\theta^2 = -a\eta,$$

$$\frac{\partial}{\partial \nu}\big(\Delta u(x)\big) = -\theta^2\,\frac{\partial u(x)}{\partial \nu} = 0.$$

Proof of (iii): Let $\mu \neq 0$ and θ satisfies $\theta J_\beta^{'}(\theta) - \beta J_\beta(\theta) = 0$. Consider the radial function

$$u(x) = v(r) := \frac{\mu}{\lambda}\left[\frac{1}{J_\beta(\theta)}\, r^{-\beta} J_\beta(\theta r) - 1\right], \quad \forall r = |x| \in (0,1]. \tag{10}$$

We have

$$v^{'}(r) = \frac{\mu}{\lambda}\left[\frac{-\beta}{J_\beta(\theta)}\, r^{-\beta-1} J_\beta(\theta r) + \frac{\theta}{J_\beta(\theta)}\, r^{-\beta} J_\beta^{'}(\theta r)\right].$$

Using (7), we obtain

$$\begin{aligned} v^{''}(r) &= (2\beta+1)r^{-1}\frac{\mu}{\lambda}\left[\frac{\beta}{J_\beta(\theta)} r^{-\beta-1} J_\beta(\theta r) - \frac{\theta}{J_\beta(\theta)} r^{-\beta} J_\beta^{'}(\theta r)\right] \\ &\quad - \theta^2 \frac{\mu}{\lambda}\frac{1}{J_\beta(\theta)} r^{-\beta} J_\beta(\theta r) \\ &= -(2\beta+1)r^{-1}v'(r) - \theta^2\left[v(r)+\frac{\mu}{\lambda}\right] = -(N-1)r^{-1}v'(r) - \theta^2\left[v(r)+\frac{\mu}{\lambda}\right]. \end{aligned}$$

Thus

$$\Delta u(x) = \frac{N-1}{r}\, v'(r) + v''(r) = -\theta^2\left[v(r)+\frac{\mu}{\lambda}\right] = -\theta^2\left[u(x)+\frac{\mu}{\lambda}\right],$$

so we have

$$\Delta^2 u(x) = -\theta^2 \Delta u(x) = \theta^4\left[u(x)+\frac{\mu}{\lambda}\right].$$

We are then led to

$$L_\tau u(x) = \Delta^2 u(x) - \tau\Delta u(x) = (\theta^4+\tau\theta^2)\Big(u(x)+\frac{\mu}{\lambda}\Big).$$

As by (4), we have $\lambda = \theta^4 + \tau\theta^2$, then $u(x) = v(r)$ given by (10) is a solution of the equation

$$L_\tau u = \lambda u + \mu \quad \text{on } B(0,1).$$

Moreover, on ∂B, we have

$$u(x) = v(r)\Big|_{r=1} = 0,$$

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x)\cdot\vec{\nu}(x) = \frac{1}{r}\, v'(r)\ \vec{x}\cdot\vec{\nu}(x)\Big|_{r=1} = \Big(\frac{\mu}{\lambda J_\beta(\theta)}\Big)\Big(\theta\, J_\beta^{'}(\theta) - \beta\, J_\beta(\theta)\Big) = 0,$$

$$\Delta u(x) = -\theta^2 u(x) - \theta^2\frac{\mu}{\lambda} = -\frac{\eta\mu}{\lambda}, \qquad \frac{\partial}{\partial \nu}\big(\Delta u(x)\big) = -\theta^2\,\frac{\partial u(x)}{\partial \nu} = 0.$$

□

3. A SYMMETRY RESULT FOR PROBLEM (P)

In this section, we prove Theorem 1.2. We start by some technical lemmas.

