THE INFLUENCE OF SOME SUBGROUPS OF PRIME POWER ORDERS ON THE STRUCTURE OF A FINITE GROUP

M. M. AL-SHOMRANI and M. EZZAT MOHAMED

Communicated by Sorin Dăscălescu

Let G be a finite group. A subgroup of G is called S-quasinormal in G if it permutes with each Sylow subgroup of G. In this paper, we investigate the structure of the finite group G when certain abelian subgroups of largest possible exponent of prime power orders are S-quasinormal in G.

AMS 2020 Subject Classification: 20D10, 20D20.

Key words: supersolvable group, ALPE-subgroup, S-quasinormal subgroup, generalized hypercenter subgroup, generalized supersolvably embedded subgroup.

1. INTRODUCTION

All groups considered in this paper will be finite.

A subgroup of a group G which permutes with each subgroup of G is called a quasinormal subgroup of G. We say, following Kegel [13], that a subgroup of G is S-quasinormal in G if it permutes with each Sylow subgroup of G.

In 2017, Ezzat et al. [11], introduced the following definition: let $\mathfrak{F}$ be a class of groups. A subgroup H of a group G is $\mathfrak{F}_{hq}$-supplemented in G if G has a quasinormal subgroup N such that HN is a Hall subgroup of G and $(H \cap N)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$, where H_G is the core of H in G and $Z_{\mathfrak{F}}(G/H_G)$ is the $\mathfrak{F}$-hypercenter of G/H_G. A 2-group is called quaternion-free if it has no section isomorphic to the quaternion group of order 8. If P is a p-group, we denote $\Omega(P) = \Omega_1(P)$ if $p > 2$ and $\Omega(P) = \langle \Omega_1(P), \Omega_2(P) \rangle$ if $p = 2$, where $\Omega_i(P) = \langle x \in P \mid |x| = p^i \rangle$. We define $D(G) = \cap\{H \mid H \lhd G$ and G/H is nilpotent$\}$ and call it the nilpotent residual of G.

Many authors have examined the structure of a finite group G under the assumption that certain subgroups of G of prime power orders are well-situated in G.

Itô [14] proved that a group G of odd order is nilpotent provided that each subgroup of G of prime order lies in the center of G. A sharpened form

MATH. REPORTS **25(75)** (2023), *3*, 455–464
doi: 10.59277/mrar.2023.25.75.3.455

of Itô's result is the following statement ([12], p. 435): if, for an odd prime p, each subgroup of G of order p lies in the center of G, then G is p-nilpotent and that if each subgroup of G of order 2 and 4 lie in the center of G, then G is 2-nilpotent.

Buckley [7], proved that a group G of odd order is supersolvable if each subgroup of G of prime order is normal in G.

Asaad et al. [6] proved that: let P be a Sylow p-subgroup of a finite group G and put $\exp \Omega(P) = p^{e_p}(e_p \geq 1)$. If each member of the family $\{H \mid H \leq \Omega(P), H' = 1, \exp H = p^{e_p}$, p ranging through each prime dividing the order of $G\}$ is normal in G, then G is supersolvable. Also, Asaad et al. [5] continued to study the influence of abelian subgroups of largest possible exponent of prime power order (they call such subgroups ALPE-subgroups) on the structure of G. Shomrani and Ezzat [2] proved that: let G be a group. If, for each Sylow subgroup P of G, the ALPE-subgroups of $\Omega(G' \cap P)$ are quasinormal in G, then G is supersolvable. Ezzat et al. [11] proved that if the maximal subgroups of the Sylow subgroups of a group G are $\mathfrak{U}_{hq}$-supplemented in G, then $G \in \mathfrak{U}$, where $\mathfrak{U}$ is the class of supersolvable groups. Recently, Ezzat et al. [10] proved that: if the cyclic subgroups of prime order or order 4 of a group G are $\mathfrak{U}_{hq}$-supplemented in G, then $G \in \mathfrak{U}$.

