IMMEDIATE EXTENSIONS OF VALUATION RINGS AND ULTRAPOWERS

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We describe the immediate extensions of a one dimensional valuation ring V which could be embedded in some separation of a ultrapower of V with respect to a certain ultrafilter. For such extensions, a kind of Artin's approximation holds.

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Let $(R, \mathbf{m})$ be a Noetherian local ring and $\tilde{R}$ the ultrapower of R with respect of a non principal ultrafilter on $\mathbf{N}$. Then $\bar{R} = \tilde{R}/\cap_{n\in \mathbf{N}} \mathbf{m^n}\tilde{\mathbf{R}}$ is a Noetherian complete local ring which is flat over R (see [10, Proposition 2.9], or [13, Theorem 2.5]). Here we try to find an analogue result in the frame of valuation rings.

Let V be a valuation ring with value group Γ containing its residue field k, K its fraction field and $\tilde{V} = \Pi_{\mathcal{U}} V$ the ultrapower of V with respect to an ultrafilter $\mathcal{U}$ on a set U (see [4], [18], [2]). Then

$$\bar{V} = \tilde{V}/\cap_{z\in V, z\neq 0}\, z\tilde{V}$$

is a valuation ring extending V, a kind of separation of $\tilde{V}$. Indeed,

$$q = \cap_{z\in V, z\neq 0} z\tilde{V}$$

is a prime ideal because if $x_1x_2 \in q$ for some $x_i \in \tilde{V}$ then $\mathrm{val}(x_1x_2) \geq \gamma$ for all $\gamma \in \Gamma$ and so one of $\mathrm{val}(x_i)$ $i = 1, 2$ must be bigger than all $\gamma \in \Gamma$, that is one of x_i belongs to q.

The goal of this paper is to describe the valuation subrings of $\bar{V}$ (given for some special ultrafilters), which are immediate extensions of V when $\dim V = 1$. If the characteristic of V is > 0 then there exist some immediate extensions which cannot be embedded in $\bar{V}$ (see Remark 10).

An inclusion $V \subset V'$ of valuation rings is an *immediate extension* if it is local as a map of local rings and induces isomorphisms between the value groups and the residue fields of V and V'. V has some maximal immediate

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extensions (see [6]). If the characteristic of the residue field of V is zero then there exists an unique maximal immediate extension of V.

Let λ be a fixed limit ordinal and $v = \{v_i\}_{i<\lambda}$ a sequence of elements in V indexed by the ordinals i less than λ. Then v is *pseudo convergent* if

$$\mathrm{val}(v_i - v_{i''}) < \mathrm{val}(v_{i'} - v_{i''})$$

(that is, $\mathrm{val}(v_i - v_{i'}) < \mathrm{val}(v_{i'} - v_{i''})$) for $i < i' < i'' < \lambda$ (see [6], [17]).

A *pseudo limit* of v is an element $z \in V$ with

$$\mathrm{val}(z - v_i) < \mathrm{val}(z - v_{i'})$$

(that is, $\mathrm{val}(z - v_i) = \mathrm{val}(v_i - v_{i'})$) for $i < i' < \lambda$.

We say that v is

1. *algebraic* if some $f \in V[T]$ satisfies $\mathrm{val}(f(v_i)) < \mathrm{val}(f(v_{i'}))$ for large enough $i < i' < \lambda$;

2. *transcendental* if each $f \in V[T]$ satisfies $\mathrm{val}(f(v_i)) = \mathrm{val}(f(v_{i'}))$ for large enough $i < i' < \lambda$,

3. *fundamental* if for any $\gamma \in \Gamma$ there exist $i < i'$ large enough such that $\mathrm{val}(v_i - v_{i'}) > \gamma$, Γ being the value group of V.

We need [14, Proposition A.6], which is obtained using [4, Theorem 6.1.4] and says in particular the following:

Proposition 1. *Let U be an infinite set with $\mathrm{card}\, U = \tau$. Then there exists an ultrafilter $\mathcal{U}$ on U such that for any valuation ring V any system of polynomial equations $(g_i((X_j)_{j\in J})_{i\in I}$ with $\mathrm{card}\, I \leq \tau$ in variables $(X_j)_{j\in J}$ with coefficients in the ultrapower $\tilde{V} = \Pi_{\mathcal{U}} V$ has a solution in $\tilde{V}$ if and only if all its finite subsystems have.*

The above proposition is trivial when $U = \mathbf{N}$. In general, the ultrafilter $\mathcal{U}$ is very special given by [4, Theorem 6.1.4].

