A THEOREM OF TYPE "PARTITION OF UNITY" WITH APPLICATIONS IN THE WEIGHTED SPACES

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In the first part of the paper, we present a theorem of type "partition of unity", and three of its consequences, i.e., for algebras, linear subspaces and convex cones. In the second part, some theorems of localizability (or density) in a weighted space are presented. We mention that the weighted spaces are classes of continuous scalar functions on a locally compact space (for example, the space of function with compact support, the space of bounded functions, the space of functions vanishing at infinity, the space of functions rapidly decreasing at infinity).

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1. INTRODUCTION

Let X be a Hausdorff locally compact space and let $\mathcal{F}$ be a subset of continuous functions on X with values in the interval [0,1]. The set $\mathcal{F}$ induces on X the following equivalence relation:

$$x \sim_{\mathcal{F}} y \Leftrightarrow f(x) = f(y), \quad \forall f \in \mathcal{F}.$$

For any $x \in X$, we denote by $[x]_{\mathcal{F}}$ the set:

$$[x]_{\mathcal{F}} = \{y \in X;\ f(y) = f(x)\},\ \forall f \in \mathcal{F}.$$

Obviously, $[x]_{\mathcal{F}}$ is a closed subset of X and any element $f \in \mathcal{F}$ is constant on this set. In fact, $[x]_{\mathcal{F}}$ is the maximal set containing x with this property and for any $x, y \in X$ we have $[x]_{\mathcal{F}} = [y]_{\mathcal{F}}$ or $[x]_{\mathcal{F}} \cap [y]_{\mathcal{F}} = \emptyset$. We denote by $X/\sim_{\mathcal{F}}$ the space of all equivalence classes, i.e.:

$$X/\sim_{\mathcal{F}} = \{[x]_{\mathcal{F}};\ x \in X\}.$$

Further, we denote by βX the Stone-Čech compactification of the locally compact space X, i.e., the compact space (uniquely determined up to a topological isomorphism) such that X is a dense subset of βX.

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The topology of X coincides with the trace topology of βX on X and any bounded continuous function on X may be extended to a unique continuous function on βX. If for any $f \in C_b(X)$ (i.e., f is a continuous bounded function on X) we shall denote by βf the continuous extension of f to βX then the map $\beta : C_b(X) \to C(\beta X)$ is an isomorphism between two Banach algebras.

The family $\beta\mathcal{F} = \{\beta f;\ f \in \mathcal{F}\}$ induces on βX the following equivalence relation:

$$u \sim_{\beta\mathcal{F}} v \Leftrightarrow (\beta f)(u) = (\beta f)(v),\ \forall f \in \mathcal{F}.$$

We denote by Z the quotient space $\beta X/\sim_{\beta\mathcal{F}}$ and by $\pi : \beta X \to Z$ the canonical mapping. For any $f \in \mathcal{F}$, the function $\tilde{f} : Z \to K$ is given by $\tilde{f}(z) = (\beta f)(y)$ if $z = \pi(y)$. Clearly, the function $\tilde{f}$ is well definite. If denote by $\tilde{\mathcal{F}} = \{\tilde{f}\}_{f\in\mathcal{F}}$ then $\tilde{\mathcal{F}} \subset C(Z)$.

A generalization of the results of this paper are possible by the generalization of the concept of maximal set of constancy with respect to a subalgebra, a vector subspace or a convex subcone of a weighted space, by the concept of antisymmetric set with respect to the above mathematical structures of a weighted space. This fact assumes a generalization of Lemma de Branges from the spaces of continuous functions on a compact space to a weighted spaces.