In the present paper, we prove the following three theorems:

Theorem 1. *Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G. Fix an ALPE-subgroup $A(P)$ of $\Omega_1(D(G) \cap P)$ having maximal order. If $p = 2$, suppose P is quaternion-free. Then the following statements are equivalent:*

(a) *G is p-nilpotent.*

(b) *the ALPE-subgroups of $A(P)$ are S-quasinormal in G.*

(c) *$\Omega_1(D(G) \cap P) \leq Z(P)$ and the ALPE-subgroups of $A(P)$ are S-quasinormal in $N_G(P)$.*

The argument which established Theorem 1 can easily be adapted to yield the following three corollaries.

Corollary 1. *Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G. Fix an ALPE-subgroup $A(P)$ of $\Omega(D(G) \cap P)$ having maximal order. If the ALPE-subgroups of $A(P)$ are S-quasinormal in G. Then G is p-nilpotent.*

Corollary 2 (Shomrani and Ezzat Mohamed [2; Lemma 3.1]). *Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G. If the ALPE-subgroups of $\Omega(G' \cap P)$ are quasinormal in G. Then G is p-nilpotent.*

Corollary 3. *Let P be a Sylow 2-subgroup of a group G. If P is quaternion-free and $\Omega_1(P) \leq Z(G)$. Then G is p-nilpotent.*

The following examples show that the quaternion-free and the condition involvnig $Z(P)$ hypothesis are necessary in Theorem 1.

Example 1. Take $G = SL(2,3)$. Then the Sylow 2-subgroup P of G is the quaternion group of order 8 and $D(G) = P$ and $\Omega_1(D(G) \cap P)$ is a cyclic subgroup of order 2 lies in $Z(P)$. Clearly, G is not 2-nilpotent.

Example 2. Take $G = S_4$, the symmetric group of degree four. Then the Sylow 2-subgroup P of G is the dihedral group $\langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ of order 8, and $D(G) = A_4$ is the alternating group of degree four. Hence $N_G(P) = P$ and $\Omega_1(D(G) \cap P)$ is not contained in $Z(P)$. Clearly, G is not 2-nilpotent.

Theorem 2. *Let G be a quaternion-free group. For each Sylow subgroup P of G, fix an ALPE-subgroup $A(P)$ of $\Omega_1(D(G) \cap P)$ having maximal order. Then G is supersolvable if one of the following conditions holds:*

(a) *the ALPE-subgroups of $A(P)$ are S-quasinormal in G.*

(b) *$\Omega_1(D(G) \cap P) \leq Z(P)$ and the ALPE-subgroups of $A(P)$ are S-quasinormal in $N_G(P)$.*

Also, the argument which established Theorem 2 can easily be adapted to yield the following three corollaries.

Corollary 4. *Let G be a group. For each Sylow subgroup P of G, fix an ALPE-subgroup $A(P)$ of $\Omega(D(G) \cap P)$ having maximal order. If the ALPE-subgroups of $A(P)$ are S-quasinormal in G, then G is supersolvable.*

Corollary 5 (Shomrani and Ezzat Mohamed [2; Theorem 3.3]). *Let G be a group. If, for each Sylow subgroup P of G, the ALPE-subgroups of $\Omega(G' \cap P)$ are quasinormal in G, then G is supersolvable.*

Corollary 6 (Buckley [7]). *Assume that G is a group of odd order and that each subgroup of G of prime order is normal in G. Then G is supersolvable.*

Theorem 2 is not true if we omit the condition $\Omega_1(D(G) \cap P) \leq Z(P)$, the symmetric group of degree four S_4 is a counterexample. The following example shows that the quaternion-free hypothesis is necessary in Theorem 2.

Example 3. Let Q be the quaternion group

$$< a, b | a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} >,$$

C_9 be a cyclic group of order 9 with generator c and the action of C_9 on Q is given by $a^c = b$, $b^c = ab$. Let the group G be the semi-direct product of Q by

C_9. Then G is a group of even order in which each subgroup of prime order is normal in G but G is not supersolvable (see Buckley, [7, Examples (iii)]).

THEOREM 3. *Let K be a normal subgroup of a quaternion-free group G such that G/K is supersolvable. For each Sylow subgroup P of K, fix an ALPE-subgroup $A(P)$ of $\Omega_1(D(G) \cap P)$ having maximal order. Then G is supersolvable if one of the following conditions holds:*

(a) *the ALPE-subgroups of $A(P)$ are S-quasinormal in G.*

(b) *$\Omega_1(D(G) \cap P) \leq Z(P)$ and the ALPE-subgroups of $A(P)$ are S-quasinormal in $N_G(P)$.*

The argument which established Theorem 3 can easily be adapted to yield the following two corollaries.

