GRÖBNER BASES FOR MODULES OVER PRÜFER DOMAINS

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Let R be a Prüfer domain of Krull dimension one. We prove the existence of Gröbner bases for finitely generated submodules of finitely generated free modules over $R[X]$, where the term order is POT, or, "position over term". In order to do this, we first prove that there is a Gröbner basis for finitely generated ideals in $R[X]$, which is a special case of the main result. The proof is based on the results from [3]. In addition to this we show, in the case of valuation domains, that every Gröbner basis is actually a strong Gröbner basis.

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1. INTRODUCTION

The notion of Gröbner basis for modules of special type was discussed in [1]. In particular, the authors consider submodules of $k[X_1, \dots, X_n]^m$, where $k[X_1, \dots, X_n]$ is the ring of polynomials in several indeterminates, k is a field and $m \geq 1$. They give a proof that there is a Gröbner basis for each finitely generated submodule of $k[X_1, \dots, X_n]^m$. This represents a generalization of the theory of Gröbner basis for ideals in $k[X_1, \dots, X_n]$, with analogous notions and constructions, like the notion of S-polynomials. When the ring of coefficients of polynomials is not a field, in the case of ideals, as well as modules, not all notions are easy to translate, for example, S-polynomials. This is because not each leading coefficient of a polynomial will be divisible by another leading coefficient. Results in the case when the base ring is a PID or a Dedekind domain (see, e.g., [2]) are known, but not much is considered in non-noetherian case. In that sense, a natural generalization of a Dedekind domain is actually a Prüfer domain.

Valuation domains are yet another subclass of Prüfer domains, with the conditions on divisibility which are very convenient for the purpose of consideration of a Gröbner basis (see Definition 1.2 below). In [6], it is proved that

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there is a Gröbner basis for any finitely generated ideal in $V[X]$, for a valuation domain V of Krull dimension one. Actually, this can be deduced directly from a result in [3], as we show in Theorem 2.5. In [6], the authors also state a corollary where they claim that the same result can be proved over semihereditary rings, using certain dynamical techniques. We shall briefly discuss this in the final section of this paper.

As the main result of this paper, we prove that for each finitely generated submodule of $R[X]^m$, where R is a dimension one Prüfer domain, there is a Gröbner basis, when we fix a specific term order on $R[X]^m$ (Theorem 3.2). First, in Theorem 2.5 it is proved that each finitely generated ideal in $R[X]$ has a Gröbner basis (the case $m = 1$) using the result in [3] about Prüfer domains.

Here, we suppose that rings are commutative, with unity and without zero divisors.

There are many equivalent definitions of a Prüfer domain (see [4], [3]), one of which is that it is a semihereditary domain. We state the one involving valuation domains, so that we easily see that each valuation domain is also a Prüfer domain, and therefore, the theory presented below holds for valuation domains too.

Definition 1.1. A domain R is called a Prüfer domain if R_P is a valuation domain for each prime ideal P in R.

As a reminder, here we give a definition of a valuation domain.

Definition 1.2. A domain V is called a valuation domain if at least one of the following relations is true: $a \mid b$ or $b \mid a$, for all $a, b \in V \setminus \{0\}$.

2. GRÖBNER BASIS OF AN IDEAL IN A PRÜFER DOMAIN

Let us present some definitions needed in the following text. For more details, see [3].

Definition 2.1. Let J be an ideal in $R[X]$, where R is an integral domain, and let $I_n, n \geq 0$, be the ideals in R of leading coefficients of polynomials of degree less than or equal to n in J. Also, let $I = \cup_{n=0}^{\infty} I_n$. The ideals I_n and I are called the associated ideals to J.

Obviously, $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots \subseteq I$.

Definition 2.2. An integral domain R satisfies KP property if for each finitely generated ideal J in $R[X]$ the associated ideals $I_n, n \geq 0$ and $I = \cup_{n=0}^{\infty} I_n$ are finitely generated. Also, R satisfies K_0P property if for each finitely generated ideal J in $R[X]$ the associated ideal $I_0 = J \cap R$ is finitely generated.

The following result is given as Theorem 6.2.10. in [3].

THEOREM 2.3. *Let R be a Prüfer domain. The following statements are equivalent:*
a) *R satisfies KP;*
b) *R satisfies K_0P;*
c) $\dim(R) \leq 1$.