2. A GENERAL THEOREM OF TYPE “PARTITION OF UNITY”

THEOREM 2.1. *Let $\mathcal{F}\subset C(X,[0,1])$ with the property that $\tilde{\mathcal{F}}=C(Z,[0,1])$. Suppose in addition that for any $x \in X$ there exists a compact subset $K_x \subset X$ such that $[x]_{\mathcal{F}} \cap K_x = \emptyset$. Then there exists a finite set $\{[x_1]_{\mathcal{F}}, [x_2]_{\mathcal{F}}, \ldots, [x_n]_{\mathcal{F}}\}$ of equivalence classes and there exists a finite number of functions*

$$f_1, f_2, \ldots, f_n \in \mathcal{F}$$

with the properties:

$$f_i\big|_{K_{x_i}} = 0,\ i = \overline{1,n}, \quad \sum_{i=1}^{n} f_i = 1.$$

Proof. We remark that for any $x \in X$ we have $[x]_{\mathcal{F}} = \pi(x) \cap X$ and if $y \in \beta X$ is such that $\pi(y) \cap X \neq \emptyset$ then $\pi(y) \cap X = [x]_{\mathcal{F}}$ for any $x \in \pi(y) \cap X$. Hence π may be seen as an injection from $\{[x]_{\mathcal{F}};\ x \in X\}$ into Z, namely:

$$\pi([x]_{\mathcal{F}}) = \pi(y),\ \forall y \in [x]_{\mathcal{F}}.$$

In fact, we have a bijection between the space $\{[x]_{\mathcal{F}};\ x \in X\}$ and the subspace $\pi(X)$ of Z. Since the set $[x]_{\mathcal{F}} = \pi(y) \cap X$ and the compact subset K_x are disjoint it follows that $\pi(x)$ does not belong to the compact subset $\pi(K_x)$ of $\pi(X)$. This implies that $\bigcap_{x\in X} \pi(K_x) = \emptyset$. Since $\pi(K_x)$ is compact we deduce that there exists a finite subset $\{x_1, \dots, x_n\} \subset X$ such that $\bigcap_{i=1}^{n} \pi(K_{x_i}) = \emptyset$ and therefore the compact space Z is covered by a finite family of open subsets of Z:

$$Z = \bigcup_{i=1}^{n} (Z \setminus \pi(K_{x_i})).$$

We consider now a partition $\sum_{i=1}^{n} g_i$ of the constant function 1, $g_i : Z \to [0,1]$, g_i vanishing outside $Z \setminus \pi(K_{x_i})$, g_i continuous for any $i = \overline{1,n}$. Since $\tilde{\mathcal{F}} = C(Z,[0,1])$ it follows that there exists $f_i \in \mathcal{F}$ such that $g_i = \tilde{f_i}$, $\forall i = \overline{1,n}$. Clearly $\tilde{f_i} \circ \pi = \beta f_i$ and that for any $x \in X$, we have:

$$f_i(x) = (\beta f_i)(x) = (\tilde{f_i} \circ \pi)(x) = g_i[\pi(x)] \geq 0;$$

$$f_i\big|_{K_{x_i}} = g_i\big|_{\pi[K_{x_i}]} = 0 \quad \text{and} \quad \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} g_i[\pi(x)] = 1.$$

□

COROLLARY 2.2. (L. Nachbin [1, Lemma 1]) *Let X be a Hausdorff locally compact space and let $\mathcal{A} \subset C_b(X)$ be a closed subalgebra which contains the constant functions (selfadjoint in the complex case). We suppose in addition that for any $x \in X$ there exists a compact subset $K_x \subset X$ such that $[x]_{\mathcal{A}} \cap K_x = \emptyset$. Then there exists a finite set $\{[x_1]_{\mathcal{A}}, [x_2]_{\mathcal{A}}, \dots, [x_n]_{\mathcal{A}}\}$ of equivalence classes and there exists a finite set of functions $\{a_1, a_2, \dots, a_n\} \subset \mathcal{A}$ with the properties:*

$$a_i \geq 0,\ a_i\big|_{K_{x_i}} = 0,\ i = \overline{1,n},\ \sum_{i=1}^{n} a_i = 1.$$

Proof. First, we remark that if we denote by $\mathcal{A}_1 = \{a \in \mathcal{A}; 0 \leq a \leq 1\}$ then $x \sim_{\mathcal{A}} y$ iff $x \sim_{\mathcal{A}_1} y$, i.e. $a(x) = a(y)$, $\forall a \in \mathcal{A}$ iff $b(x) = b(y)$, $\forall b \in \mathcal{A}_1$. It is sufficient to show that $b(x) = b(y)$, $\forall b \in \mathcal{A}_1$ implies $a(x) = a(y)$, $\forall a \in \mathcal{A}$. Indeed, if $a \in \mathcal{A}_+$ then $b = \dfrac{a}{||a||} \in \mathcal{A}_1$ and so $a(x) = a(y)$. On the other hand, we remark that $\mathcal{A}$ is a lattice since it is a closed subalgebra containing the constant functions.

