CHARACTERIZATIONS OF COMMUTATORS OF SINGULAR INTEGRAL OPERATORS ON MORREY TRIEBEL-LIZORKIN SPACES

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In this paper, we obtain the characterizations of Morrey Triebel-Lizorkin spaces by two families of operators. Applying the characterizations of Morrey Triebel-Lizorkin spaces, it is proved that b is a Lipschitz function if and only if the commutator $[b, T]$ is bounded from Morrey spaces to Morrey Triebel-Lizorkin spaces, where T is singular integral operator or Riesz potential operator.

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1. INTRODUCTION

In this paper, for $\beta > 0$, the Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x,h\in\mathbb{R}^n,\ h\neq 0} \frac{|f(x+h) - f(x)|}{|h|^\beta} < \infty.$$

For a linear operator T and a locally integrable function $b \in \dot{\Lambda}_\beta$, define a commutator operator by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

In 1978, Janson [9] proved that $b \in \dot{\Lambda}_\beta$ if and only if the commutator of Calderón-Zygmund singular integral operator T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < q < \infty$ and $1/p - 1/q = \beta/n$. In 1995, for $0 < \beta < 1$ and $1 < p < \infty$, Paluszynski [13] showed that

$$b \in \dot{\Lambda}_\beta \Leftrightarrow [b, T] : L^p(\mathbb{R}^n) \to F_p^{\beta,\infty}(\mathbb{R}^n). \tag{1}$$

It is worth noting that the result of Paluszynski depends on the following characterization of the Triebel-Lizorkin space $F_p^{\beta,\infty}(\mathbb{R}^n)$,

$$\|f\|_{F_p^{\beta,\infty}(\mathbb{R}^n)} \approx \Big\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q| \Big\|_{L^p(\mathbb{R}^n)}, \tag{2}$$

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where Q denotes a cube with sides parallel to the axes and f_Q is the mean value of f on cube Q. The equivalent relation of (2) is proved by Seeger in [20, Theorem 1].

In this paper, replacing Triebel-Lizorkin spaces by Morrey Triebel-Lizorkin spaces in (1) and (2), we prove that b belongs to Lipschitz space if and only if the commutator $[b, T]$ of singular integral operator T is bounded from Morrey spaces to Morrey Triebel-Lizorkin spaces. Meanwhile, we show that the commutator of Riesz potential operator has corresponding result as well. To this end, let us recall the definitions of Morrey spaces and Morrey Triebel-Lizorkin spaces as follows.

The classical Morrey space was originally introduced by Morrey [12] in 1938 to study the local behavior of solutions to second order elliptic partial differential equations. Now, we give the definition of Morrey spaces on $\mathbb{R}^n$.

Definition 1.1. Let $1 \leq p < \infty$ and $0 \leq \lambda < n$. Define Morrey spaces $M_{p,\lambda}$ to be the collection of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B_r(x))} < \infty,$$

where $B_r(x) = B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ for $x \in \mathbb{R}^n$ and $r > 0$.

Let's review the Fourier analytical approach to Triebel-Lizorkin spaces. The set $\mathcal{S}$ denotes the usual Schwartz class of infinitely differentiable rapidly decreasing complex-valued functions, $\mathcal{S}'$ is the dual of $\mathcal{S}$. The Fourier transform of a tempered distribution f is denoted by $\hat{f}$ while its inverse transform is denoted by $\check{f}$. Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0 \geq 0$ and satisfying

$$\varphi_0(x) = \begin{cases} 1, & |x| \leq 1; \\ 0, & |x| \geq 2. \end{cases}$$

For any $x \in \mathbb{R}^n$, if $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and $\varphi_j(x) = \varphi(2^{-j}x)$ with $j \in \mathbb{N}$, then we call $\{\varphi_j\}_{j \in \mathbb{N}_0}$ a smooth dyadic resolution of unity. It follows that

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1.$$

Definition 1.2. Let $\{\varphi_j\}_{j \in \mathbb{N}_0}$ be a smooth dyadic resolution of unity, $0 < \beta, q \leq \infty$, $0 < p < \infty$. The set

$$\Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \ \Big\| \Big(\sum_{j=0}^{\infty} |2^{j\beta} (\varphi_j * \widehat{f})^{\vee}|^q \Big)^{1/q} \Big\|_{M_{p,\lambda}(\mathbb{R}^n)} < \infty \Big\}$$

is called the Morrey Triebel-Lizorkin spaces and denoted by $\mathcal{E}^{\beta}_{p\lambda q}(\mathbb{R}^n)$. The quasi-norm of f in this space is denoted by

$$\|f\|_{\mathcal{E}^{\beta}_{p\lambda q}(\mathbb{R}^n)} = \Big\| \Big(\sum_{j=0}^{\infty} |2^{j\beta}(\varphi_j * \widehat{f})^{\vee}|^q \Big)^{1/q} \Big\|_{M_{p,\lambda}(\mathbb{R}^n)}.$$

For more research on Morrey spaces and Triebel-Lizorkin spaces, see [2, 6, 8, 12, 16, 17, 18, 20, 24, 25].

Remark 1.3. (i) For $1 \le p < \infty$ and $0 \le \lambda < n$, Mazzucato proves $\mathcal{E}^0_{p\lambda 2}(\mathbb{R}^n) = M_{p,\lambda}(\mathbb{R}^n)$ in [11, Proposition 4.1].

