SINGULARITIES OF QUOTIENT SURFACES OF $K3$ SURFACES WITH AN AUTOMORPHISM OF MAXIMUM ORDER

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We study cyclic quotient singularities given by automorphisms of maximum order on $K3$ surfaces. In particular, we describe fixed loci of such automorphisms, and provide types of these singularities.

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1. INTRODUCTION

We will work over $\mathbb{C}$, the field of complex numbers, throughout this paper. Let G be a finite cyclic group acting on an algebraic surface S as an automorphism and φ a fixed generator of G. It is well known that singularities of S/G relates to properties of φ in this case. On the other hand, properties of automorphisms gives not only local properties (e.g., singularities) but also global properties of quotients surfaces S/G. For instance, a quotient surface of a $K3$ surface by an automorphism which satisfies certain conditions (see also [13]) gives an example of a log Enriques surface. Then the index of the log Enriques surface relates to the order of the automorphism ([5], [15], [16]).

This paper is devoted to a study of singularities of quotient surfaces given by automorphisms of $K3$ surfaces. The main result is the following.

Main Theorem. *Let X be a $K3$ surface with an automorphism g of maximum order 66. Then the following hold:*

(1) *The fixed loci of g, g^2 and g^3 consist of exactly 3 points, 5 points and 6 points, respectively.*

(2) *Each quotient surfaces $X/\langle g\rangle$, $X/\langle g^2\rangle$ and $X/\langle g^3\rangle$ have one point of type $\frac{1}{11}(1,5)$ and one point of type $\frac{1}{11}(1,8)$ as singularities. In particular, these surfaces are log Enriques surfaces of index 11.*

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The proof of Main Theorem is given by Section 3 and Section 4.

Remark 1.1. Remaining automorphisms g^6, g^{11}, g^{22} and g^{33} are not the subject of this paper, but are mentioned in Proposition 3.1 and Lemma 4.1.

For an algebraic surface S, if its canonical line bundle K_S is trivial and $H^1(S, \mathcal{O}_S) = 0$ then it is called a *$K3$ surface*. Since K_S is trivial, there exists a nowhere vanishing holomorphic 2-form on S which is unique up to constants. Let φ be an automorphism on S of finite order I. It is called *symplectic*, respectively, *(purely) non-symplectic*, if and only if it satisfies $\varphi^*\omega_S = \omega_S$, respectively, $\varphi^*\omega_S = \zeta_I\omega_S$ where ζ_I is a (primitive) I-th root of unity.

Let (x, y) be a local coordinate centered at a point of a $K3$ surface S. If an automorphism φ acts on the points as mapping (x, y) to $(\alpha x, \beta y)$ then the action of φ for $\omega_S(= dx \wedge dy)$ is multiplication by $\alpha\beta$, hence $\varphi^*\omega_S = \alpha\beta\omega_S$. Thus if φ is symplectic (resp., non-symplectic) then a group generated by φ is a subgroup of $SL(2, \mathbb{C})$ (resp., $GL(2, \mathbb{C})$). (See also Lemma 2.1 (2).) This implies that it is important to focus on (non-)symplecticity in the study of singularities.

Automorphisms of $K3$ surfaces were widely studied in the last years. In this paper, we study cyclic quotient singularities given by automorphisms of maximum order on $K3$ surfaces. The following results are known for such automorphisms.

Theorem 1.2 ([7, Main Theorem and Lemma 4.2], [9, Theorem], [10, Main Theorem 1 (1)]). *The following hold:*

(1) *The maximum finite order of an automorphism on $K3$ surfaces is 66.*

(2) *An automorphism of order 66 is non-symplectic.*

(3) *A pair of a $K3$ surface and a non-symplectic automorphism of order 66 is isomorphic to Example 1.3.*

Example 1.3 ([9, (3.0.1)]). Put

$$X : y^2 = x^3 + t(t^{11} - 1), \; g(x, y, t) = (\zeta_{66}^2 x, \zeta_{66}^3 y, \zeta_{66}^6 t).$$

Then X is a $K3$ surface and g is a non-symplectic automorphism of order 66. Note that the Jacobian elliptic fibration $\pi : X \to \mathbb{P}^1$ has 12 singular fibers of type II over $t = 0$ and $t^{11} = 1$.