To review some basic notions, let $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]$, where $a_n \neq 0$. Then the leading term of f is denoted by $\mathrm{LT}(f) := a_nX^n$, and the leading coefficient by $\mathrm{LC}(f) := a_n$. Also, for an ideal $J \lhd R[X]$, $\mathcal{LT}(J) := \{\mathrm{LT}(f) \,|\, f \in J\}$ and $\mathrm{LT}(J) := \langle \mathcal{LT}(J) \rangle$. Let us recall a definition of a Gröbner basis for an ideal in $R[X]$ for a commutative ring R with identity (see, e.g., [1]).

Definition 2.4. Let J be an ideal in $R[X]$. A subset $G = \{g_1, \ldots, g_r\}$ of J is a Gröbner basis for J if $\mathrm{LT}(J) = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_r) \rangle$.

Now we prove the existence of a Gröbner basis for finitely generated ideals in the ring of polynomials in one indeterminate over one-dimensional Prüfer domains.

THEOREM 2.5. *Let R be a Prüfer domain of Krull dimension one. If J is a finitely generated ideal in $R[X]$, then there exists a Gröbner basis G for J.*

Proof. Let us denote by I_n and I the associated ideals of J. According to Theorem 2.3, these are all finitely generated. Let $I = \langle c_1, \ldots, c_s \rangle$. There are $f_1, \ldots, f_s \in J$ such that $\mathrm{LT}(f_i) = c_iX^{k_i}$, for $1 \leq i \leq s$. Let $j \in \{1, \ldots, s\}$ be such that $k_j \geq k_i$, for all $i \in \{1, \ldots, s\}$. If we put $n = k_j$, and denote by $g_{i,n}$ the polynomials $f_iX^{n-k_i}$ of degree n, then we see that $I = I_n$. Namely, if $c \in I$ is the leading coefficient of a degree l polynomial in J, and $l > n$, then $c = a_1c_1 + \cdots + a_sc_s$, for $a_1, \ldots, a_s \in R$. Therefore, polynomial $a_1g_{1,n} + \cdots + a_sg_{s,n}$ is a degree n polynomial whose leading coefficient is c, so $c \in I_n$.

Now, let $I_m = \langle \mathrm{LC}(g_{1,m}), \ldots, \mathrm{LC}(g_{s_m,m}) \rangle$ for $0 \leq m \leq n-1$, where $g_{1,m}, \ldots, g_{s_m,m}$ are degree m polynomials that belong to J. Let

$$G = \{g_{j,m} \,|\, 1 \leq j \leq s_m, 0 \leq m \leq n-1\} \cup \{g_{1,n}, \ldots, g_{s,n}\}.$$

To prove that G is a Gröbner basis for J, let $f \in J$ be a polynomial of degree m. If $m \geq n$, then $\mathrm{LC}(f) \in I$ and $\mathrm{LC}(f) = \alpha_1c_1 + \cdots + \alpha_sc_s$, for $\alpha_i \in R$. So,

$$\mathrm{LT}(f) = (\alpha_1c_1 + \cdots + \alpha_sc_s)X^m = \alpha_1X^{m-n}\mathrm{LT}(g_{1,n}) + \cdots + \alpha_sX^{m-n}\mathrm{LT}(g_{s,n}).$$

If $m < n$, then $\mathrm{LC}(f) \in I_m$ and

$$\mathrm{LC}(f) = \beta_1\mathrm{LC}(g_{1,m}) + \cdots + \beta_{s_m}\mathrm{LC}(g_{s_m,m}), \qquad \beta_i \in R.$$

Hence, $\mathrm{LT}(f) = \beta_1 \mathrm{LT}(g_{1,m}) + \cdots + \beta_{s_m} \mathrm{LT}(g_{s_m,m})$.

It follows that in each case $\mathrm{LT}(f) \in \langle \mathrm{LT}(g) \mid g \in G \rangle$. □

Let us remind the reader of the notion of a strong Gröbner basis.

Definition 2.6. Let J be an ideal in $R[X]$. A subset $G = \{g_1, \ldots, g_r\}$ of J is a strong Gröbner basis for J if for any $f \in J$, there exists $g_i \in G$ such that $\mathrm{LT}(g_i) \mid \mathrm{LT}(f)$.

It is clear that, if it exists, any strong Gröbner basis for J is also a Gröbner basis. Let us prove that, in the case of valuation domains, these two notions actually coincide.