(ii) By [8, Theorem 3.1], for any $\beta_i \in \mathbb{R}$, $0 < q_i \le \infty$ and $0 < \lambda_i \le p_i < \infty$ $(i = 1, 2)$, if

$$\lambda_0 \le \lambda_1 \text{ and } \frac{p_1}{\lambda_1} \le \frac{p_0}{\lambda_0},$$
$$\beta_0 - \frac{n}{\lambda_0} > \beta_1 - \frac{n}{\lambda_1},$$
$$\text{or } \beta_0 - \frac{n}{\lambda_0} = \beta_1 - \frac{n}{\lambda_1} \text{ and } \lambda_0 \ne \lambda_1,$$
$$\text{or } \beta_0 = \beta_1, \lambda_0 = \lambda_1 \text{ and } q_0 \le q_1,$$

then

$$\mathcal{E}^{\beta_0}_{p_0\lambda_0 q_0}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^{\beta_1}_{p_1\lambda_1 q_1}(\mathbb{R}^n).$$

Thus, when $q_0 = \infty$, $\beta_1 = 0$ and $q_1 = 2$, it is obvious that $\mathcal{E}^{\beta_0}_{p_0\lambda_0\infty}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^0_{p_0\lambda_0 2}(\mathbb{R}^n) = M_{p_1,\lambda_1}(\mathbb{R}^n)$. This fact will be used in Section 3.

Based on Definition 1.1 and Definition 1.2, we can now give the following characterization of Morrey Triebel-Lizorkin space $\mathcal{E}^{\beta}_{p\lambda\infty}(\mathbb{R}^n)$.

THEOREM 1.4. *Let* $0 < \beta < 1$ *and* $1 < \lambda \le p < \infty$, *then*

$$\|f\|_{\mathcal{E}^{\beta}_{p\lambda\infty}(\mathbb{R}^n)} \approx \Big\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q| \Big\|_{M_{p,\lambda}(\mathbb{R}^n)}.$$

The above theorem will be proved in next section.

From now on, it is assumed that the singular integral operator we are working with

$$Tf(x) = p \cdot v \cdot \int_{\mathbb{R}^n} K(x-y) f(y) \mathrm{d}y$$

is regular, which means that the kernel satisfies

$$\big|K(x)\big| \le C\big|x\big|^{-n}, \quad x \ne 0 \tag{3}$$

and

$$\big|K(x-y) - K(x'-y)\big| + \big|K(y-x) - K(y-x')\big| \le c_1 \Big(\frac{|x-x'|}{|x-y|}\Big)^{\varepsilon} |x-y|^{-n} \tag{4}$$

whenever $0 < \varepsilon \le 1$ and $2|x - x'| < |x - y|$. Note that Difazio and Ragusa [7] studied the boundedness of the singular integral operator in Morrey spaces.

Theorem 1.5. *Let $0 < \beta < 1$ and $1 < \lambda \le p < \infty$. Then, the following conditions are equivalent:*

(i) $b \in \dot{\Lambda}_\beta$;

(ii) $[b, T]$ *is a bounded operator from* $M_{p,\lambda}(\mathbb{R}^n)$ *to* $\mathcal{E}^\beta_{p\lambda\infty}(\mathbb{R}^n)$.

For $0 < \alpha < n$, the Riesz potential operator I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y.$$

Spanne (published by Peetre [14]) and Adams [1] studied the boundedness of Riesz potential operator in Morrey spaces. In 2006, Shirai [21] showed that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from $M_{p,\lambda}(\mathbb{R}^n)$ to $M_{q,\lambda}(\mathbb{R}^n)$ or from $M_{p,\lambda}(\mathbb{R}^n)$ to $M_{q,\mu}(\mathbb{R}^n)$ for some appropriate indices p, q, λ, μ, and β. Inspired by the above conclusions, we give the following theorem for Riesz potential operator.

Theorem 1.6. *Let $0 < \alpha$, $0 < \beta < 1$ and $1 < p < q < \infty$. Then, the following conditions are equivalent:*

(i) $b \in \dot{\Lambda}_\beta$;

(ii) $[b, I_\alpha]$ *is a bounded operator from* $M_{p,\lambda}(\mathbb{R}^n)$ *to* $\mathcal{E}^\beta_{q\mu\infty}(\mathbb{R}^n)$ *with* $1/p - 1/q = \alpha/n < 1$, $0 < \lambda < n - \alpha p$ *and* $\lambda/p = \mu/q$;

(iii) $[b, I_\alpha]$ *is a bounded operator from* $M_{p,\lambda}(\mathbb{R}^n)$ *to* $\mathcal{E}^\beta_{q\lambda\infty}(\mathbb{R}^n)$ *with* $1 < \lambda < \infty$ *and* $1/p - 1/q = \alpha/(n-\lambda) < n/(n-\lambda)$.

Note that the proofs of Theorem 1.5 and Theorem 1.6 are postponed until Section 3.

Throughout the whole article, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Use C as a generic positive constant, and denote simply by $A \lesssim B$ if there exists a constant $C > 0$ such that $A \le CB$. Further, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. THE CHARACTERIZATIONS OF MORREY TRIEBEL-LIZORKIN SPACES

In this section, using two families of operators, we discuss the characterizations of Morrey Triebel-Lizorkin spaces $\mathcal{E}^\beta_{p\lambda q}(\mathbb{R}^n)$ in Theorem 2.5, which implies Theorem 1.4.

To show the characterizations of $\mathcal{E}^{\beta}_{p\lambda q}(\mathbb{R}^n)$, we need the property of Peetre maximal operator on $\mathcal{E}^{\beta}_{p\lambda q}(\mathbb{R}^n)$ and the boundedness of Hardy-Littlewood maximal operator on sequences function spaces. To this end, let us recall the following Peetre maximal operator, introduced in [15].