Lemma 2. *Let U be an infinite set with $\mathrm{card}\, U = \tau$, V a valuation ring with value group Γ and $\mathrm{card}\, \Gamma \leq \tau$ and λ be an ordinal with $\mathrm{card}\, \lambda \leq \tau$. Let $\mathcal{U}$ be the ultrafilter on U given by the above proposition, $\tilde{V} = \Pi_{\mathcal{U}} V$ the ultrafilter of V with respect to $\mathcal{U}$ and $\bar{V}$ its separation introduced above. Then any pseudo convergent sequence $\bar{v} = (\bar{v}_i)_{i<\lambda}$ over V has a pseudo limit in $\bar{V}$.*

Proof. Let $\mathcal{S}$ be the system of polynomial equations over $\bar{V}$

$$S_i := X - \bar{v}_i - Y_i(\bar{v}_{i+1} - \bar{v}_i); Y_i Y_i' - 1,\ i < \lambda.$$

For each $\gamma \in \Gamma_+$ choose an element $z_\gamma \in V$ with $\mathrm{val}(z_\gamma) = \gamma$ and lift $\bar{v}_i$ to some elements $\tilde{v}_i \in \tilde{V}$. Let $\mathcal{S}'$ be the system of polynomial equations

$$S'_{i\gamma} := X - \tilde{v}_i - Y_i(\tilde{v}_{i+1} - \tilde{v}_i) - z_\gamma Z_\gamma;$$

$$Y_i Y'_i - 1, \quad \text{for} \ \ i < \lambda, \ \gamma \in \Gamma_+,$$

and some variables X, Y_i, Y'_i, Z_γ.

Then $\mathcal{S}'$ has a solution in $\tilde{V}$ if and only if $\mathcal{S}$ has a solution modulo $z_\gamma \tilde{V}$ for all $\gamma \in \Gamma$, that is if $\mathcal{S}$ has a solution in $\bar{V}$, which happens if and only if $(\tilde{v}_i)_{i<\lambda}$ has a pseudo limit in $\tilde{V}$. Note that the cardinal of the system $\mathcal{S}'$ is $\leq \tau$. By the above proposition, $\mathcal{S}'$ has solutions in $\tilde{V}$ if and only if every finite subsystem $\mathcal{T}$ of $\mathcal{S}'$ has a solution in $\tilde{V}$. We may enlarge $\mathcal{T}$ such that it has the form

$$(S'_{i\gamma})_{i=i_1,\ldots,i_e;\gamma=\gamma_1,\ldots,\gamma_e}$$

for some $i_1 < \ldots < i_e < \lambda$ and $\gamma_1, \ldots, \gamma_e \in \Gamma_+$. But then $x = \tilde{v}_{i_e+1}$ induces a solution of $\mathcal{T}$ in $\tilde{V}$ because

$$\mathrm{val}(\tilde{v}_{i_e+1} - \tilde{v}_{i_j}) = \mathrm{val}(\tilde{v}_{i_{j+1}} - \tilde{v}_{i_j})$$

for $1 \leq j \leq e$ and so there exist some units $y_j \in \tilde{V}$ such that

$$\tilde{v}_{i_e+1} - \tilde{v}_{i_j} - y_j(\tilde{v}_{i_{j+1}} - \tilde{v}_{i_j}) \in \cap_{z\in V, z\neq 0} z\tilde{V},$$

for $1 \leq j \leq e$. Thus $(\bar{v}_i)_{i<\lambda}$ has a pseudo limit x in $\bar{V}$. □

Remark 3. Let K be the fraction field of V. If in the above proof v is transcendental then $\mathrm{val}(x) \in \Gamma$ and even the extension $K \subset K(x)$ is immediate (see [6, Theorem 2]). If v is algebraic then $\mathrm{val}(x)$ could be in $\tilde{\Gamma} \setminus \Gamma$, $\tilde{\Gamma}$ being the value group of $\tilde{V}$.