The Example 1.3 suggests that we may study the fixed locus directly, by using the elliptic fibration and the action of the automorphism. We will give two methods: one using Lefschetz's formulas and the other by using the elliptic fibration.

2. PRELIMINARIES

In this section, we collect some basic results which will be frequently applied in this article. For the details, see e.g. [11] and [2]. For a $K3$ surface X, we denote by S_X and T_X the Néron-Severi lattice and the transcendental lattice, respectively.

LEMMA 2.1. *Let f be a non-symplectic automorphism of order I on X. Then*

(1) *The eigenvalues of $f^* \mid T_X$ are the primitive I-th roots of unity, hence $f^* \mid T_X \otimes \mathbb{C}$ can be diagonalized as:*

$$\begin{pmatrix} \zeta_I E_q & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \zeta_I^n E_q & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \zeta_I^{I-1} E_q \end{pmatrix},$$

where E_q is the identity matrix of size q and $1 \le n \le I-1$ is co-prime with I.

(2) *Let $P^{i,j}$ be an isolated fixed point of f on X. Then f^* can be written as*

$$\begin{pmatrix} \zeta_I^i & 0 \\ 0 & \zeta_I^j \end{pmatrix} \qquad (i + j \equiv 1 \mod I)$$

under some appropriate local coordinates around $P^{i,j}$.

(3) *Let C be an irreducible curve in X^f and Q a point on C. Then f^* can be written as*

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta_I \end{pmatrix}$$

under some appropriate local coordinates around Q. In particular, fixed curves are non-singular.

Lemma 2.1 (1) implies that $\Phi(I)$ divides $\operatorname{rk} T_X$, where Φ is the Euler function. Hence if $I = 66$ then $\operatorname{rk} T_X = 20$ and $\operatorname{rk} S_X = 2$. Lemma 2.1 (2) and (3) imply that the fixed locus of f is either empty or the disjoint union of non-singular curves and isolated points:

$$X^f = \{P_1^{i_1,j_1}, \ldots, P_M^{i_M,j_M}\} \amalg C_1 \amalg \cdots \amalg C_N,$$

where $P_k^{i_k,j_k}$ is an isolated fixed point and C_l is a non-singular curve.

A fixed locus plays an essential role for studying singularities given by f. In particular, local actions for isolated fixed points are very important.

PROPOSITION 2.2 (Lefschetz formulae). *Let f be a non-symplectic automorphism of order I on X. The fixed locus of f satisfies the following:*

(1) $\sum_{k=0}^{4}(-1)^k tr(f^*|H^k(X,\mathbb{R})) = \chi(X^f)$,

(2) $\sum_{k=0}^{2} tr(f^*|H^k(X,\mathcal{O}_X)) = \sum_{i+j=I+1}^{M} a(P^{i,j}) + \sum_{l=1}^{N} b(C_l)$,

where $\chi(X^f)$ is the Euler characteristic of X^f, $a(P^{i,j}) = 1/((1-\zeta_I^i)(1-\zeta_I^j))$ and $b(C_l) = (1-g(C_l))/(1-\zeta_I) - \zeta_I C_l^2/(1-\zeta_I)^2$.

(1) and (2) are called the topological Lefschetz formula and the holomorphic Lefschetz formula, respectively. See also [3, page 542] and [4, page 567] for details. We frequently study the local action of a non-symplectic automorphism by using Proposition 2.2. The following Remark is important too.

Remark 2.3. For positive integers m and n, let f be an automorphism of order mn and $P^{\alpha,\beta}$ be an isolated fixed point of f. If $f^n(P^{\alpha,\beta}) = P^{\alpha',\beta'}$, hence $P^{\alpha',\beta'}$ is an isolated fixed point of f^n, then we have $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$ mod m.