Therefore for any $a \in \mathcal{A}$, we have $a_+ = \dfrac{|a| + a}{2} \in \mathcal{A}_+$, $a_- = \dfrac{|a| - a}{2} \in A_+$ and hence:

$$a(x) = a_+(x) - a_-(x) = a_+(y) - a_-(y) = a(y).$$

Now the proof of the Corollary 2.2 follows from Theorem 2.1 for $\mathcal{A} \equiv \mathcal{A}_1 = \{a \in \mathcal{A}; 0 \leq a \leq 1\}$. Indeed, $\tilde{\mathcal{A}}$ is a closed subalgebra which contains the constant functions and separates the points of the compact set Z. Therefore, by Stone-Weierstrass theorem, we have $\tilde{\mathcal{A}} = C(Z)$. Obviously, $\tilde{\mathcal{A}}_1 = C(Z, [0,1])$. □

Definition 2.3. We shall say that a subset $\mathcal{M} \subset C(X, [0,1])$ has the property (VN) if:

$$f \cdot g + (1 - f) \cdot h \in \mathcal{M}, \quad \forall f, g, h \in \mathcal{M}.$$

COROLLARY 2.4. *Let X be a Hausdorff locally compact space and let $\mathcal{M} \subset C(X, [0,1])$ be a closed subset with the property (VN) which contains the constant functions $0, 1$, and at least a constant function $0 < c < 1$.*

We suppose in addition that for any $x \in X$ there exists a compact subset $K_x \subset X$ such that $[x]_{\mathcal{M}} \cap K_x = \emptyset$.

Then there exists a finite set $\{[x_1]_{\mathcal{M}}, [x_2]_{\mathcal{M}}, \ldots, [x_n]_{\mathcal{M}}\}$ of equivalence classes and there exists a finite set of functions $m_1, m_2, \ldots, m_n \in \mathcal{M}$ with the properties:

$$m_i\big|_{K_{x_i}} = 0,\ i = \overline{1,n}, \quad \sum_{i=1}^{n} m_i = 1.$$

Proof. The proof follows from Theorem 2.1 for $\mathcal{F} \equiv \mathcal{M}$. Indeed, the set $\tilde{\mathcal{M}}$ is a closed subset of $C(Z, [0,1])$ which separates the points of the compact set $Z = \beta X / \sim_{\beta\mathcal{M}}$ and contains the constant functions $0, 1$, and at least a constant function $0 < c < 1$. From [2, Theorem 4.18] it follows that $\tilde{\mathcal{M}} = C(Z, [0,1])$. □

COROLLARY 2.5. *Let X be a Hausdorff locally compact space and let $\mathcal{C} \subset C_b^+(X)$ be a closed convex cone containing the constant functions $0, 1$, and has the property: for any $u, v \in \beta X$, $\pi(u) \neq \pi(u)$ there is some $\varphi \in C(X, [0,1])$ such that $\varphi \cdot f + (1 - \varphi) \cdot h \in \mathcal{C}$, $\forall f, h \in \mathcal{C}$ and $(\beta\varphi)(u) \neq (\beta\varphi)(v)$. Suppose also that that for any $x \in X$ there exists a compact subset $K_x \subset X$ such that $[x]_{\mathcal{C}} \cap K_x = \emptyset$. Then there exists a finite number of equivalence classes $\{[x_1]_{\mathcal{C}}, [x_2]_{\mathcal{C}}, \ldots, [x_n]_{\mathcal{C}}\}$ and an equal number of functions $h_1, h_2, \ldots, h_n \in \mathcal{C}$ with the properties:*

$$h_i\big|_{K_{x_i}} = 0,\ i = \overline{1,n}, \quad \sum_{i=1}^{n} h_i = 1.$$

Proof. The proof follows from Theorem 2.1 for $\mathcal{F} \equiv \mathcal{C}_1 = \{h \in \mathcal{C};\ h \leq 1\}$.