LEMMA 4. *Let U, τ, $\mathcal{U}$, V, Γ be as in Lemma 2. Then the extension $V \subset \bar{V}$ factors through the completion of V.*

Proof. By the above lemma, any fundamental sequence over V has a limit in $\bar{V}$. The limits of the fundamental sequences over V form a valuation subring $\hat{V}$ which must be separate because $\cap_{z\in V, z\neq 0} z\bar{V} = 0$. Hence $\hat{V}$ is the completion of V. □

LEMMA 5. *Let $V, \Gamma, \mathcal{U}, U, \tau, \tilde{V}, \bar{V}$ be as in Lemma 2, a an element of V with* $\mathrm{val}(a) > 0$ *and B a finitely presented V-algebra. Assume V is Henselian and the completion inclusion $V \subset \hat{V}$ is separable. Then any V-morphism $B \to \bar{V}$ could be lifted modulo $a\bar{V}$ to a V-morphism $B \to \tilde{V}$.*

Proof. The proof is similar to the proof of [10, Corollary 2.7] or part of the proof of [13, Theorem 2.9] (see also [3]). Let $B \cong V[Y]/(f)$, $Y = (Y_1, \ldots, Y_n)$,

$f = (f_1, \dots, f_m)$ and $\bar{w} : B \to \bar{V}$ given by $Y \to \bar{y} \in \hat{V}^n$, let us say that $\bar{y}$ is induced by $\tilde{y} = [(y_u)_{u \in U}] \in \tilde{V}$. Set $\gamma = \mathrm{val}(a)$. By [8, Theorem 1.2] applied to V, there exist a positive integer N and $\nu \in \Gamma_+$ such that if $z \in V$ and $\mathrm{val}(f(z)) \geq N\gamma + \nu$ then there exists $z' \in V$ such that $f(z') = 0$ and $\mathrm{val}(z - z') \geq \gamma$.

By construction, we have in particular $\mathrm{val}(f((y_u)) \geq N\gamma + \nu$ for all u from a set $\delta \in \mathcal{U}$. So there exists $y'_u \in V$ such that $f(y'_u) = 0$ and $\mathrm{val}(y_u - y'_u) > \gamma$. Define $y'_t = 0$ if $t \notin \delta$ and let $\tilde{y}' = [(y'_u)_u \in \tilde{V}]$. Then $f(\tilde{y}') = 0$ in $\tilde{V}$ and the V-morphism $B \to \tilde{V}$ given by $Y \to \tilde{y}'$ lifts $\bar{w}$ modulo $a\hat{V}$. □

Lemma 6. *Let $V, \Gamma, \mathcal{U}, U, \tau, \tilde{V}, \bar{V}$ be as in Lemma 2 and $V' \subset \bar{V}$ a valuation subring, which is an immediate extension of V. Assume V is Henselian and the completion inclusion $V \subset \hat{V}$ is separable. Then any algebraic pseudo convergent sequence of V which has a pseudo limit in V' has one also in V.*

Proof. Let $v = (v_j)_{j<\lambda}$ be an algebraic pseudo convergent sequence of V which has a pseudo limit x in V'. Let $h \in V[X]$ be a polynomial of minimal degree among the polynomials $f \in V[Y]$ such that $\mathrm{val}(f(v_i)) < \mathrm{val}(f(v_j))$ for large $i < j < \lambda$. Set $h^{(i)} = \partial^i h/\partial X^i$, $0 \leq i \leq \deg h$ with $h^{(i)} \neq 0$. By [12, Proposition 6.5] there exists an ordinal $\nu < \lambda$ such that an element $z \in V$ is a pseudo limit of v if and only if $\mathrm{val}(h^{(i)}(z)) = \mathrm{val}(h^{(i)}(v_\nu))$ where $h^{(i)} \neq 0$ and $\mathrm{val}(h(z)) > val(h(v_\rho))$ for $\nu \leq \rho < \lambda$. In particular, we have $\mathrm{val}(h^{(i)}(x)) = \mathrm{val}(h^{(i)}(v_\nu))$ where $h^{(i)} \neq 0$ and $\mathrm{val}(h(x)) > \mathrm{val}(h(v_\rho))$ for $\nu \leq \rho < \lambda$. Thus if $z \in V$ satisfies $\mathrm{val}(h^{(i)}(z)) = \mathrm{val}(h^{(i)}(x))$ for all $0 \leq i \leq \deg h$ with $h^{(i)} \neq 0$ then z is a pseudo limit of v.