Definition 2.1. Given a sequence of functions $\{\Psi_j\}_j \subset \mathcal{S}(\mathbb{R}^n)$, a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ and a positive number $a > 0$, the Peetre's maximal functions are defined as

$$(\Psi_j^* f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|\Psi_j * f(y)|}{1+|2^j(x-y)|^a} \quad \text{for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}_0.$$

The proof of the following lemma may be found in [19, Theorem 4.1].

LEMMA 2.2. *Let* $0 < \lambda \le p < \infty$, $0 < q \le \infty$ *and* $\beta \in \mathbb{R}$. *If* $f \in \mathcal{E}^{\beta}_{p\lambda q}(\mathbb{R}^n)$, *then*

$$\|f\|_{\mathcal{E}^{\beta}_{p\lambda q}(\mathbb{R}^n)} \sim \left\| \left(\sum_{j=0}^{\infty} \left(2^{j\beta} (\Psi_j^* f)_a \right)^q \right)^{1/q} \right\|_{M_{p,\lambda}(\mathbb{R}^n)}$$

$$\sim \left\| \left(\sum_{j=0}^{\infty} \left(2^{j\beta} (\Psi_j * f) \right)^q \right)^{1/q} \right\|_{M_{p,\lambda}(\mathbb{R}^n)}.$$

For the case $q = \infty$, *the above facts have the usual modification and replace integrations by sup-norms.*

For any $f \in L^1_{loc}(\mathbb{R}^n)$, the standard Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B_r(x)} |f(y)| \mathrm{d}y.$$

Chiarenza and Frasca [3] showed that the operator M is bounded on $M_{p,\lambda}(\mathbb{R}^n)$ with $1 < p < \infty$ and $0 \le \lambda < n$.

The boundedness of Hardy-Littlewood maximal operator on vector-valued function spaces is given as follows.

LEMMA 2.3. *If* $1 < \lambda \le p < \infty$ *and* $1 < q \le \infty$, *then there is a constant* C *such that for all sequences* $\{f_j\}_{j=0}^{\infty}$ *of locally integrable functions,*

$$\Big\| \Big(\sum_{j=0}^{\infty} (Mf_j)^q \Big)^{1/q} \Big\|_{M_{p,\lambda}(\mathbb{R}^n)} \le C \Big\| \Big(\sum_{j=0}^{\infty} f_j^q \Big)^{1/q} \Big\|_{M_{p,\lambda}(\mathbb{R}^n)}.$$

Now, we recall two families of operators.

Definition 2.4. Let $\beta > 0$, $1 < q \leq \infty$, $1 \leq r < \infty$ and $m \in \mathbb{N}$. Assume that $Q_x(t) = Q(x,t)$ denotes a cube centered at x, with side length t, sides parallel to the axes.

(i) Consider the family of operators $S^\beta_{q,r,m}$, defined by

$$S^\beta_{q,r,m} f(x) = \Big(\int_0^\infty \big(\frac{1}{|Q_0(t)|} \int_{Q_0(t)} |\Delta^m_h f(x)|^r \mathrm{d}h \big)^{q/r} \frac{\mathrm{d}t}{t^{1+\beta q}} \Big)^{1/q},$$

where Δ^m_h is difference operator, that is

$$\Delta^1_h f(x) = \Delta_h f(x) = f(x+h) - f(x),$$
$$\Delta^{m+1}_h f(x) = \Delta^k_h f(x+h) - \Delta^k_h f(x), \quad m \geq 1.$$

(ii) For a fixed cube $Q = Q_x(t)$, we define the oscillation

$$\operatorname{osc}^m_r(f,Q) = \operatorname{osc}^m_r(f,x,t) = \inf_{P \in \mathcal{P}_m} \Big(\frac{1}{|Q|} \int_Q |f(y) - P(y)|^r \mathrm{d}y \Big)^{1/r},$$

where $\mathcal{P}_m$ denotes the space of all polynomials on $\mathbb{R}^n$ of degree up to order m. Further, we define the family of operators

$$\mathcal{D}^\beta_{q,r,m} f(x) = \Big(\int_0^\infty \big(\operatorname{osc}^{m-1}_r (f,x,t) \big)^q \frac{\mathrm{d}t}{t^{1+\beta q}} \Big)^{1/q}.$$

For $q = \infty$ or $r = \infty$, we have the usual modifications and replace integrations by sup-norms. Some properties of the above two families of operators can be found in [4] and [5].

From now on, we assume that ν is the trace of a matrix (see [20, p. 391]). Note that the following theorem implies Theorem 1.4 in the case $q = \infty$.

THEOREM 2.5. *Let* $1 < \lambda \leq p < \infty$, $1 < q \leq \infty$, $r \geq 1$, $m \in \mathbb{N}$ *and* $\nu \in \mathbb{R}$. *If there exists a positive constant* a_0 *such that* $m > \beta/a_0$ *and*

$$\beta > \sigma_{p,q,r} = \max \big\{ 0, \nu(\frac{1}{p} - \frac{1}{r}), \nu(\frac{1}{q} - \frac{1}{r}) \big\},$$

then

$$\|f\|_{\mathcal{E}^\beta_{p\lambda q}(\mathbb{R}^n)} \sim \|S^\beta_{q,r,m} f\|_{M_{p,\lambda}(\mathbb{R}^n)} \sim \|\mathcal{D}^\beta_{q,r,m} f\|_{M_{p,\lambda}(\mathbb{R}^n)}.$$

Proof. The proof follows the ideas in [20]. Firstly, by $S^\beta_{q,r,m} f \lesssim \mathcal{D}^\beta_{q,r,m} f$, we can deduce that $\|S^\beta_{q,r,m} f\|_{M_{p,\lambda}(\mathbb{R}^n)} \lesssim \|\mathcal{D}^\beta_{q,r,m} f\|_{M_{p,\lambda}(\mathbb{R}^n)}$. In addition, according to the argument of Triebel [23], it is obvious that

$$\|f\|_{\mathcal{E}^\beta_{p\lambda q}(\mathbb{R}^n)} \lesssim \|S^\beta_{q,r,m} f\|_{M_{p,\lambda}(\mathbb{R}^n)}.$$