3. FIXED LOCI OF AUTOMORPHISMS

Put $f_I = g^{66/I}$, so f_I is an automorphism of order I on X. First, we shall study fixed loci of f_{22}, f_{33} and g via Proposition 2.2 (Lefschetz formulae). Next, we check over the fixed locus of g by a direct calculation of Example 1.3.

We remark that f_I acts trivially on the Néron-Severi lattice S_X because g acts trivially on S_X by [10, Lemma (2.4)]. Cases $I = 2$, 3, 6 and 11 then are well- known. We refer to [12, Theorem 4.2.2], [1, Theorem 2.2], [14, Theorem 1.1], [2, Theorem 7.3] and [6, Theorem 4.1].

PROPOSITION 3.1. *The following hold:*

(1) *The fixed locus of f_2 consists of one smooth curve of genus 10 and one rational curve.*

(2) *The fixed locus of f_3 consists of one smooth curve of genus 5 and one rational curve.*

(3) *The fixed locus of f_6 consists of 12 isolated points and one rational curve, hence it is of the form $X^{f_6} = \{P^{3,4} \times 12\} \amalg \mathbb{P}^1$.*

(4) *The fixed locus of f_{11} consists of 2 isolated points and one smooth curve of genus 1, hence it is of the form $X^{f_{11}} = \{P^{2,10}, P^{5,7}\} \amalg C^{(1)}$.*

3.1. **The case of $I = 22$**

LEMMA 3.2. *The fixed locus of f_{22} consists of 6 isolated points.*

Proof. If $X^{f_{22}}$ contains a smooth curve then it belongs to both X^{f_2} and $X^{f_{11}}$. But it contradicts Proposition 3.1 (1) and (4). Thus, the Euler characteristic of $X^{f_{22}}$ is equal to the number of isolated fixed points.

By Proposition 2.2 (1) and Lemma 2.1 (1), we have

$$\begin{aligned}\chi(X^{f_{22}}) &= \sum_{k=0}^{4}(-1)^k \mathrm{tr}(f_{22}^*|H^k(X,\mathbb{R}))\\ &= 1 - 0 + \mathrm{tr}(f_{22}^*|S_X) + \mathrm{tr}(f_{22}^*|T_X) - 0 + 1\\ &= 4 + \mathrm{tr}(f_{22}^*|T_X)\\ &= 4 + (\zeta_{22} + \zeta_{22}^3 + \zeta_{22}^5 + \zeta_{22}^7 + \zeta_{22}^9 + \zeta_{22}^{13} + \zeta_{22}^{15} + \zeta_{22}^{17} + \zeta_{22}^{19} + \zeta_{22}^{21}) \times 2\\ &= 4 - ((1 + \zeta_{22}^2 + \zeta_{22}^4 + \cdots + \zeta_{22}^{20}) + \zeta_{22}^{11}) \times 2\\ &= 4 - (0 + (-1)) \times 2\\ &= 6.\end{aligned}$$

□

To study details of local actions on isolated fixed points, we apply Proposition 2.2 (2) for $X^{f_{22}}$.

PROPOSITION 3.3. *The fixed locus of f_{22} is of the form*

$$X^{f_{22}} = \{P^{5,18}, P^{10,13}, P^{11,12}, P^{11,12}, P^{11,12}, P^{11,12}\}.$$

Proof. Let $m_{i,j}$ be the number of isolated fixed points of type $P^{i,j}$. Since $X^{f_{22}}$ does not have a smooth curve by Lemma 3.2, we have

$$\sum_{k=0}^{2} \mathrm{tr}(f^*|H^k(X,\mathcal{O}_X)) = \sum_{i+j=23}^{M} \frac{m_{i,j}}{(1-\zeta_{22}^i)(1-\zeta_{22}^j)}.$$

Using the Serre duality $H^2(X,\mathcal{O}_X) \simeq H^0(X,\mathcal{O}_X(K_X))^\vee$, we calculate the left-hand side as $1 + \zeta_{22}^{21}$. Since isolated points contained in $X^{f_{11}}$ are only $P^{2,10}$ and $P^{5,7}$, we have $m_{2,21} + m_{10,13} \leq 1$, $m_{5,18} + m_{7,16} \leq 1 \ldots (\diamondsuit)$ and $m_{3,20} = m_{4,19} = m_{6,17} = m_{8,15} = m_{9,14} = 0$.