Let $d_i \in V$ such that $\mathrm{val}(d_i) = \mathrm{val}(h^{(i)}(x))$ for $0 \leq i \leq \deg h$ with $h^{(i)} \neq 0$, let us say $h^{(i)}(x) = d_i t_i$ for some invertible $t_i \in V'$, and g the system of equations $h^{(i)}(Z) - d_i U_i$, $U_i U'_i - 1$. If $z, (u_i)_i$ is a solution of g in V then z is a pseudo limit of $(v_j)_{j<\lambda}$. But the map $B := V[Z, (U_i)_i]/(g) \to \bar{V}$ given by $(Z, (U_i), (U'_i)) \to (x, (t_i), (t_i^{-1}))$ could be lifted by Lemma 5 to a map $B \to \tilde{V}$, that is g has a solution in $\tilde{V}$ and so in V as well. This ends the proof. □

Remark 7. If V is Henselian and V' is a filtered direct limit of smooth V-algebras we get as above that any algebraic pseudo convergent sequence of V which has a pseudo limit in V' has also one in V. Indeed, let $x, (v_j)_{j<\lambda}$, h, (d_i), (t_i), g be as above. Then the solution $(x, (t_i), (t_i^{-1}))$ of g in V' comes from a solution of g in a smooth V-algebra C. But there exists a V-morphism ρ from C to V because V is Henselian. Thus we get a solution of g in V via ρ, so $(v_j)_{j<\lambda}$ has a pseudo limit in V.

Lemma 8. *Let $V, \Gamma, \mathcal{U}, U, \tau, \tilde{V}, \bar{V}$ be as in Lemma 2 and $V'' \subset V' \subset \bar{V}$ some valuation subrings such that $V \subset V''$, $V'' \subset V'$ are immediate extensions.*

Assume V'' is Henselian and the completion inclusion $V'' \subset \hat{V}''$ is separable. Then any algebraic pseudo convergent sequence of V'' which has a pseudo limit in V' has one also in V''.

Proof. Let $\bar{V}''$ be given from V as $\bar{V}$ from V and $(v_j)_j$ an algebraic pseudo convergent sequence over V'' which has a pseudo limit in $V' \subset \bar{V} \subset \bar{V}''$. By Lemma 6 applied to V'' it has one in V''. Note that the construction of $\bar{V}$, $\bar{V}''$ are done with the same U and $\mathcal{U}$. □

THEOREM 9. *Let $V \subset V'$ be an immediate extension of one dimensional valuation rings and $\Gamma, \mathcal{U}, U, \tau, \tilde{V}, \bar{V}$ be as in Lemma 2. Assume V' is complete and card $U \geq$card Γ. The following statements are equivalents:*

1. *the extension $V \subset \bar{V}$ factors through V',*
2. *for any valuation subring $V'' \subset V'$ such that $V \subset V''$ and $V'' \subset V'$ are immediate extensions any algebraic pseudo convergent sequence of V'' which is not fundamental and has a pseudo limit in V' has one also in V''.*

Moreover, if $V \subset V'$ is separable and one from (1), (2) *holds then V' is a filtered direct limit of smooth V-algebras.*

Proof. Suppose (1) holds and let V'' be as in (2). Then the completion $\hat{V}''$ of V'' is contained in $\bar{V}''$ by Lemma 4. An algebraic pseudo convergent sequence v over V'' which has a pseudo limit in $V' \subset \bar{V} \subset \bar{V}''$ must have a pseudo limit in $\hat{V}''$ by Lemma 8 because $\hat{V}''$ is Henselian since $\dim V'' = 1$. Then v has a pseudo limit in V'' too by [12, Lemma 2.5], for example.