Thus, we only need to verify

$$\|\mathcal{D}^\beta_{q,r,m} f\|_{M_{p,\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{E}^\beta_{p\lambda q}(\mathbb{R}^n)}.$$

By the proof of [20, Theorem 1], for $0<\tau<\min(1,q,p)$, it follows that $\|\mathcal{D}^{\beta}_{q,r,m}f\|_{M_{p,\lambda}(\mathbb{R}^n)}\lesssim\|\mathcal{D}^{\beta}_{q,\tau,m}f\|_{M_{p,\lambda}(\mathbb{R}^n)}$. Now we decompose

$$f=f_{0,t}+f_{1,t}\quad\text{and}\quad f_{0,t}=\sum_{2^kt\geq1}\Psi_k*f,$$

where Ψ_k is as in Definition 2.1. Then

$$\begin{aligned}
&\mathcal{D}^{\beta}_{q,\tau,m}f(x)\\
&=\Big(\int_0^\infty\big(\mathrm{osc}^{m-1}_{\tau}(f,x,t)\big)^q\frac{\mathrm{d}t}{t^{1+\beta q}}\Big)^{1/q}\\
&\leq\Big(\int_0^\infty\big(\mathrm{osc}^{m-1}_{\tau}(f_{0,t},x,t)\big)^q\frac{\mathrm{d}t}{t^{1+\beta q}}\Big)^{1/q}+\Big(\int_0^\infty\big(\mathrm{osc}^{m-1}_{\tau}(f_{1,t},x,t)\big)^q\frac{\mathrm{d}t}{t^{1+\beta q}}\Big)^{1/q}\\
&=:I+II.
\end{aligned}$$

To estimate I, based on $\mathrm{osc}^{m-1}_{\tau}(f_{0,t},x,t)\leq\big(M(f^{\tau}_{0,t})\big)^{1/\tau}$ with $\tau<q$, we use Lemma 2.3 and Lemma 2.2 to derive

$$\begin{aligned}
\|I\|_{M_{p,\lambda}(\mathbb{R}^n)}&\lesssim\Big\|\Big(\int_0^\infty\big(\sum_{2^kt\geq1}\Psi_k*f\big)^q\frac{\mathrm{d}t}{t^{1+\beta q}}\Big)^{1/q}\Big\|_{M_{p,\lambda}(\mathbb{R}^n)}\\
&\lesssim\Big\|\Big(\sum_k2^{k\beta q}|\Psi_k*f|^q\Big)^{1/q}\Big\|_{M_{p,\lambda}(\mathbb{R}^n)}\\
&\lesssim\|f\|_{\mathcal{E}^{\beta}_{p\lambda q}(\mathbb{R}^n)}.
\end{aligned}$$

To estimate II, for $a>0$, we have

$$\mathrm{osc}^{m-1}_{\tau}(f_{1,t},x,t)\lesssim\Big(\int_{Q_x(t)}|\sum_{2^kt\leq1}(2^kt)^{ma_0}(\Psi^*_kf)_a(z)|^{\tau}\mathrm{d}z\Big)^{1/\tau},$$

where $Q_x(t)$ is a cube with x as its center and t as its edge. Due to $m>\beta/a_0$, it follows that

$$\begin{aligned}
\|II\|_{M_{p,\lambda}(\mathbb{R}^n)}&\lesssim\Big\|\Big(\int_0^\infty|\sum_{2^kt\leq1}(2^kt)^{ma_0-\beta}2^{k\beta}(\Psi^*_kf)_a|^q\frac{\mathrm{d}t}{t}\Big)^{1/q}\Big\|_{M_{p,\lambda}(\mathbb{R}^n)}\\
&\lesssim\Big\|\Big(\sum_k2^{k\beta q}(\Psi^*_kf)^q_a\Big)^{1/q}\Big\|_{M_{p,\lambda}(\mathbb{R}^n)}\\
&\lesssim\|f\|_{\mathcal{E}^{\beta}_{p\lambda q}(\mathbb{R}^n)},
\end{aligned}$$

where the last estimate follows by choosing $a>\nu\min\{p,q\}$ in Lemma 2.2. This ends the proof of Theorem 2.5. □

3. PROOFS OF THEOREM 1.5 AND THEOREM 1.6

In this section, by means of some lemmas, we give the proofs of Theorem 1.5 and Theorem 1.6.

LEMMA 3.1. *Let* $0<\beta<1$ *and* $1<q\le\infty$. *Then*

$$\|f\|_{\dot{\Lambda}_\beta}\approx\sup_Q\frac{1}{|Q|^{1+\beta/n}}\int_Q|f-f_Q|\approx\sup_Q\frac{1}{|Q|^{\beta/n}}\Big(\frac{1}{|Q|}\int_Q|f-f_Q|^q\Big)^{1/q}.$$

For $q=\infty$, *the formula should be interpreted appropriately.*

The proof of the Lemma 3.1 may be found in [5, p. 14, p. 38] and [10].

Proof of Theorem 1.5. Let $0<\beta<1$ and $\frac{1}{p}-\frac{1}{q}=\frac{\beta}{n}$. Fix a cube $Q=Q(x_Q,t)$ and assume that $x\in Q$. For $f\in L^p(\mathbb{R}^n)$, we write $f^0=f\chi_{2Q}$ and $f^\infty=f-f^0$.