Indeed, $\tilde{\mathcal{C}}$ is a closed convex cone in $C^+(Z)$ which contains the constant functions $0, 1$, separates the points of the compact set Z and has the property for any $z_1, z_2 \in Z$, $z_1 \neq z_2$, there is a multiplier $\tilde{\varphi} \in C(Z,[0,1])$, i.e. $\tilde{\varphi} \cdot \tilde{f} + (1-\tilde{\varphi}) \cdot \tilde{h} \in \tilde{\mathcal{C}}$, $\forall \tilde{f}, \tilde{h} \in \tilde{\mathcal{C}}$ with the property $\tilde{\varphi}(z_1) = (\beta\varphi)(u) \neq (\beta\varphi)(v) = \tilde{\varphi}(z_2)$ where $z_1 = \pi(u)$, $z_2 = \pi(v)$. From [3, Theorem 2, Corollary 1] it follows that $\tilde{\mathcal{C}} = C^{+}(Z)$ and hence that

$$\tilde{\mathcal{C}}_1 = C(Z,[0,1]).$$

□

3. SOME APPROXIMATION THEOREM IN WEIGHTED SPACES

Definition 3.1. A family $\mathcal{V}$ of upper semicontinuous, non-negative functions on the Hausdorff locally compact space X such that for any $v_1, v_2 \in \mathcal{V}$ and any $\lambda \in \mathbb{R}$, $\lambda > 0$ there exists $w \in \mathcal{V}$ such that:

$$v_i(x) \leq \lambda \cdot w(x),\ \forall x \in X,\ i = 1,2$$

is called Nachbin family. Any element of $\mathcal{V}$ will be called a *weight.*

We shall denote by $C\mathcal{V}_0(X,\mathbb{R})$ or by $C\mathcal{V}_0(X)$ the set of all continuous functions f on X such that the function $f \cdot v$ vanishes at infinity for all $v \in \mathcal{V}$.

Any weight $v \in \mathcal{V}$ generates a seminorm $p_v : C\mathcal{V}_0(X) \to \mathbb{R}_+$ defined by:

$$p_v(f) = \sup\{v(x) \cdot |f(x)|;\ x \in X\},\ \forall f \in C\mathcal{V}_0(X).$$

The locally convex topology defined by this family of seminorms is denoted by $\omega_{\mathcal{V}}$ and it will be called the weighted topology on $C\mathcal{V}_0(X)$. The space $C\mathcal{V}_0(X)$ endowed with the topology $\omega_{\mathcal{V}}$ is called weighted space.

If for any $x \in X$ there exists a weight $v_x \in \mathcal{V}$ such that $v_x(x) > 0$, then $(C\mathcal{V}_0(X), \omega_{\mathcal{V}})$ is a Hausdorff locally convex space. Further, we suppose that $C\mathcal{V}_0(X)$ is Hausdorff.

Further, we mention some particular weighted spaces:

a) If $\mathcal{V} = \{1\}$ then $C\mathcal{V}_0(X) = C_0(X)$-the space of continuous functions vanishing at infinity and the weighted topology $\omega_{\mathcal{V}}$ coincide with the uniform convergence topology.

b) If $\mathcal{V} = C_0^{+}(X)$ then $C\mathcal{V}_0(X) = C_b(X)$ – the space of continuous bounded functions on X and the weighted topology $\omega_{\mathcal{V}}$ coincide with the strict topology β.