LEMMA 3.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega$. Let $u \in C^4(\bar{\Omega})$ be a solution of problem (P) with $\frac{\partial(\Delta u)}{\partial \nu} = d$ on $\partial\Omega$. We have the following Pohozaev-type identity*

(11)
$$\int_\Omega (\Delta u)^2 \mathrm{d}x + \lambda \int_\Omega u^2 \mathrm{d}x + \frac{(N+2)}{2}\mu \int_\Omega u \mathrm{d}x = \frac{N|\Omega|}{2}\Big[\lambda a^2 + \tau b^2 - c^2 + 2\mu a - 2bd\Big] + \frac{|\partial\Omega|}{2}\Big[Ncb - (N-2)(d-\tau b)a\Big] + c\sum_{i,j=1}^N \int_{\partial\Omega} x_i u_{x_i x_j} \nu_j \mathrm{d}\sigma.$$

If $b = 0$, then

$$\int_\Omega (\Delta u)^2 \mathrm{d}x + \lambda \int_\Omega u^2 \mathrm{d}x + \frac{(N+2)}{2}\mu \int_\Omega u \mathrm{d}x = \frac{1}{2}\Big[\lambda a^2 + c^2 + 2\mu a\Big] N|\Omega| + \frac{1}{2}\Big[(2-N)ad\Big]|\partial\Omega|, \tag{12}$$

where $|\Omega|$ is the N-dimensional volume of Ω, and $|\partial\Omega|$ is the $(N-1)$-dimensional surface area of $\partial\Omega$.

Proof. We multiply equation $L_\tau u = \Delta^2 u - \tau\Delta u = \lambda u$ by $x \cdot \nabla u$ and integrate over Ω,

$$\int_\Omega \Delta^2 u(x\cdot\nabla u)\mathrm{d}x - \tau \int_\Omega \Delta u(x\cdot\nabla u)\mathrm{d}x = \lambda \int_\Omega (x\cdot\nabla u)u\mathrm{d}x + \mu \int_\Omega (x\cdot\nabla u)\mathrm{d}x. \tag{13}$$

Using integrations by parts and the boundary conditions, we have

$$\int_\Omega (x\cdot\nabla u)\mathrm{d}x = Na|\Omega| - N\int_\Omega u\mathrm{d}x, \tag{14}$$

$$\int_\Omega (x\cdot\nabla u)u\mathrm{d}x = \frac{Na^2|\Omega|}{2} - \frac{N}{2}\int_\Omega u^2\mathrm{d}x, \tag{15}$$

$$\int_\Omega \Delta u(x\cdot\nabla u)\mathrm{d}x = \frac{Nb^2|\Omega|}{2} + \frac{N-2}{2}\int_\Omega |\nabla u|^2\mathrm{d}x, \tag{16}$$

and
(17)

$$\begin{aligned}
\int_\Omega \Delta^2 u(x\cdot\nabla u)\mathrm{d}x &= \sum_{i,j=1}^N \int_\Omega x_i u_{x_i}(\Delta u)_{x_jx_j}\mathrm{d}x\\
&= \sum_{i,j=1}^N \int_{\partial\Omega} x_i u_{x_i}(\Delta u)_{x_j}\nu_j \mathrm{d}\sigma - \sum_{i,j=1}^N \int_{\partial\Omega} x_i u_{x_ix_j}\Delta u\,\nu_j\mathrm{d}\sigma\\
&\quad - \int_{\partial\Omega}\frac{\partial u}{\partial\nu}\Delta u\mathrm{d}\sigma + 2\int_\Omega(\Delta u)^2\mathrm{d}x + \sum_{i,j=1}^N\int_\Omega x_i u_{x_jx_jx_i}\Delta u\mathrm{d}x\\
&= \frac{(4-N)}{2}\int_\Omega(\Delta u)^2\mathrm{d}x - c\sum_{i,j=1}^N\int_{\partial\Omega} x_i u_{x_ix_j}\nu_j\mathrm{d}\sigma + \frac{Nc^2|\Omega|}{2}\\
&\quad + bdN|\Omega| - bc|\partial\Omega|.
\end{aligned}$$

On the other hand, multiplying equation $L_\tau u = lu$ by u, integrating by parts and using the boundary conditions, we obtain

$$\int_\Omega(\Delta u)^2\mathrm{d}x+\tau\int_\Omega|\nabla u|^2\mathrm{d}x = (bc-ad+\tau ab)\,|\partial\Omega|+\lambda\int_\Omega u^2\mathrm{d}x+\mu\int_\Omega u\mathrm{d}x. \tag{18}$$

Finally, combining equalities (13)-(18), we obtain (11).

Suppose that $b=0$.