(i)⇒(ii): According to $\big[b,T\big]f=\big[b-b_Q,T\big]f$, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q\Big|\big[b,T\big]f-\big(\big[b,T\big]f\big)_Q\Big|\\
&=\frac{1}{|Q|^{1+\beta/n}}\int_Q\Big|\big[b-b_Q,T\big]f-\big(\big[b-b_Q,T\big]f\big)_Q\Big|\\
&\lesssim\frac{1}{|Q|^{1+\beta/n}}\int_Q\Big|\big[b-b_Q,T\big]f-T\big((b-b_Q)f^\infty\big)(x_Q)\Big|\\
&\lesssim\frac{1}{|Q|^{1+\beta/n}}\int_Q\Big|(b-b_Q)Tf\Big|\\
&\quad+\frac{1}{|Q|^{1+\beta/n}}\int_Q\Big|T((b-b_Q)f)-T\big((b-b_Q)f^\infty\big)(x_Q)\Big|\\
&\lesssim\frac{1}{|Q|^{1+\beta/n}}\int_Q\Big|(b-b_Q)Tf\Big|\\
&\quad+\frac{1}{|Q|^{1+\beta/n}}\int_Q\Big|T((b-b_Q)f^0)\Big|\\
&\quad+\frac{1}{|Q|^{\beta/n}}\sup_{y\in Q}\Big|T\big((b-b_Q)f^\infty\big)(y)-T\big((b-b_Q)f^\infty\big)(x_Q)\Big|\\
&=:D_1+D_2+D_3.
\end{aligned}$$

Firstly, to estimate D_1, we utilize Lemma 3.1 to derive

$$\begin{aligned}
\frac{1}{|Q|^{1+\beta/n}}\int_Q\Big|(b-b_Q)Tf\Big|&\lesssim\frac{1}{|Q|^{\beta/n}}\sup_{y\in Q}|b(y)-b_Q|\Big(\frac{1}{|Q|}\int_Q|Tf|\Big)\\
&\lesssim\|b\|_{\dot{\Lambda}_\beta}M\big(Tf\big)(x),
\end{aligned}$$

which allows us to obtain that $D_1 \lesssim \|b\|_{\dot{\Lambda}_\beta} M\big(Tf\big)(x)$.

To estimate D_2, for $0 < t < p$, the boundedness of T implies that

$$\begin{aligned}
D_2 &= \frac{1}{|Q|^{1+\beta/n}} \int_Q \Big|T((b-b_Q)f^0)\Big| \\
&\lesssim \frac{1}{|Q|^{1+\beta/n}} \bigg(\int_Q \Big|T\big((b-b_Q)f^0\big)\Big|^t \bigg)^{1/t} \big|Q\big|^{1-1/t} \\
&\lesssim |Q|^{-\beta/n-1/t} \Big(\int_Q |(b-b_Q)f^0|^t \Big)^{1/t} \\
&\lesssim |Q|^{-\beta/n} \sup_{y\in Q} |b(y)-b_Q| \Big(\frac{1}{|Q|} \int_Q |f|^t \Big)^{1/t} \\
&\lesssim \|b\|_{\dot{\Lambda}_\beta} \big(M(|f|^t)\big)^{1/t}(x).
\end{aligned}$$

In view of D_3, we need the following well-known fact:

$$|b_{Q^*} - b_Q| \lesssim C\|b\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \quad \text{for } Q^* \subset Q.$$

From this and Lemma 3.1, it follows that

$$\begin{aligned}
&\Big|T\big((b-b_Q)f^\infty\big)(y) - T\big((b-b_Q)f^\infty\big)(x_Q)\Big| \\
&= \Big| \int_{R^n} \big(K(y-z) - K(x_Q-z)\big)(b(z)-b_Q)f^\infty(z)\mathrm{d}z \Big| \\
&\lesssim \int_{(2Q)^c} \frac{|y-x_Q|}{|x_Q-z|^{n+1}} |b(z)-b_Q||f(z)|\mathrm{d}z \\
&\lesssim \sum_{m=2}^{\infty} \int_{2^mQ\setminus 2^{m-1}Q} 2^{-m}|2^mQ|^{-1}(|b(z)-b_{2^kQ}| + |b_{2^kQ}-b_Q|)|f(z)|\mathrm{d}z \\
&\lesssim \sum_{m=2}^{\infty} 2^{-m}|2^mQ|^{\beta/n}\|b\|_{\dot{\Lambda}_\beta} M(f)(x) + \sum_{m=2}^{\infty} 2^{-m}|2^mQ|^{\beta/n}\|b\|_{\dot{\Lambda}_\beta} M(f)(x) \\
&\lesssim \|b\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \sum_{m=2}^{\infty} 2^{-m+\beta m} M(f)(x) \\
&\lesssim \|b\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M(f)(x),
\end{aligned}$$

which implies that $D_3 \lesssim \|b\|_{\dot{\Lambda}_\beta} M(f)(x)$.