Thus Proposition 2.2 (2) implies

$$\begin{aligned}1 + \zeta_{22}^{21} = \frac{m_{2,21}}{(1-\zeta_{22}^2)(1-\zeta_{22}^{21})} + \frac{m_{5,18}}{(1-\zeta_{22}^5)(1-\zeta_{22}^{18})} + \frac{m_{7,16}}{(1-\zeta_{22}^7)(1-\zeta_{22}^{16})}\\ + \frac{m_{10,13}}{(1-\zeta_{22}^{10})(1-\zeta_{22}^{13})} + \frac{m_{11,12}}{(1-\zeta_{22}^{11})(1-\zeta_{22}^{12})}.\end{aligned}$$

It induces the following equations:

$$\begin{cases} 4m_{2,21} - m_{5,18} + m_{7,16} + 2m_{10,13} & = 1, \\ 5m_{2,21} + m_{5,18} + 3m_{7,16} + m_{10,13} & = 2, \\ 6m_{2,21} + 2m_{5,18} + 2m_{7,16} - 2m_{10,13} + m_{11,12} & = 4. \end{cases}$$

We have $m_{7,16} = m_{10,13} = 1$ and $m_{11,12} = 4$ from these and $(\diamondsuit)$. $\square$

3.2. **The case of $I = 33$**

Lemma 3.4. *The fixed locus of f_{33} consists of 5 isolated points.*

Proof. If $X^{f_{33}}$ contains a smooth curve then it belongs to both X^{f_3} and $X^{f_{11}}$. But it contradicts Proposition 3.1 (2) and (4). Thus, the Euler characteristic of $X^{f_{33}}$ is equal to the number of isolated fixed points.

By Proposition 2.2 (1) and Lemma 2.1 (1), we have

$$\begin{aligned} \chi(X^{f_{33}}) &= \sum_{k=0}^{4}(-1)^k \operatorname{tr}(f_{33}^*|H^k(X,\mathbb{R})) \\ &= 4 + \operatorname{tr}(f_{33}^*|T_X) \\ &= 4 + (\zeta_{33} + \zeta_{33}^2 + \zeta_{33}^4 + \zeta_{33}^5 + \zeta_{33}^7 + \zeta_{33}^8 + \zeta_{33}^{10} + \zeta_{33}^{13} + \zeta_{33}^{14} + \zeta_{33}^{16} \\ &\qquad + \zeta_{33}^{17} + \zeta_{33}^{19} + \zeta_{33}^{20} + \zeta_{33}^{23} + \zeta_{33}^{25} + \zeta_{33}^{26} + \zeta_{33}^{28} + \zeta_{33}^{29} + \zeta_{33}^{31} + \zeta_{33}^{32}) \\ &= 4 - ((1 + \zeta_{33}^3 + \zeta_{33}^6 + \cdots + \zeta_{33}^{30}) + (\zeta_{33}^{11} + \zeta_{33}^{22})) \\ &= 5. \end{aligned}$$

$\square$

Proposition 3.5. *The fixed locus of f_{33} is of the form*

$$X^{f_{33}} = \{P^{7,27}, P^{12,22}, P^{12,22}, P^{12,22}, P^{13,21}\}.$$

Proof. It is essentially the same as Proposition 3.3. Let $m_{i,j}$ be the number of isolated fixed points of type $P^{i,j}$. Note that X^{f_3} does not have isolated fixed points, and $X^{f_{11}}$ has just one $P^{2,10}$ and one $P^{5,7}$. Then we have $m_{10,24} + m_{13,21} \leq 1$, $m_{7,27} + m_{16,18} \leq 1$ and other $m_{i,j}$ are 0 except $m_{12,22}$. Thus these and Proposition 2.2 (2):