LEMMA 2.7. *Let $G = \{g_1, \ldots, g_r\}$ be a Gröbner basis for an ideal J in $V[X]$, where V is a valuation domain. Then G is also a strong Gröbner basis for J.*

Proof. Let $f \in J$. Since G is a Gröbner basis, there exists polynomials $p_1(X)$,..., $p_r(X)$ such that

$$\mathrm{LT}(f) = p_1(X)\mathrm{LT}(g_1) + \cdots + p_r(X)\mathrm{LT}(g_r).$$

Let $p_i(X) = b_0^{(i)} + b_1^{(i)}X + \cdots + b_{s_i}^{(i)}X^{s_i}$, $\mathrm{LT}(g_i) = a_iX^{n_i}$ and $\mathrm{LT}(f) = aX^n$. Hence

$$aX^n = \sum_{i=1}^{r}\sum_{j=0}^{s_i} a_iX^{n_i}b_j^{(i)}X^j.$$

So, for some subset $K \subseteq \{1, \ldots, r\}$, we have

$$aX^n = \sum_{k\in K} a_kX^{n_k}b_{n-n_k}^{(k)}X^{n-n_k} = \left(\sum_{k\in K} a_kb_{n-n_k}^{(k)}\right)X^n,$$

and therefore, $a = \sum_{k\in K} a_kb_{n-n_k}^{(k)}$. Since V is a valuation domain, there is $k_0 \in K$ such that $a_{k_0} \mid a_k$ for all $k \in K$. Then we have that $a_{k_0}X^{n_{k_0}} \mid aX^n$ and we are done. □

This allows us to conclude that the following theorem holds.

THEOREM 2.8. *Let V be a dimension one valuation domain. If J is a finitely generated ideal in $R[X]$, then there exists a strong Gröbner basis G for J.*

3. SUBMODULES OF FINITELY GENERATED FREE $R[X]$-MODULES

Following [1], let us review some notions concerning Gröbner bases for modules of the specific type. Let R be a ring and M a finitely generated $R[X]$-submodule of $R[X]^m$, $m \geq 1$. Each element of M can be represented using the standard basis

$$e_1 = (1, 0, \ldots, 0), \ \ldots, \ e_m = (0, \ldots, 0, 1).$$

In module $R[X]^m$ elements of the form $X^r e_i$, $r \geq 0$, are called monomials. Also, we say that monomial $X^r e_i$ divides monomial $X^s e_j$ if $i = j$ and $r \leq s$, and we write $X^r e_i \,|\, X^s e_j$. In that case, the quotient in this division is X^{s-r}. Elements of the form $\alpha X^r e_i$, for $\alpha \in R$, are called terms. Similarly, the term $\alpha X^r e_i$ divides $\beta X^s e_j$ if $\alpha \,|\, \beta$ and $X^r e_i \,|\, X^s e_j$.

Next, by a term order on $R[X]^m$ we mean a total order $<$ on monomials which satisfies these conditions:

1) $X^r e_i < X^s X^r e_i$, for each monomial $X^r e_i$ and for each $s \geq 1$;

2) If $X^r e_i < X^s e_j$ for monomials $X^r e_i, X^s e_j$, then $X^t X^r e_i < X^t X^s e_j$ for each $t \geq 0$.

We can introduce different examples of term orders, but the one that will be of interest here is POT, or "position over term". POT is defined in the following way:

$$X^r e_i < X^s e_j \Leftrightarrow i > j \quad \text{or} \quad i = j, r \leq s.$$

It is easily checked that this is a term order on $R[X]^m$.

When we fix a term order $<$ on $R[X]^m$, then each element $f \in R[X]^m$ can be represented as

$$f = \alpha_1 X^{s_1} e_{k_1} + \alpha_2 X^{s_2} e_{k_2} + \cdots + \alpha_l X^{s_l} e_{k_l},$$

where $\alpha_i \in R$ and $X^{s_i} e_{k_i}$ are monomials, such that

$$X^{s_1} e_{k_1} > X^{s_2} e_{k_2} > \cdots > X^{s_l} e_{k_l}.$$

Then, the leading term of f is denoted by $\mathrm{LT}(f) := \alpha_1 X^{s_1} e_{k_1}$ and the leading coefficient of f by $\mathrm{LC}(f) := \alpha_1$. If M is a submodule of $R[X]^m$, then $\mathcal{LT}(M) := \{\mathrm{LT}(f) \,|\, f \in M\}$ and $\mathrm{LT}(M) := \langle \mathcal{LT}(M) \rangle$.