Summarizing all the estimates of D_1 through D_3, we obtain that

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q \Big| [b,T]f - \big([b,T]f\big)_Q \Big| \\
&\lesssim \|b\|_{\dot{\Lambda}_\beta} \Big(M\big(Tf\big)(x) + \big(M(|f|^t)\big)^{1/t}(x) + Mf(x) \Big).
\end{aligned}$$

Further, taking the supremum over all Q such that $x\in Q$, and the norm of $M_{p,\lambda}(\mathbb{R}^n)$ on the both sides of the inequality above. By Theorem 1.4 and the boundedness of M, we conclude that

$$\begin{aligned}&\Big\|[b,T]f\Big\|_{\mathcal{E}^{\beta}_{p\lambda\infty}(\mathbb{R}^n)}\\&\lesssim \|b\|_{\dot{\Lambda}_\beta}\Big(\big\|M(Tf)\big\|_{M_{p,\lambda}(\mathbb{R}^n)}+\big\|\big(M(|f|^t)\big)^{1/t}\big\|_{M_{p,\lambda}(\mathbb{R}^n)}+\big\|Mf\big\|_{M_{p,\lambda}(\mathbb{R}^n)}\Big)\\&\lesssim \|b\|_{\dot{\Lambda}_\beta}\|f\|_{M_{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

(ii)⇒(i): Note that $K(y-z)$ is a homogeneous kernel of degree $-n$ on some ball. Choose $z_0\in\mathbb{R}^n$, $Q\big(z_0,\delta\sqrt{n}\big)\subset\mathbb{R}^n$ and take $|z_0|>\sqrt{n}$, $\delta<1$ small such that $\bar{Q}\bigcap\{0\}=\varnothing$. Thus, we can express $\frac{1}{K(x-y)}$ as an absolutely convergent Fourier series on the ball, of the form

$$\frac{1}{K(x-y)}=\sum_{m=0}^{\infty}a_m e^{i\langle\nu_m,(x-y)\rangle}, \tag{5}$$

where above and in what follows, for the specific vectors $\nu_m\in\mathbb{R}^n$,

$$\sum_{m=0}^{\infty}|a_m|<\infty.$$

For $x_0\in\mathbb{R}^n$ and $t>1$, define $Q=Q(x_0,t)$ and $Q^0=Q(x_0+z_0t,t)$. Let $x\in Q$, $y\in Q^0$ with $(y-x)/t\in Q\big(z_0,\delta\sqrt{n}\big)$. Take on $s(x)=\overline{\operatorname{sgn}(b(x)-b_{Q^0})}$. By (5) and Remark 1.3, we see that

$$\begin{aligned}&\frac{1}{|Q|^{1+\beta/n}}\int_Q|(b(x)-b_Q)|\mathrm{d}x\\&\lesssim\frac{1}{|Q|^{1+\beta/n}}\int_Q\big|b(x)-b_{Q^0}\big|\mathrm{d}x\\&\lesssim\frac{1}{|Q|^{1+\beta/n}}\frac{1}{|Q^0|}\int_Q s(x)\bigg(\int_{Q^0}\Big(b(x)-b(y)\Big)\mathrm{d}y\bigg)\mathrm{d}x\\&\approx\frac{1}{t^{2n+\beta}}\int_Q s(x)\bigg(\int_{Q^0}\Big(b(x)-b(y)\Big)\frac{K(x-y)}{K(x-y)}\mathrm{d}y\bigg)\mathrm{d}x\\&\approx\frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty}a_m\int_Q s(x)\bigg(\int_{Q^0}\Big(b(x)-b(y)\Big)K(x-y)e^{i\langle\nu_m,(y-x)/t\rangle}\mathrm{d}y\bigg)\mathrm{d}x\\&\lesssim\frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty}|a_m|\int_{R^n}\Big([b,T]\big(\chi_{Q^0}e^{i\langle\nu_m,\cdot/t\rangle}\big)(x)\Big)\Big(\chi_Q(x)e^{-i\langle\nu_m,x/t\rangle}s(x)\Big)\mathrm{d}x\\&\lesssim\frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty}|a_m|\Big\|[b,T]\big(\chi_{Q^0}e^{i\langle\nu_m,\cdot/t\rangle}\big)\Big\|_{M_{q,\lambda}(\mathbb{R}^n)}\big\|\chi_Q\big\|_{M_{q',\lambda}(\mathbb{R}^n)}\end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty}|a_m|\Big\|[b,T]\big(\chi_{Q^0}e^{i\langle\nu_m,\cdot/t\rangle}\big)\Big\|_{\mathcal{E}^{\beta}_{p\lambda\infty}(\mathbb{R}^n)}\big\|\chi_Q\big\|_{M_{q',\lambda}(\mathbb{R}^n)}\\
&\lesssim \frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty}|a_m|\big\|[b,T]\big\|_{M_{p,\lambda}(\mathbb{R}^n)\to\mathcal{E}^{\beta}_{p\lambda\infty}(\mathbb{R}^n)}\|\chi_{Q^0}\|_{M_{p,\lambda}(\mathbb{R}^n)}\big\|\chi_Q\big\|_{M_{q',\lambda}(\mathbb{R}^n)}\\
&\lesssim \big\|[b,T]\big\|_{M_{p,\lambda}(\mathbb{R}^n)\to\mathcal{E}^{\beta}_{p\lambda\infty}(\mathbb{R}^n)},
\end{aligned}$$

which implies that (ii)⇒(i) holds. This ends the proof of Theorem 1.5. □

We now turn to the commutator of Riesz potential operator for the above problem. Referring to the proof in [5, pp. 71-72], replacing $L^p(\mathbb{R}^n)$ by $M_{p,\lambda}(\mathbb{R}^n)$, we can obtain the following lemma.

Lemma 3.2. *Let* $1<\lambda\le p<q<\infty$ *and* $1/p-1/q=\alpha/n$. *Suppose that, for each cube* Q, *there exists a function* h^Q *defined on* Q. *Then, for* $0\le\gamma$,

$$\Big\|\sup_Q\frac{1}{|Q|^{1+\gamma/n}}\int_Q|h^Q|\Big\|_{M_{q,\lambda}(\mathbb{R}^n)}\le C\Big\|\sup_Q\frac{1}{|Q|^{1+\gamma/n+\alpha/n}}\int_Q|h^Q|\Big\|_{M_{p,\lambda}(\mathbb{R}^n)},$$

where the constant C *depends only on* p,q,α *and* n.