Furthermore, thanks to the Dirichlet boundary conditions, we know that (see, for example, [11])

$$\frac{\partial^2 u}{\partial x_i x_j} = \frac{\partial^2 u}{\partial \nu^2}\nu_i\nu_j \quad \text{on } \partial\Omega.$$

We have

$$\sum_{i,j=1}^N\int_{\partial\Omega} x_i u_{x_ix_j}\,\nu_j\mathrm{d}\sigma = c\int_{\partial\Omega} x\cdot\nu\,\mathrm{d}\sigma = cN|\Omega|. \tag{19}$$

Then from (11) and (19), we deduce that (12). $\square$

We recall the following result (see, for example, Miranda [10]).

Remark 3.1 ([10]). Let Ω be a bounded smooth domain in $\mathbb{R}^N$ $(N\geq 2)$, with $\partial\Omega$ of class $C^{2+k,\gamma}$ for some $k\geq 0$, $\gamma\in(0,1]$. If $u\in C^2(\Omega)\cap C(\bar\Omega)$ satisfies

$$\begin{cases} -\Delta u = \lambda u+\mu & \text{in} \quad \Omega,\\ \quad u=0 & \text{on} \quad \partial\Omega,\end{cases}$$

then $u\in C^{2+k,\gamma}(\bar\Omega)$.

Using a result due to Dalmasso [6], Lemma 3.1 and Remark 3.1, we have the following lemma.

LEMMA 3.2. *Let* Ω *be a bounded smooth domain in* $\mathbb{R}^N$ $(N \geq 2)$*, with* $\partial\Omega$ *of class* $C^{4,\gamma}$ *for some* $\gamma \in (0,1]$*. Let* $\lambda > 0$*,* $\tau > 0$ *and* η *given by* (4)*. The following statements are equivalent.*

(i) *There exists* $u \not\equiv 0$ *in* $C^{4,\gamma}(\bar{\Omega})$ *satisfying*

$$L_\tau u = \lambda u \quad in \quad \Omega, \tag{20}$$

$$u = \Delta u = 0, \quad \frac{\partial u}{\partial \nu} = b, \quad \frac{\partial(\Delta u)}{\partial \nu} = -b\eta \ on \quad \partial\Omega. \tag{21}$$

(ii) *There exists* $u \not\equiv 0$ *in* $C^2(\Omega) \cap C(\bar{\Omega})$ *satisfying*

$$-\Delta u = \eta u \quad in \quad \Omega, \tag{22}$$

$$u = 0, \quad \frac{\partial u}{\partial \nu} = b \quad on \quad \partial\Omega. \tag{23}$$

Proof. (ii)$\Longrightarrow$(i): Let $u \not\equiv 0$ in $C^2(\Omega) \cap C(\bar{\Omega})$ satisfying (22)-(23) with η given by (4). Using Remark 3.1, we deduce that u in $C^{4,\gamma}(\bar{\Omega})$. We have u satisfies $-\Delta u = \eta u$ in Ω, then

$$L_\tau u = \Delta^2 u - \tau \Delta u = \eta(\eta + \tau)u = \lambda u \quad \text{in}\, \Omega.$$

As $u = 0$ and $\dfrac{\partial u}{\partial \nu} = b$ on $\partial\Omega$, then $\Delta u = 0$ and $\dfrac{\partial(\Delta u)}{\partial \nu} = -\eta b$ on $\partial\Omega$.

(i)$\Longrightarrow$(ii): Suppose that there exists $u \not\equiv 0$ in $C^{4,\gamma}(\bar{\Omega})$ satisfying (20)-(21). In order to prove (ii), in view of Theorem 4.2, we just need to prove that for all $w \in C^{2,\gamma}(\bar{\Omega})$ satisfying

$$-\Delta w = \eta w \quad \text{in} \quad \Omega, \tag{24}$$

we have

$$\int_{\partial\Omega} w \mathrm{d}\sigma = 0.$$

Let $w \in C^{2,\gamma}(\bar{\Omega})$ satisfying (24). Multiplying (20) by w, integrating by parts over Ω and using the boundary conditions (21), we have

$$\begin{aligned} \lambda \int_\Omega u w \mathrm{d}x = \int_\Omega w L_\tau u \mathrm{d}x = \int_\Omega (\Delta^2 w) u \mathrm{d}x + b \int_{\partial\Omega} \Delta w \mathrm{d}\sigma - \eta b \int_{\partial\Omega} w \mathrm{d}\sigma \\ - \tau \int_\Omega u \Delta w \mathrm{d}x - b\tau \int_{\partial\Omega} w \mathrm{d}\sigma. \end{aligned} \tag{25}$$