Assume (2) holds. Let $V'' \subset V'$ be a valuation subring such that $V \subset V''$ and $V'' \subset V'$ are immediate and K'' its fraction field. Applying Zorn's Lemma, we may suppose that V'' is maximal for inclusion among those immediate extensions $W \subset V'$ of V such that $V \subset \bar{V}$ factors through W. Assume that $V'' \neq V'$. Let $x \in V' \setminus V''$ and v be a pseudo convergent sequence over V'' having x as a pseudo limit but with no pseudo limit in V'' (see [6, Theorem 1]). Then v is either fundamental or transcendental by (2). If v is transcendental then $K'' \subset K''(x)$ is the extension constructed in [6, Theorem 2] for v. By Lemma 2, we see that v has a pseudo limit z in $\bar{V}''$. Actually, the proof of Lemma 8 gives that z could be taken in the completion $\hat{V}''$ of V''. Then the unicity given by [6, Theorem 2] shows that $K''(x) \cong K''(z)$ and so the extension $V'' \subset \bar{V}$ factors through $V_1 = V' \cap K''(x)$ because $z \in \hat{V}'' \subset V' \subset \bar{V}$ since V' is complete. If v is fundamental then as above $x \in \hat{V}'' \subset \bar{V}$. In both cases, the extension $V'' \subset \bar{V}$ factors through $V_1 = V' \cap K(x)$. These contradict that V'' is maximal by inclusion, that is V'' must be V'.

Now suppose $V \subset V'$ is separable and (2) holds. We reduce to the case when the fraction field extension $K \subset K'$ of $V \subset V'$ is of finite type because V' is a filtered direct union of $V' \cap L$ for some subfields $L \subset K'$, which are finite type field extensions of K. By induction on the number of generators of L over K we may reduce to the case when $K' = K''(x)$ for some element $x \in V'$, K'' being the fraction field of a valuation subring $V'' \subset V'$ as in (2). As K'/K is separable of finite type we may arrange that K'/K'' is still separable. Then x is a pseudo limit of a pseudo convergent sequence v from V'' which has no pseudo limit in K'' (see [6, Theorem 1]). Assume that v is not a fundamental sequence. Then v is transcendental by (2) and so V' is a filtered direct union of localizations of polynomial V''-subalgebras of V' in one variable by [11, Theorem 3.2] (see also [14, Lemma 15]). If v is fundamental then $V'' \subset V'$ is dense and separable and we may apply a theorem of type Néron-Schappacher (see e.g. [12, Theorem 4.1]). □

Remark 10. If $V \subset V'$ is the valuation ring extension given in [12, Example 3.13] then V' is not a filtered direct limit of smooth V-algebras (see [12, Remark 6.10]) and so cannot be embedded in $\bar{V}$ by Theorem 9.

The following corollary is a kind of Artin approximation (see [1], [13], [5], [16]) in the frame of valuation rings. Its statement extends the idea of [7, Corollary 8, Theorem 11] and [15, Theorem 14] replacing the order by the valuation.

COROLLARY 11. *Let $V \subset V'$ be a separable immediate extension such that for any valuation subring $V'' \subset V'$ such that $V \subset V''$ and $V'' \subset V'$ are immediate extensions any algebraic pseudo convergent sequence of V'' which has a pseudo limit in V' has one also in V''. Let f be a finite system of polynomials equations from $V[Y]$, $Y = (Y_1, \ldots, Y_n)$, which has a solution in V'. Assume V is complete and* $\dim V = 1$. *Then f has a solution in V. Moreover, if $y' = (y'_1, \ldots, y'_n)$ is a solution of f in V' then there exists a solution $y = (y_1, \ldots, y_n)$ of f in V such that* $\mathrm{val}(y_i) = \mathrm{val}(y'_i)$ *for $1 \leq i \leq n$.*

Proof. Let y' be a solution of f in V', $B = V[Y]/(f)$ and $w : B \to V'$ be the map given by $Y \to y'$. Let $\Gamma, \mathcal{U}, U, \tilde{V}, \bar{V}$ be as in Theorem 9. Then the extension $V \subset \bar{V}$ factors through V' and $\bar{w}$ the composite map $B \xrightarrow{w} V' \to \bar{V}$ could be lifted to a map $\tilde{w} : B \to \tilde{V}$ by Lemma 5. Thus f has in $\tilde{V}$ the solution $\tilde{w}(Y)$ and so it has also a solution in V.