Lemma 3.3 (Adams [1]). *Let* $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $0<\lambda<n-\alpha p$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$. *Then the operator* I_α *is bounded from* $M_{p,\lambda}(\mathbb{R}^n)$ *to* $M_{q,\lambda}(\mathbb{R}^n)$.

Lemma 3.4 (Spanne, but published by Peetre [14]). *Let* $0<\alpha<n$, $1<p<\frac{n}{\alpha}$ *and* $0<\lambda<n-\alpha p$. *Assume that* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$ *and* $\frac{\lambda}{p}=\frac{\mu}{q}$. *Then the operator* I_α *is bounded from* $M_{p,\lambda}(\mathbb{R}^n)$ *to* $M_{q,\mu}(\mathbb{R}^n)$.

Proof of Theorem 1.6. Let $0<\beta<1$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$ and $\frac{1}{p}-\frac{1}{r}=\frac{\alpha+\beta}{n}$ with $\frac{1}{p}>\frac{\alpha+\beta}{n}$. Fix x_Q as the center of a cube Q. Let $g\in L^p(\mathbb{R}^n)$. Define $g^0=g\chi_{2Q}$ and $g^\infty=g-g^0$.

(i)⇒(ii): From Theorem 1.4 and Lemma 3.2, it follows that

$$\begin{aligned}
&\|[b,I_\alpha](g)\|_{\mathcal{E}^{\beta}_{p\mu\infty}(\mathbb{R}^n)}\\
&\lesssim\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q\big|[b,I_\alpha](g)-\big([b,I_\alpha](g)\big)_Q\big|\Big\|_{M_{q,\mu}(\mathbb{R}^n)}\\
&\lesssim\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q\big|[b-b_Q,I_\alpha](g)-I_\alpha\big((b-b_Q)g^\infty\big)(x_Q)\big|\Big\|_{M_{q,\mu}(\mathbb{R}^n)}\\
&\lesssim\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q\big|(b-b_Q)I_\alpha(g)\big|\Big\|_{M_{q,\mu}(\mathbb{R}^n)}\\
&\quad+\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+(\alpha+\beta)/n}}\sup_{y\in Q}\big|I_\alpha\big((b-b_Q)g^0\big)(y)\big|\Big\|_{M_{p,\mu}(\mathbb{R}^n)}
\end{aligned}$$

$$+\left\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{\frac{\alpha+\beta}{n}}}\sup_{y\in Q}\left|I_\alpha\big((b-b_Q)g^\infty\big)(y)-I_\alpha\big((b-b_Q)g^\infty\big)(x_Q)\right|\right\|_{M_{p,\mu}(\mathbb{R}^n)}$$
$$=: E_1+E_2+E_3.$$

Firstly, we estimate E_1. For each $x\in Q$, by Lemma 3.1, it follows that

$$\begin{aligned}\frac{1}{|Q|^{\beta/n}}\frac{1}{|Q|}\int_Q\left|(b-b_Q)I_\alpha(g)\right| &\lesssim \frac{1}{|Q|^{\beta/n}}\sup_{y\in Q}|(b(y)-b_Q)|\Big(\frac{1}{|Q|}\int_Q|I_\alpha(g)|\Big)\\ &\lesssim \|b\|_{\dot{\Lambda}_\beta}M\big(I_\alpha(g)\big)(x).\end{aligned}$$

Thus, the boundedness of M and Lemma 3.4 imply that

$$E_1\lesssim\|b\|_{\dot{\Lambda}_\beta}\big\|I_\alpha(g)\big\|_{M_{q,\mu}(\mathbb{R}^n)}\lesssim\|b\|_{\dot{\Lambda}_\beta}\|g\|_{M_{p,\lambda}(\mathbb{R}^n)}.$$

To estimate E_2, choose r $(1<r<p)$ and $\bar r$ such that $1/r-1/\bar r=\alpha/n$. Such $\bar r$ exists, by $r<p<n/\alpha$, we deduce that

$$\begin{aligned}&\frac{1}{|Q|^{1+(\alpha+\beta)/n}}\int_Q\left|I_\alpha((b-b_Q)g^0)\right|\\ &\lesssim\frac{1}{|Q|^{1+(\alpha+\beta)/n}}\big\|I_\alpha((b-b_Q)g^0)\big\|_{L^{\bar r}(\mathbb{R}^n)}|Q|^{1/\bar r'}\\ &\lesssim|Q|^{-1-(\alpha+\beta)/n+1-1/\bar r}\|(b-b_Q)g^0\|_{L^r(\mathbb{R}^n)}\\ &\lesssim\|b\|_{\dot{\Lambda}_\beta}\big(M(|g|^r)\big)^{1/r},\end{aligned}$$

which implies $E_2\lesssim\|b\|_{\dot{\Lambda}_\beta}\|g\|_{M_{p,\mu}(\mathbb{R}^n)}\lesssim\|b\|_{\dot{\Lambda}_\beta}\|g\|_{M_{p,\lambda}(\mathbb{R}^n)}$.