Definition 3.1. Let M be a finitely generated submodule of $R[X]^m$, where R is a ring. A subset $G = \{g_1, \ldots, g_r\}$ of M is a Gröbner basis for M if $\mathrm{LT}(M) = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_r) \rangle$.

Now, we present the main theorem regarding existence of the Gröbner basis for finitely generated submodules and with the base ring a Prüfer domain of dimension one.

Theorem 3.2. *Let M be a finitely generated submodule of $R[X]^m$, where R is a Prüfer domain of Krull dimension one and let us fix the POT term order on $R[X]^m$. Then, there is a Gröbner basis G for M.*

Proof. Let $M = \langle (f_{11}, f_{12}, \dots, f_{1m}), \dots, (f_{s1}, \dots, f_{sm})\rangle$. If we set

$$I = \{f \in R[X] \,|\, (f(X), f_2(X), \dots, f_m(X)) \in M, \text{ for some } f_2, \dots, f_m \in R[X]\},$$

then I is an ideal in $R[X]$ that is finitely generated. Actually, $I = \langle f_{11}, \dots, f_{s1}\rangle$.

According to Theorem 2.5, there is a Gröbner basis for I. Let that be $\{g_{11}, \dots, g_{k1}\}$. Since these polynomials belong to I, there are $(m-1)$-tuples $(g_{12}, \dots, g_{1m})$,…, $(g_{k2}, \dots, g_{km})$ such that

$$(g_{11}, g_{12}, \dots, g_{1m}), \dots, (g_{k1}, g_{k2}, \dots, g_{km}) \in M.$$

Now, let

$$M_1 = \{(f_2(X), \dots, f_m(X)) \in R[X]^{m-1} \,|\, (0, f_2(X), \dots, f_m(X)) \in M\}.$$

To prove that M_1 is a finitely generated $R[X]$-module, we first notice that, by Corrollary 7.3.4. of [5], $R[X]$ is a coherent domain, i.e., a coherent $R[X]$-module. Then, $R[X]^m$ is also a coherent module. Since $\{0\} \times M_1$ is the intersection of M and submodule $\langle e_2, \dots, e_m\rangle$ of $R[X]^m$, which are both finitely generated, it follows that $\{0\} \times M_1$ is also finitely generated, that is, M_1 is finitely generated (for the results on coherency, see, e.g., [5]).

We proceed by induction. M_1 is a finitely generated submodule of $R[X]^{m-1}$, so there is a Gröbner basis G_1 for M_1. The Gröbner basis G for M is given by

$$\{(0, h_2, \dots, h_m) \,|\, (h_2, \dots, h_m) \in G_1\} \cup \{(g_{11}, g_{12}, \dots, g_{1m}), \dots, (g_{k1}, g_{k2}, \dots, g_{km})\}.$$

The case $m = 1$ is contained in Theorem 2.5.

Namely, let $f = (f_1, \dots, f_m) \in M$. If $f_1 \neq 0$, then $\mathrm{LT}(f) = \mathrm{LT}(f_1)e_1$. Since $\mathrm{LT}(f_1) = p_1\mathrm{LT}(g_{11}) + \cdots + p_k\mathrm{LT}(g_{k1})$, for $p_i \in R[X]$, multiplying by e_1 we get that

$$\mathrm{LT}(f) = p_1\mathrm{LT}((g_{11}, g_{12}, \dots, g_{1m})) + \cdots + p_k\mathrm{LT}((g_{k1}, g_{k2}, \dots, g_{km})).$$

Therefore, $\mathrm{LT}(f) \in \langle \mathrm{LT}(g) \,|\, g \in G\rangle$.

If $f_1 = 0$, then $(f_2, \dots, f_m) \in M_1$ and consequently

$$\mathrm{LT}((f_2, \dots, f_m)) \in \langle \mathrm{LT}(h) \,|\, h \in G_1\rangle.$$

It follows that

$$\mathrm{LT}(f) = \mathrm{LT}((0, f_2, \dots, f_m)) \in \langle \mathrm{LT}((0, h_2, \dots, h_m)) \,|\, (h_2, \dots, h_m) \in G_1\rangle.$$

□

As before, in the case of valuation domains, the existing Gröbner basis is actually strong. We give the definition first.