c) Let $X = \mathbb{R}^n$, and let $\mathcal{P}_n$ be the set of all polynomials defined on $\mathbb{R}^n$ with values in $\mathbf{K}$. If $\mathcal{V} = \{|p|;\forall p \in \mathcal{P}_n\}$, then $C\mathcal{V}_0(\mathbb{R}^n)$ coincides with the space of functions rapidly decreasing at infinity. It is not difficult to show that $C\mathcal{V}_0(\mathbb{R}^n) = C\mathcal{W}_0(\mathbb{R}^n)$ where:

$$\mathcal{W} = \{w_k;\ w_k : \mathbb{R}^n \to \mathbb{R}_+,\ k \in \mathbb{N}\},\ w_k(x) = (1 + \|x\|)^k.$$

Definition 3.2. A linear subspace $\mathcal{W}$ of $C\mathcal{V}_0(X)$ is called localizable with respect to the family $\mathcal{F}$ of $C_b(X)$ if:

$$\overline{\mathcal{W}} = f \in C\mathcal{V}_0(X); f\big|_{[x]_\mathcal{F}} \in \overline{\mathcal{W}\big|_{[x]_\mathcal{F}}},\ \forall x \in X\}.$$

Remark 3.3. The linear subspace $\mathcal{W} \subset C\mathcal{V}_0(X)$ is dense in $C\mathcal{V}_0(X)$ if the following conditions are satisfied:

(a) $\mathcal{W}$ is localizable with respect to $\mathcal{F}$,

(b) $\mathcal{F}$ separates the points of X,

(c) for any $x \in X$ there exists $w \in \mathcal{W}$ such that $w(x) \neq 0$.

THEOREM 3.4. *Let $\mathcal{F} \subset C(X;[0,1])$ be a subset with the property $\tilde{\mathcal{B}} = C(Z,[0,1])$, where $\mathcal{B} = \overline{\mathcal{F}}$ is the closure of $\mathcal{F}$ in $C(X;[0,1])$. If $\mathcal{W} \subset C\mathcal{V}_0(X)$ is a linear subspace with the property $\mathcal{F} \cdot \mathcal{W} \subset \mathcal{W}$ then $\mathcal{W}$ is localizable with respect to $\mathcal{F}$, i.e.*

$$\overline{\mathcal{W}} = \{f \in C\mathcal{V}_0(X);\ f\big|_{[x]_\mathcal{F}} \in \overline{\mathcal{W}\big|_{[x]_\mathcal{F}}},\ \forall x \in X\}.$$

Proof. Since it is obvious that the set on the left of equality belongs to the set on the right side of this, it is sufficient to prove the inverse inclusion.

Let $g \in C\mathcal{V}_0(X)$ be such that $g\big|_{[x]_\mathcal{F}} \in \overline{\mathcal{W}\big|_{[x]_\mathcal{F}}}$, $\forall x \in X$. We shall prove that $g \in \overline{\mathcal{W}}$.

Let $v \in \mathcal{V}$ and $\varepsilon > 0$ be arbitrary and fixed. Then for any $x \in X$ there exists $w_x \in \mathcal{W}$ such that:

$$v(y)|g(y) - w_x(y)| < \varepsilon,\ \forall y \in [x]_\mathcal{F}.$$

If we denote by $K_x = \{y \in X;\ v(y)|g(y) - w_x(y)| \geq \varepsilon\}$, then K_x is a compact set and:

$$[x]_\mathcal{F} \cap K_x = \emptyset,\ [x]_\mathcal{F} \cup K_x = X.$$

Since by hypothesis $\mathcal{B} = C(Z,[0,1])$ where $\mathcal{B} = \overline{\mathcal{F}}$, from Theorem 2.1 it follows that there exists a finite number $[x_1]_\mathcal{F}, [x_2]_\mathcal{F}, \ldots, [x_n]_\mathcal{F}$ of equivalence classes and there exists also a finite number of functions $b_1, b_2, \ldots, b_n \in \mathcal{B}$ with the properties:

$$b_i\big|_{K_{x_i}} = 0,\ i = \overline{1,n},\ \sum_{i=1}^n b_i = 1.$$

Further, for any $i \in \{1, 2, \ldots, n\}$ we have:

$$b_i(y) \cdot v(y) \cdot |g(y) - w_{x_i}(y)| \leq \varepsilon \cdot b_i(y),\ \forall y \in X. \tag{1}$$