To estimate E_3, referring to the estimates of D_3, we see that

$$\begin{aligned}&\frac{1}{|Q|^{\alpha/n+\beta/n}}\left|I_\alpha\big((b-b_Q)g^\infty\big)(y)-I_\alpha\big((b-b_Q)g^\infty\big)(x_Q)\right|\\ &\lesssim\frac{1}{|Q|^{\alpha/n+\beta/n}}\int_{(2Q)^c}\frac{|y-x_Q||b(z)-b_Q||g(z)|}{|x_Q-z|^{n+1-\alpha}}\mathrm{d}z\\ &\lesssim\frac{1}{|Q|^{\alpha/n+\beta/n}}\sum_{k=2}^\infty\int_{2^kQ\setminus2^{k-1}Q}2^{-k}|2^kQ|^{-1+\alpha/n}|g(z)||b(z)-b_Q|\mathrm{d}z\\ &\lesssim\sum_{k=2}^\infty2^{-k+k\alpha+k\beta}\frac{1}{|2^kQ|^{\beta/n}}\frac{1}{|2^kQ|}\int_{2^kQ}|b(z)-b_{2^kQ}||g(z)|\mathrm{d}z\\ &\quad+\sum_{k=2}^\infty2^{-k+k\alpha}\frac{1}{|Q|^{\beta/n}}|2^kQ|^{\beta/n}\|b\|_{\dot{\Lambda}_\beta}\frac{1}{|2^kQ|}\int_{2^kQ}|g(z)|\mathrm{d}z\\ &\lesssim\|b\|_{\dot{\Lambda}_\beta}M(g)(x).\end{aligned}$$

So,

$$E_3 \lesssim \|b\|_{\dot{\Lambda}_\beta}\|g\|_{M_{p,\mu}(\mathbb{R}^n)} \lesssim \|b\|_{\dot{\Lambda}_\beta}\|g\|_{M_{p,\lambda}(\mathbb{R}^n)}.$$

This allows us to obtain that (i)⇒(ii) holds.

(ii)⇒(iii): According to Remark 1.3, we have the following inequality

$$\|[b, I_\alpha]f\|_{\mathcal{E}^\beta_{q\lambda\infty}(\mathbb{R}^n)} \lesssim \|[b, I_\alpha]f\|_{\mathcal{E}^\beta_{p\mu\infty}(\mathbb{R}^n)} \lesssim \|f\|_{M_{p,\lambda}(\mathbb{R}^n)}.$$

(iii)⇒(i): We know that $\frac{1}{(|x-y|)^{n-\alpha}}$ is a homogeneous kernel of degree $-n+\alpha$. Choose $x_0 \in \mathbb{R}^n, t > 0$, let $Q = Q(x_0, t)$ and $Q^0 = Q(x_0 + z_1 t, t)$. By (5), we can get that for any $x \in Q$ and $y \in Q^0$,

$$\frac{1}{|x-y|^{n-\alpha}} = \sum_{m=0}^{\infty} a_m e^{i\langle \nu_m, x-y\rangle}.$$

Applying the above formula and Remark 1.3, we deduce that

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |(b(x) - b_Q)|\mathrm{d}x \\
&\lesssim \frac{1}{|Q|^{1+\beta/n}}\frac{1}{|Q^0|}\int_Q s(x)\Big(\int_{Q^0}\Big(b(x) - b(y)\Big)\mathrm{d}y\Big)\mathrm{d}x \\
&\approx \frac{1}{t^{2n+\beta}}\int_Q s(x)\Big(\int_{Q^0}\Big(b(x) - b(y)\Big)\frac{|x-y|^{n-\alpha}}{|x-y|^{n-\alpha}}\mathrm{d}y\Big)\mathrm{d}x \\
&\approx \frac{1}{t^{n+\beta+\alpha}}\sum_{m=0}^{\infty} a_m \int_Q s(x)\Big(\int_{Q^0}\Big(b(x) - b(y)\Big)|x-y|^{n-\alpha}e^{i\langle \nu_m,(y-x)/t\rangle}\mathrm{d}y\Big)\mathrm{d}x \\
&\lesssim \frac{1}{t^{n+\beta+\alpha}}\sum_{m=0}^{\infty} |a_m| \int_{R^n}\Big(\big[b, I_\alpha\big]\big(\chi_{Q^0}e^{i\langle \nu_m,\cdot/t\rangle}\big)(x)\Big)\Big(\chi_Q(x)e^{-i\langle \nu_m,x/t\rangle}s(x)\Big)\mathrm{d}x \\
&\lesssim \frac{1}{t^{n+\beta+\alpha}}\sum_{m=0}^{\infty} |a_m|\Big\|\big[b, I_\alpha\big]\big(\chi_{Q^0}e^{i\langle \nu_m,\cdot/t\rangle}\big)\Big\|_{M_{r,\lambda}(\mathbb{R}^n)}\|\chi_Q\|_{M_{r',\lambda}(\mathbb{R}^n)} \\
&\lesssim \frac{1}{t^{n+\beta+\alpha}}\sum_{m=0}^{\infty} |a_m|\Big\|\big[b, I_\alpha\big]\big(\chi_{Q^0}e^{i\langle \nu_m,\cdot/t\rangle}\big)\Big\|_{\mathcal{E}^\beta_{q\lambda\infty}(\mathbb{R}^n)}\|\chi_Q\|_{M_{r',\lambda}(\mathbb{R}^n)} \\
&\lesssim \frac{1}{t^{n+\beta+\alpha}}\sum_{m=0}^{\infty} |a_m|\big\|\big[b, I_\alpha\big]\big\|_{M_{p,\lambda}(\mathbb{R}^n)\to\mathcal{E}^\beta_{q\lambda\infty}(\mathbb{R}^n)}\|\chi_{Q^0}\|_{M_{p,\lambda}(\mathbb{R}^n)}\|\chi_Q\|_{M_{r',\lambda}(\mathbb{R}^n)} \\
&\lesssim \big\|\big[b, I_\alpha\big]\big\|_{M_{p,\lambda}(\mathbb{R}^n)\to\mathcal{E}^\beta_{q\lambda\infty}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, by the fact that $\big[b, I_\alpha\big]$ is a bounded operator from $M_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{E}^\beta_{q\lambda\infty}(\mathbb{R}^n)$, we conclude that (iii)⇒(i) holds. This ends the proof of Theorem 1.6. □

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