$$\begin{aligned} 1 + \zeta_{33}^{32} = \frac{m_{7,27}}{(1-\zeta_{33}^{7})(1-\zeta_{33}^{27})} + \frac{m_{10,24}}{(1-\zeta_{33}^{10})(1-\zeta_{33}^{24})} + \frac{m_{12,22}}{(1-\zeta_{33}^{12})(1-\zeta_{33}^{22})} \\ + \frac{m_{13,21}}{(1-\zeta_{33}^{13})(1-\zeta_{33}^{21})} + \frac{m_{16,18}}{(1-\zeta_{33}^{16})(1-\zeta_{33}^{18})} \end{aligned}$$

indicates $m_{7,27} = m_{13,21} = 1$, $m_{12,22} = 3$ and $m_{10,24} = m_{16,18} = 0$. $\square$

3.3. **The case of $I = 66$**

This is the case we want to know. We study the fixed locus of $g = f_{66}$. But techniques are same as above.

LEMMA 3.6. *The fixed locus of g consists of 3 isolated points.*

Proof. It follows from Proposition 3.2 and Proposition 3.4 that X^g does not contain a smooth curve. Thus, the Euler characteristic of X^g is equal to the number of isolated fixed points.

By Lemma 2.1 (1), we have

$$\begin{aligned}\mathrm{tr}(g^*|T_X) &= \zeta_{66} + \zeta_{66}^5 + \zeta_{66}^7 + \zeta_{66}^{13} + \zeta_{66}^{17} + \zeta_{66}^{19} + \zeta_{66}^{23} + \zeta_{66}^{25} + \zeta_{66}^{29} + \zeta_{66}^{31} \\ &\quad + \zeta_{66}^{35} + \zeta_{66}^{37} + \zeta_{66}^{41} + \zeta_{66}^{43} + \zeta_{66}^{47} + \zeta_{66}^{49} + \zeta_{66}^{53} + \zeta_{66}^{59} + \zeta_{66}^{61} + \zeta_{66}^{65} \\ &= -((1 + \zeta_{66}^2 + \zeta_{66}^4 + \cdots + \zeta_{66}^{64}) + (\zeta_{66}^3 + \zeta_{66}^9 + \cdots + \zeta_{66}^{64}) + (\zeta_{66}^{11} + \zeta_{66}^{55})) \\ &= 0 - (\zeta_{22} + \zeta_{22}^3 + \cdots + \zeta_{22}^{21}) - (\zeta_6 + \zeta_6^5)) \\ &= -1.\end{aligned}$$

We apply Proposition 2.2 (1) for X^g:

$$\chi(X^g) = \sum_{k=0}^{4} (-1)^k \mathrm{tr}(g^*|H^k(X,\mathbb{R})) = 4 + \mathrm{tr}(g^*|T_X) = 4 - 1 = 3.$$

Therefore, X^g consists of 3 isolated points. □

PROPOSITION 3.7. *The fixed locus of g is of the form*

$$X^g = \{P^{12,55}, P^{13,54}, P^{27,40}\}.$$

Proof. Let $m_{i,j}$ be the number of isolated fixed points of type $P^{i,j}$. It is easy to see that $m_{12,55} \leq 3$, $m_{13,54} \leq 1$, $m_{27,40} \leq 1$ and other $m_{i,j}$ are 0 by Proposition 3.3 and Proposition 3.5.

Then Proposition 2.2 (2):

$$1 + \zeta_{66}^{65} = \frac{m_{12,55}}{(1-\zeta_{66}^{12})(1-\zeta_{66}^{55})} + \frac{m_{13,54}}{(1-\zeta_{66}^{13})(1-\zeta_{66}^{54})} + \frac{m_{27,40}}{(1-\zeta_{66}^{27})(1-\zeta_{66}^{40})}$$

gives $m_{12,55} = m_{13,54} = m_{27,40} = 1$. □

3.4. **Another observation**

We treat Example 1.3 directly. We consider elliptic fibration $\pi : X \to \mathbb{P}^1$. By replacing a primitive 66-th root of unity (we take ζ_{66}^{53} instead of ζ_{66}), we may assume $g(x, y, t) = (\zeta_{66}^{40}x, \zeta_{66}^{27}y, \zeta_{66}^{54}t)$.