On the other hand, we have

$$L_\tau w = \Delta^2 w - \tau \Delta w = \eta(\eta+\tau) w = \lambda w \quad \text{in} \quad \Omega, \tag{26}$$

$$\Delta w = -\eta w \quad \text{on} \quad \partial\Omega. \tag{27}$$

Using (25),(26) and (27), we finally get

$$\lambda \int_\Omega uw \mathrm{d}x = \int_\Omega u L_\tau w \mathrm{d}x - b(\tau+2\eta)\int_{\partial\Omega} w \mathrm{d}x = \lambda \int_\Omega uw \mathrm{d}x - b(\tau+2\eta)\int_{\partial\Omega} w \mathrm{d}x,$$

then

$$b(\tau+2\eta)\int_{\partial\Omega} w \mathrm{d}x = 0. \tag{28}$$

On the other hand, as u satisfies (20)-(21), then using the Pohozaev identity (11) given by Lemma 3.1 with $\mu = a = c = 0$ and $d = -\eta b$, we obtain

$$\int_\Omega (\Delta u)^2 \mathrm{d}x + \lambda \int_\Omega u^2 \mathrm{d}x = \frac{N}{2} b^2(\tau+2\eta)|\Omega|. \tag{29}$$

As we have $u \not\equiv 0$ in Ω, $\lambda > 0$, $\eta > 0$, $\tau > 0$, we deduce from (29) that $b \neq 0$, and so (28) gives

$$\int_{\partial\Omega} w \mathrm{d}x = 0.$$

□

Using a result due to Dalmasso [6] and Remark 3.1, we have the following lemma.

Lemma 3.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$, with $\partial\Omega$ of class $C^{4,\gamma}$ for some $\gamma \in (0,1]$. Let $a \neq 0$, $b \in \mathbb{R}$, $\lambda > 0$, $\tau > 0$ and η given by (4). The following statements are equivalent.*

(i) *There exists u in $C^{4,\gamma}(\bar{\Omega})$ satisfying*

$$L_\tau u = \lambda u \quad in \quad \Omega, \tag{30}$$

$$u = a, \quad \frac{\partial u}{\partial \nu} = b, \quad \Delta u = -a\eta, \quad \frac{\partial(\Delta u)}{\partial \nu} = -b\eta \quad on \quad \partial\Omega. \tag{31}$$

(ii) *There exists u in $C^2(\Omega) \cap C(\bar{\Omega})$ satisfying*

$$-\Delta u = \eta u \quad in \quad \Omega, \tag{32}$$

$$u = a, \quad \frac{\partial u}{\partial \nu} = b \quad on \quad \partial\Omega. \tag{33}$$

Proof. (ii) $\Longrightarrow$ (i): Let u in $C^2(\Omega)\cap C(\bar{\Omega})$ satisfying (32)-(33) with η given by (4). Using Remark 3.1, we deduce that u in $C^{4,\gamma}(\bar{\Omega})$. We have u satisfies $-\Delta u = \eta u$ in Ω, then

$$L_\tau u = \Delta^2 u - \tau\Delta u = \eta(\eta+\tau)u = \lambda u \quad \text{in}\, \Omega.$$

As $u = a$ and $\dfrac{\partial u}{\partial \nu} = b$ on $\partial\Omega$, then $\Delta u = -a\eta$ and $\dfrac{\partial(\Delta u)}{\partial \nu} = -b\eta$ on $\partial\Omega$.