Indeed, if $y \in [x_i]_\mathcal{F}$ then we have $v(y) \cdot |g(y) - w_{x_i}(y)| < \varepsilon$ and if $x \notin [x_i]_\mathcal{F}$ then $b_i(y) = 0$. From (1) it follows:

$$v(y) \cdot \sum_{i=1}^n b_i(y) \cdot |g(y) - w_{x_i}(y)| < \varepsilon \cdot \sum_{i=1}^n b_i(y) = \varepsilon,\ \forall y \in X.$$

Further, we have:

$$v(y)\cdot\left|g(y)-\sum_{i=1}^{n}b_i(y)\cdot w_{x_i}(y)\right| = v(y)\cdot\left|\sum_{i=1}^{n}b_i(y)\cdot[g(y)-w_{x_i}(y)]\right| \leq$$

$$\leq v(y)\cdot\sum_{i=1}^{n}b_i(y)\cdot|g(y)-w_{x_i}(y)| \leq \varepsilon,\ \forall x\in X.$$

Since $b_1, b_2, \ldots, b_n \in \overline{\mathcal{F}}$, it follows that for any $i \in \{1, \ldots, n\}$ and any $\delta > 0$ there is $f_i \in \mathcal{F}$, such that:

$$|b_i(y)-f_i(y)| < \delta,\ \forall y\in X.$$

Since the functions $v \cdot w_{x_i}$ vanish at infinity it follows that these are bounded on X and therefore there exist:

$$\alpha_i = \sup\{v(y)\cdot|w_{x_i}|;\ y\in X\},\ i\in\overline{1,n}.$$

Further we have:

$$v(y)\cdot\left|\sum_{i=1}^{n}f_i(y)\cdot w_{x_i}(y)-g(y)\right| \leq v(y)\cdot\left|\sum_{i=1}^{n}f_i(y)\cdot w_{x_i}(y)-\sum_{i=1}^{n}b_i(y)\cdot w_{x_i}(y)\right|+$$

$$+v(y)\cdot\left|\sum_{i=1}^{n}b_i(y)\cdot w_{x_i}(y)-g(y)\right| \leq$$

$$\leq \sum_{i=1}^{n}|f_i(y)-b_i(y)|\cdot v(y)\cdot|w_{x_i}(y)]+\varepsilon \leq \delta\cdot\sum_{i=1}^{n}\alpha_i+\varepsilon.$$

If we suppose that:

$$\delta < \frac{\varepsilon}{\sum_{i=1}^{n}\alpha_i},$$

then:

$$v(y)\cdot\left|\sum_{i=1}^{n}f_i(y)\cdot w_{x_i}(y)-g(y)\right| \leq 2\cdot\varepsilon,\ \forall y\in X.$$

Finally, if we denote by $w = \sum_{i=1}^{n} f_i \cdot w_{x_i}$, then $w \in \mathcal{F}\cdot\mathcal{W} \subset \mathcal{W}$ and so the proof is finished. □

COROLLARY 3.5. *If we suppose in addition that $\mathcal{F}$ separates the points of X and that for any $x \in X$ there exists $w \in \mathcal{W}$ such that $w(x) \neq 0$, then $\mathcal{W}$ is dense in $C\mathcal{V}_0(X)$, i.e.*

$$\overline{\mathcal{W}} = C\mathcal{V}_0(X).$$

The proof follows from Theorem 3.4 and Remark 3.3.