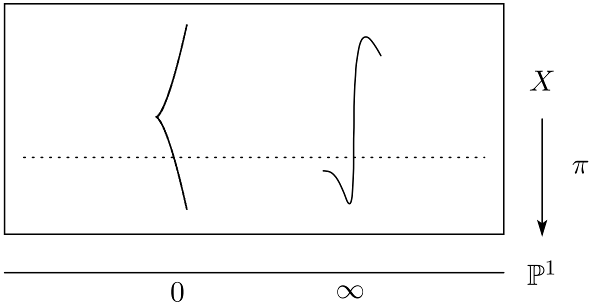

Figure 1 – The elliptic fiberation of Example 1.3.

Since the automorphism g acts on the base of π with order 11, g induces an automorphism of order 6 on fibers $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$. Hence g has fixed points on these fibers.

Since g also acts on the 0-section (the dotted line of Figure 1) of π, two fixed points of g are contained in it. Note that an automorphism of order 6 on an elliptic curve has exactly one fixed point. Thus $\pi^{-1}(\infty)$ has just one fixed point of g, since it is a smooth elliptic curve. On the other hand, $\pi^{-1}(0)$ has two fixed points of g. Because g induces an automorphism on $\mathbb{P}^1$ which is obtained by blow-ups of the cuspidal curve $\pi^{-1}(0)$, it has exactly two isolated fixed points.

Thus there exist three fixed points of g, hence the cusp of $\pi^{-1}(0)$ and intersection points of the 0-section of π and $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$.

Proposition 3.8. *Fixed points $P^{13,54}$ and $P^{27,40}$ lie on $\pi^{-1}(0)$, and $P^{12,55}$ is on $\pi^{-1}(\infty)$.*

Proof. First we see a fixed point at the cusp of $\pi^{-1}(0)$, hence $(x, y, t) = (0, 0, 0)$. Since (x, y) is a local coordinate centered at the point and the automorphism g maps (x, y) to $(\zeta_{66}^{40}x, \zeta_{66}^{27}y)$, the fixed point of g is $P^{27,40}$.

Next, we study the intersection point of the 0-section of π and $\pi^{-1}(0)$, that is the infinite point of $\pi^{-1}(0)$. We may take a local coordinate $(x/y, t)$ centered at the point. Since the automorphism g maps $(x/y, t)$ to $(\zeta_{66}^{13}x/y, \zeta_{66}^{54}t)$, the point is $P^{13,54}$.

Finally, we study the intersection point of the 0-section of π and $\pi^{-1}(\infty)$. Set $x/t^4 = x_1, y/t^6 = y_1, t = t_1^{-1}$ (See also [9, §3]). Then we may take a local coordinate $(x_1/y_1, t_1)$ centered at the point and the automorphism g maps $(x_1/y_1, t_1)$ to $(\zeta_{66}^{55}x_1/y_1, \zeta_{66}^{12}t_1)$. This implies the point is $P^{12,55}$. □

Remark 3.9. Let $[x : y : z]$ be a homogeneous coordinate of $\mathbb{P}^2$. Then X

is given by the following equations in $\mathbb{P}^2 \times \mathbb{C}$:

$$\begin{cases} zy^2 = x^3 + t(t^{11} - 1)z^3, \\ z_1 y_1^2 = x_1^3 + (1 - t_1^{11})z_1^3. \end{cases}$$

Note that the action of g at $t = \infty$ ($t_1 = 0$) is

$$([x_1 : y_1 : z_1], t_1) \mapsto ([\zeta_{66}^{22} x_1 : \zeta_{66}^{33} y_1 : z_1], \zeta_{66}^{12} t_1).$$

It is easy to see

$$\begin{aligned} X^{g^2} &= \{([x : y : z], t) = ([0 : 0 : 1], 0), ([0 : 1 : 0], 0)\} \\ &\quad \amalg \{([x_1 : y_1 : z_1], t_1) = ([0 : 1 : 0], 0), ([0 : 1 : 1], 0), ([0 : 1 : -1], 0)\} \end{aligned}$$

and

$$\begin{aligned} X^{g^3} &= \{([x : y : z], t) = ([0 : 0 : 1], 0), ([0 : 1 : 0], 0)\} \\ &\quad \amalg \{([x_1 : y_1 : z_1], t_1) = ([0 : 1 : 0], 0), ([1 : 0 : \zeta_6], 0), ([1 : 0 : -1], 0), ([1 : 0 : \zeta_6^5], 0)\}. \end{aligned}$$

In the same way as above, we can recheck Proposition 3.3 and Proposition 3.5.