(i) $\Longrightarrow$ (ii): Suppose that there exists u in $C^{4,\gamma}(\bar{\Omega})$ satisfying (30)-(31). In order to prove (ii), in view of Theorem 4.1, we just need to prove that there exists a constant d such that $\int_\Omega w\mathrm{d}x = d\int_{\partial\Omega} w\mathrm{d}\sigma$, for all $w \in C^{2,\gamma}(\bar{\Omega})$ satisfying

$$-\Delta w = \eta w \quad \text{in} \quad \Omega. \tag{34}$$

Let $w \in C^{2,\gamma}(\bar{\Omega})$ satisfying (34). Multiplying (30) by w, integrating by parts over Ω and using the boundary conditions (31), we have

$$\begin{aligned}\lambda\int_\Omega uw\mathrm{d}x = \int_\Omega wL_\tau u\mathrm{d}x = &\int_\Omega(\Delta^2 w)u\mathrm{d}x + b\int_{\partial\Omega}\Delta w\mathrm{d}\sigma\\ &- a\int_{\partial\Omega}\frac{\partial\Delta w}{\partial\nu}\mathrm{d}\sigma - b\eta\int_{\partial\Omega} w + a\eta\int_{\partial\Omega}\frac{\partial\ w}{\partial\nu}\mathrm{d}\sigma\\ &-\tau\int_\Omega u\Delta w\mathrm{d}x\mathrm{d}\sigma - \tau b\int_{\partial\Omega} w\mathrm{d}\sigma + \tau a\int_{\partial\Omega}\frac{\partial\ w}{\partial\nu}\mathrm{d}\sigma.\end{aligned} \tag{35}$$

Using $w \in C^{2,\gamma}(\bar{\Omega})$ satisfying (34), we have

$$L_\tau w = \Delta^2 w - \tau\Delta w = \eta(\eta+\tau)w = \lambda w \quad \text{in}\ \ \Omega, \tag{36}$$

$$\Delta w = -\eta w \quad \text{on} \quad \partial\Omega. \tag{37}$$

On the other hand, using the divergence theorem, (34) and (37), we have

$$\int_{\partial\Omega}\frac{\partial\ w}{\partial\nu}\mathrm{d}\sigma = \int_\Omega\Delta w\mathrm{d}x = -\eta\int_\Omega w\mathrm{d}x. \tag{38}$$

$$\int_{\partial\Omega}\frac{\partial\ (\Delta w)}{\partial\nu}\mathrm{d}\sigma = \int_\Omega\Delta^2 w\mathrm{d}x = \eta^2\int_\Omega w\mathrm{d}x. \tag{39}$$

$$\int_{\partial\Omega}\Delta w\mathrm{d}\sigma = -\eta\int_{\partial\Omega} w\mathrm{d}x. \tag{40}$$

Using (35)-(36) and (38)-(39)-(40), we finally get

$$\lambda\int_\Omega uw\mathrm{d}x = \lambda\int_\Omega uw\mathrm{d}x - b(2\eta+\tau)\int_{\partial\Omega} w\mathrm{d}\sigma - a\eta(2\eta+\tau)\int_\Omega w\mathrm{d}x.$$

Then

$$-b(2\eta+\tau)\int_{\partial\Omega} w\mathrm{d}\sigma = a\eta(2\eta+\tau)\int_{\Omega} w\mathrm{d}x. \tag{41}$$

As $(2\eta+\tau)\neq 0$. Then there exists a constant $d=-\frac{b}{a\alpha}$ such that

$$\int_{\Omega} w\mathrm{d}x = d\int_{\partial\Omega} w\mathrm{d}\sigma, \tag{42}$$

for all $w\in C^{2,\gamma}(\bar{\Omega})$ satisfying $-\Delta w=\eta w$ in Ω. □

Proof of Theorem 1.2 completed. (i) By Lemma 3.2 and Lemma 3.3, we know that there exists u in $C^2(\Omega)\cap C(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta u=\eta u & \text{in} \quad \Omega,\\ u=a, \quad \dfrac{\partial u}{\partial \nu}=b & \text{on} \quad \partial\Omega, \end{cases}$$

with η given by (4). As $\lambda=\lambda_\tau=\eta_1(\eta_1+\tau)$, then $\eta=\eta_1$ and using Proposition 4.1 in the Appendix, we deduce that Ω is a ball. As $ab>0$ and using Proposition 4.1, we deduce that Ω is a ball.