Theorem 3.6 (Nachbin). *Let $\mathcal{A}$ be a subalgebra of $C_b(X)$ containing the constant function 1, self-adjoint in the complex case, and let $\mathcal{W} \subset C\mathcal{V}_0(X)$ be a linear subspace such that $\mathcal{A} \cdot \mathcal{W} \subset \mathcal{W}$. Then $\mathcal{W}$ is localizable with respect to $\mathcal{A}$, i.e.*

$$\overline{\mathcal{W}} = \{f \in C\mathcal{V}_0(X); f\big|_{[x]_{\mathcal{A}}} \in \overline{\mathcal{W}\big|_{[x]_{\mathcal{A}}}}, \ \forall x \in X\}.$$

The proof follows from Theorem 3.4 and Corollary 2.2 for $\mathcal{F} = \mathcal{A}_1 = \{a \in \mathcal{A}; 0 \leq a \leq 1\}$.

Corollary 3.7. *If we suppose in addition that $\mathcal{A}$ separates the points of X and that for any $x \in X$ there exists a $w \in \mathcal{W}$ such that $w(x) \neq 0$, then $\mathcal{W}$ is dense in $C\mathcal{V}_0(X)$, i.e.*

$$\overline{\mathcal{W}} = C\mathcal{V}_0(X).$$

Theorem 3.8. *Let $\mathcal{M} \subset C(X; [0,1])$ be a subset with (VN) property which contains the constant functions $0, 1$ and at least a constant function $0 < c < 1$. If $\mathcal{W} \subset C\mathcal{V}_0(X)$ is a linear subspace with the property $\mathcal{M} \cdot \mathcal{W} \subset \mathcal{W}$, then $\mathcal{W}$ is localizable with respect to $\mathcal{M}$, i.e.*

$$\overline{\mathcal{W}} = \{f \in C\mathcal{V}_0(X); f\big|_{[x]_{\mathcal{M}}} \in \overline{\mathcal{W}\big|_{[x]_{\mathcal{M}}}}, \ \forall x \in X\}.$$

The proof follows from Theorem 3.4 and Corollary 2.4 for $\mathcal{F} = \mathcal{M}$.

Corollary 3.9. *If we suppose in addition that $\mathcal{M}$ separates the points of X and that for any $x \in X$ there exists a $w \in \mathcal{W}$ such that $w(x) \neq 0$, then $\mathcal{W}$ is dense in $C\mathcal{V}_0(X)$, i.e.*

$$\overline{\mathcal{W}} = C\mathcal{V}_0(X).$$

Theorem 3.10. *Let X be a Hausdorff locally compact space and let $\mathcal{C} \subset C_b^+(X)$ be a convex cone containing the constant functions $0, 1$ and has the property for any $u, v \in \beta X$, $\pi(u) \neq \pi(u)$ there is a multiplier $\varphi \in C(X, [0,1])$ (i.e. $\varphi \cdot f + (1 - \varphi) \cdot h \in \mathcal{C}$, $\forall f, h \in \mathcal{C}$) and $(\beta\varphi)(u) \neq (\beta\varphi)(v)$ and let $\mathcal{W} \subset C\mathcal{V}_0(X)$ be a linear subspace such that $\mathcal{C} \cdot \mathcal{W} \subset \mathcal{W}$. Then $\mathcal{W}$ is localizable with respect to $\mathcal{C}$, i.e.*

$$\overline{\mathcal{W}} = \{f \in C\mathcal{V}_0(X); f\big|_{[x]_{\mathcal{C}}} \in \overline{\mathcal{W}\big|_{[x]_{\mathcal{C}}}}, \ \forall x \in X\}.$$

The proof follows from Theorem 3.4 and Corollary 2.5 for $\mathcal{F} = \mathcal{C}_1 = \{h \in \mathcal{C};\ h \leq 1\}$.

Corollary 3.11. *Let $\mathcal{C} \subset C_b^+(X)$ and $\mathcal{W} \subset C\mathcal{V}_0(X)$ be as in Theorem* 3.10*. If we suppose in addition that $\mathcal{C}$ separates the points of X and for any $x \in X$ there exists $w \in \mathcal{W}$ such that $w(x) \neq 0$, then $\mathcal{W}$ is dense in $C\mathcal{V}_0(X)$, i.e.*

$$\overline{\mathcal{W}} = C\mathcal{V}_0(X).$$

The assertion follows from Theorem 3.10 and Remark 3.3.

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