Definition 3.3. Let M be a finitely generated submodule of $R[X]^m$. A subset $G = \{g_1, \dots, g_r\}$ of M is a strong Gröbner basis for M if for any $f \in M$, there exists $g_i \in G$ such that $\mathrm{LT}(g_i) \mid \mathrm{LT}(f)$.

As in the case of ideals in $V[X]$, for a valuation domain V, these notions are equivalent.

LEMMA 3.4. *Let $G = \{h_1, \dots, h_r\}$ be a Gröbner basis for finitely generated submodule M in $V[X]^m$, where V is a valuation domain. Then G is also a strong Gröbner basis for M.*

Proof. Let $f = (f_1, \dots, f_m) \in M$ and let $i = \min\{1 \le j \le m \,|\, f_j \ne 0\}$. Since G is a Gröbner basis, we have that

$$\mathrm{LT}(f) = p_1\mathrm{LT}((g_{11}, \dots, g_{1m})) + \cdots + p_r\mathrm{LT}((g_{r1}, \dots, g_{rm})),$$

where $h_j = (g_{j1}, \dots, g_{jm})$ and $p_j \in V[X]$, for $1 \le j \le r$. Obviously, some of p_j may be zero and also, some of these summands may cancel out. In any case, $\mathrm{LT}(f) = \mathrm{LT}(f_i)e_i$ and there are $s_1, \dots, s_t \in \{1, \dots, r\}$ such that

$$\mathrm{LT}(f_i) = p_{s_1}\mathrm{LT}(g_{s_1 i}) + \cdots + p_{s_t}\mathrm{LT}(g_{s_t i}).$$

So, we are now in the same situation as in Lemma 2.7. Hence, there exists $l \in \{s_1, \dots, s_t\}$ such that $\mathrm{LT}(g_{li}) \,|\, \mathrm{LT}(f_i)$. Since $\mathrm{LT}(h_l) = \mathrm{LT}(g_{li})e_i$, we have that $\mathrm{LT}(h_l) \mid \mathrm{LT}(f)$ and we are done. □

From here, the following holds:

THEOREM 3.5. *Let M be a finitely generated submodule of $V[X]^m$, where V is a valuation domain of Krull dimension one and let us fix the POT term order on $V[X]^m$. Then, there is a strong Gröbner basis G for M.* □

4. CONCLUDING REMARKS

The primary purpose of this short paper is a discussion of the existence of Gröbner bases for finitely generated submodules of $R[X]^m$ where R is a Prüfer domain. So, we deal only with polynomial rings in one indeterminate. In particular, we generalize the result from the paper [6] from valuation domains to Prüfer domains. The authors in that paper suggest an extension of their results to Prüfer domains using dynamical techniques. These techniques involve using convenient localizations of the ring of coefficients and in this way, one obtains *dynamical Gröbner bases*. However, dynamical Gröbner bases of an ideal is a collection of Gröbner bases for the extension of this ideal in the polynomial ring over several localizations of the ring of coefficients and these localizations depend on the given ideal (for the precise definition of dynamical Gröbner bases, see, e. g. [8]). Although these Gröbner bases are useful in, for example, solving the ideal membership problem, they are not proper Gröbner bases over the original ring of coefficients and we are interested in the existence of proper Gröbner bases over Prüfer domains.

It is also a natural question to generalize this result from univariate to multivariate case. In the general case of the arbitrary monomial order, the answer to this question seems to be hard to obtain. However, in the paper [7] the author makes that generalization for a valuation domain V, from the polynomial ring $V[X]$ to the polynomial ring $V[X_1, \ldots, X_n]$ with the lexicographical term order. In that paper, the existence of one polynomial whose leading coefficient is equal to 1 allows the author to use the suitable change of variables in order to get the polynomial whose leading term is just a power of X_n and consequently, the problem is reduced to the case when the use of the induction is possible.

In the case of Prüfer domains, as we see in Theorem 2.5, we get not one, but the set of polynomials whose leading coefficients generate the ideal of all leading coefficients. Thus, in the case of multiple indeterminates, it becomes challenging to track all the ideals of coefficients in different multidegrees. One can prove that, in the case of multiple indeterminates over a Prüfer domain, the ideal of all leading coefficients is finitely generated, but this does not resolve the problem of finding the Gröbner basis, since this involves the leading terms. It remains open to see whether a different kind of a direct method or induction could be applied.

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