(ii) Suppose there exists $u\in C^{4,\gamma}(\bar{\Omega})$ solution of (P) with $a=b=0$, $c=-\frac{\mu\eta}{\lambda}$ and $\frac{\partial(\Delta u)}{\partial\nu}=0$ on $\partial\Omega$. Then $v=u+\frac{\mu}{\lambda}\not\equiv 0$ is in $C^{4,\gamma}(\bar{\Omega})$ and satisfies

$$\begin{cases} L_\tau v=\lambda v \quad \text{in} \quad \Omega,\\ v=\dfrac{\mu}{\lambda}, \quad \dfrac{\partial v}{\partial\nu}=0, \quad \Delta u=-\eta\dfrac{\mu}{\lambda}, \quad \dfrac{\partial(\Delta u)}{\partial\nu}=0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Then by Lemma 3.3, we know that there exists v in $C^2(\Omega)\cap C(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta v=\eta v & \text{in} \quad \Omega,\\ v=\dfrac{\mu}{\lambda}, \quad \dfrac{\partial v}{\partial\nu}=0 & \text{on} \quad \partial\Omega, \end{cases}$$

with η given by (4). If $\lambda=\lambda_\tau=\eta_1(\eta_1+\tau)$, then $\eta=\eta_1$ and using Proposition 4.1, we deduce that Ω is a ball. □

4. APPENDIX

Let $\Omega\subset\mathbb{R}^N$, $N\geq 2$, be a smooth bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider the following overdetermined boundary value problem for the Laplacian

$$(P_1) \qquad \begin{cases} -\Delta u=\eta u & \text{in } \Omega,\\ u=a & \text{on } \partial\Omega,\\ \dfrac{\partial u}{\partial\nu}=b \text{ (const.)} & \text{on } \partial\Omega, \end{cases}$$

where $\eta > 0$, $a, b \in \mathbb{R}$ and $\frac{\partial}{\partial \nu}$ is the outward normal derivative.

PROPOSITION 4.1 ([3, Proposition 1]). *Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a non-constant solution of the overdetermined problem (P_1), and that either*

(i) *$ab > 0$, or*

(ii) *$\eta = \eta_1$.*

Then the domain Ω, must be an N-dimensional ball.

THEOREM 4.1 ([6, Theorem 1.1]). *Let $\eta > 0$, $a \neq 0$. Assume that $\partial\Omega \in C^{3,\gamma}$ for some $\gamma \in (0, 1]$. Then the following statements are equivalent*

(i) *There exists u in $C^{2,\gamma}(\bar{\Omega})$ satisfying (P_1).*

(ii) *There exists a constant d such that*

$$\int_{\Omega} w \mathrm{d}x = d \int_{\partial\Omega} w \mathrm{d}\sigma,$$

for all $w \in C^{2,\gamma}(\bar{\Omega})$ satisfying $-\Delta w = \eta w$ in Ω.
Moreover, $b = 0$ if and only if $d = 0$.

THEOREM 4.2 ([6, Theorem 1.2]). *Let $\eta > 0$, $a = 0$. Assume that $\partial\Omega \in C^{3,\gamma}$, for some $\gamma \in (0, 1]$. Then the following statements are equivalent*

(i) *There exists $u \not\equiv 0$ in $C^{2,\gamma}(\bar{\Omega})$ satisfying (P_1).*

(ii)

$$\int_{\partial\Omega} w \mathrm{d}\sigma = 0,$$

for all $w \in C^{2,\gamma}(\bar{\Omega})$ satisfying $-\Delta w = \eta w$ in Ω.

LEMMA 4.1 ([6]). *Let $\partial\Omega \in C^k$ for some $k \geq 1$. If $u \in C^k(\bar{\Omega})$ is such that*

$$u = const \qquad \text{on } \partial\Omega,$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial^2 u}{\partial \nu^2} = ... = \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = 0 \quad \text{on } \partial\Omega, \quad \text{if } k \geq 2,$$

then

$$\frac{\partial^k u}{\partial x_{j_1}..\partial x_{j_k}} = \frac{\partial^k u}{\partial \nu^k} \nu_{j_1}..\nu_{j_k} \quad \text{on } \partial\Omega$$

for $j_1, .., j_k \in \{1, ..., N\}$.

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