Now, let $a = (a_1, \ldots, a_n) \in V^n$ be such that $\mathrm{val}(a_i) = \mathrm{val}(y'_i)$ for $1 \leq i \leq n$. Then there exists an unit $z'_i \in V'$ such that $y'_i = a_i z'_i$ and the system

g obtained by adding to f the equations $Y_i - a_iZ_i$, $Z_iT_i - 1$, $1 \le i \le n$ has in V' the solution y', $z' = (z'_1, \ldots, z'_n)$, $t' = (t'_1, \ldots, t'_n)$, the last ones are given by the inverses of (z'_i). So g has a solution y, z, t in V and it follows that $\text{val}(y_i) = val(y'_i)$, $1 \le i \le n$. □

Remark 12. Another proof of the above corollary could be done using that V' is a filtered direct limit of smooth V-algebras (see Theorem 9).

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REFERENCES

[1] M. Artin, *Algebraic approximation of structures over complete local rings*. Publ. Math. Inst. Hautes Études Sci. **36** (1969), *1*, 23–58.

[2] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven, *Asymptotic differential algebra and model theory of transseries*. Annals of Mathematics Studies **195**, Princeton Univ. Press, Princeton, NJ, 2017.

[3] J. Becker, J. Denef, L. Lipshitz, and L. van der Dries, *Ultraproducts and approximation in local rings I*. Invent. Math. **51** (1979), 189-203.

[4] C.C. Chang and H.J. Keisler, *Model Theory*. Studies in Logic and the Foundations of Mathematics **73**, North-Holland Publishing Co., Amsterdam, 1990.

[5] H. Hauser, *The classical Artin approximation theorems*. Bull. Amer. Math. Soc. **54** (2017), 595-633.

[6] I. Kaplansky, *Maximal fields with valuations*. Duke Math. J. **9** (1942), 303-321.

[7] Z. Kosar and D. Popescu, *Nested Artin strong approximation property*. J. Pure Appl. Algebra **222** (2018), *4*, 818-827.

[8] L. Moret-Bailly, *An extension of Greenberg's theorem to general valuation rings*. Manuscripta Math. **139** (2012), *1-2*, 153-166.

[9] A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper*. Math. Z. **39** (1935), *1*, 321-404.

[10] D. Popescu, *Algebraically pure morphisms*. Rev. Roumaine Math. Pures Appl. **24** (1979), 947-977.

[11] D. Popescu, *On Zariski's uniformization theorem*. In: *Algebraic Geometry* (Proc. Int. Conf., Bucharest, 1982). Lecture Notes in Math. **1056**, Springer, Berlin, 264-296 (1984).

[12] D. Popescu, *Algebraic extensions of valued fields*. J. Algebra **108** (1987), 513-533.

[13] D. Popescu, *Artin Approximation*. In: M. Hazewinkel (Ed.), *Handbook of Algebra* **2**. North-Holland Publishing Co., Amsterdam, 321-356 (2000).

[14] D. Popescu, *Néron desingularization of extensions of valuation rings with an Appendix by K. Cesnavicius*. In: A. Bostan and K. Rasche (Eds.), *Transcendence in Algebra, Combinatorics, Geometry and Number Theory*. Springer Proceedings in Mathematics and Statistics **373**, 275-307 (2021). arxiv/AC:1910.09123v4

[15] D. Popescu and G. Rond, *Remarks on Artin approximation with constraints*. Osaka J. Math. **56** (2019), 431-440.

[16] G. Rond, *Artin approximation*. J. Singul. **17** (2018), 108-192.

[17] O.F.G. Schilling, *The Theory of Valuations*. Mathematical Surveys **IV**, Amer. Math. Soc., New York, 1950.

[18] H. Schoutens, *The Use of Ultraproducts in Commutative Algebra*. Lect. Notes in Math. **1999**, Springer, Dordrecht, 2010.

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