4. QUOTIENT SINGULARITIES

Let G_{66} be an abelian group generated by g. Similarly, we put $G_{22} := \langle f_{22} \rangle$ and $G_{33} := \langle f_{33} \rangle$. Then G_{66}, G_{22} and G_{33} which are abelian groups of order 66, of order 22 and of order 33 act on X as automorphism groups, respectively. In this section, we study quotient surfaces X/G_{66}, X/G_{22} and X/G_{33}. For the details about general theory of quotient singularities, see e.g. [8, Chapter 2].

Lemma 4.1. *Set $G_2 := \langle f_2 \rangle, G_3 := \langle f_3 \rangle$ and $G_6 := \langle f_6 \rangle$. The quotient surfaces X/G_2, X/G_3 and X/G_6 are smooth.*

Proof. Note that fixed loci of f_2 and f_3 do not include isolated points by Proposition 3.1 (1) and (2). This means that G_2 and G_3 are quasi-reflection groups, hence X/G_2 and X/G_3 are smooth.

By Proposition 3.1 (3), fixed locus X^{f_6} has 12 isolated fixed points of type $P^{3,4}$. It is sufficient for us to see the action on neighborhoods of such points. Since G_2 and G_3 are quasi-reflection subgroups of G_6, quotient singularity $\mathbb{C}^2/G_6 = (\mathbb{C}^2/G_2)/(G_6/G_2)$ is isomorphic to $\mathbb{C}^2/G_3$ which is smooth. Thus X/G_6 is smooth. $\square$

Proposition 4.2. *Each quotient surfaces X/G_{66}, X/G_{22} and X/G_{33} have one point of type $\frac{1}{11}(1, 5)$ and one point of type $\frac{1}{11}(1, 8)$ as singularities.*

Proof. Put $G_{11} := \langle f_{11} \rangle$. Since G_6 is quasi-reflection subgroup of G_{66}, $\mathbb{C}^2/G_{66} = (\mathbb{C}^2/G_6)/(G_{66}/G_6)$ is isomorphic to $\mathbb{C}^2/G_{11}$. Hence X/G_{66} has the same singularities as X/G_{11}.

By Proposition 3.1 (4), X/G_{11} has one singular point of $\frac{1}{11}(2,10)$ and one singular point of type $\frac{1}{11}(5,7)$. We remark that a singular point of type $\frac{1}{11}(2,10)$ can be identified with $\frac{1}{11}(1,5)$ and a singular point of type $\frac{1}{11}(5,7)$ can be identified with $\frac{1}{11}(1,8)$ by by replacing a primitive 11-th root of unity.

Similarly, since G_2 and G_3 are also quasi-reflection subgroups of G_{22} and G_{33} respectively, singularities of X/G_{22} and X/G_{33} are same as X/G_{11}. □

Let Z be a normal algebraic surface with at worst log terminal singularities, Z is called *log Enriques* if the irregularity $\dim H^1(Z, \mathcal{O}_Z) = 0$ and a positive multiple IK_Z of a canonical Weil divisor K_Z is linearly equivalent to zero. The smallest integer $I > 0$ satisfying $IK_Z \sim 0$ is called the *index* of Z.

COROLLARY 4.3. *Quotient surfaces X/G_{66}, X/G_{22} and X/G_{33} are log Enriques surfaces of index 11.*

Proof. By Proposition 4.2, it is enough to see that X/G_{11} is a log Enriques surface of index 11. It follows from Lemma 1.7 and its proof of [13].
□

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