COROLLARY 7. *Let K be a normal subgroup of a group G such that G/K is supersolvable. For each Sylow subgroup P of K, fix an ALPE-subgroup $A(P)$ of $\Omega(D(G) \cap P)$ having maximal order. If the ALPE-subgroups of $A(P)$ are S-quasinormal in G, then G is supersolvable.*

COROLLARY 8 (Shomrani and Ezzat Mohamed [2; Theorem 3.4]). *Let K be a normal subgroup of a group G such that G/K is supersolvable. If, for each Sylow subgroup P of K, the ALPE-subgroups of $\Omega(G' \cap P)$ are quasinormal in G, then G is supersolvable.*

2. PRELIMINARIES

In this section, we collect some definitions and results that are needed in the sequel.

Recall here that the quasicenter $Q(G)$ of a group G is the subgroup generated by all elements x of G such that $< x >$ is quasinormal in G. The hyperquasicenter $Q_\infty(G)$ is the largest term of the series

$$1 = Q_0(G) \leq Q_1(G) = Q(G) \leq Q_2(G) \leq \dots ,$$

where $Q_{i+1}(G)/Q_i(G) = Q(G/Q_i(G))$ for all $i > 0$.

We say that a normal subgroup H of G is supersolvably embedded in G, if each chief factor of G which lies in H has prime order. It is easily verified that if H, K are normal subgroups of G with both H, K supersolvably embedded in G, then HK is supersolvably embedded in G. From this, a group G has a unique maximal supersolvably embedded subgroup.

The hyperquasicenter $Q_\infty(G)$ is the largest supersolvably embedded subgroup of G (see Weinstein [15; p. 33]).

The generalized center $genz(G)$ of G is the subgroup generated by all elements x of G such that $< x >$ is S- quasinormal in G. The generalized hypercenter $genz_\infty(G)$ is the largest term of the series

$$1 = genz_0(G) \leq genz_1(G) = genz(G) \leq genz_2(G) \leq \dots ,$$

where $genz_{i+1}(G)/genz_i(G) = genz(G/genz_i(G))$ for all $i > 0$. It is known that $Q_\infty(G) \leq genz_\infty(G)$ (see Weinstein, [15; pp. 33–34]). Asaad and Ezzat Mohamed [4] gave a new characterization of $genz_\infty(G)$ by introducing the following definition: A normal subgroup H of a group G is a generalized supersolvably embedded (GSE) in G if there exists a series

$$1 = H_0 \leq H_1 \leq H_2 \leq \dots ,$$

such that H_i is S-quasinormal in G and $|H_{i+1} : H_i|$ is a prime, where $0 \leq i \leq n-1$. It is easily verified that if H and K are normal GSE subgroups of G, then HK is GSE in G. From this, a group G has a unique maximal generalized supersolvably embedded subgroup of G and it is denoted by $GSE(G)$. The generalized hypercenter $genz_\infty(G)$ is the maximal generalized supersolvably embedded $GSE(G)$ (see Asaad and Ezzat Mohamed, [4; p. 2243]).

LEMMA 1. *Let P be a normal p-subgroup of a group G. Fix an ALPE-subgroup $A(P)$ of P having maximal order.*

(a) *If the ALPE-subgroups of $A(P)$ are normal in G, then P is supersolvably embedded in G, i.e., $P \leq Q_\infty(G)$.*

(b) *If the ALPE-subgroups of $A(P)$ are S-quasinormal in G, then P is GSE in G, i.e., $P \leq genz_\infty(G)$.*

Proof. See [5; Theorem 3.1 and Theorem 3.2]. □

LEMMA 2. *Let P be a Sylow p-subgroup of G. If $p = 2$, suppose P is quaternion-free. Then the following statements are equivalent:*

(a) *G is p-nilpotent.*

(b) *$\Omega_1(D(G) \cap P) \leq Z(N_G(P))$.*

(c) *$N_G(P)$ is p-nilpotent and $\Omega_1(D(G) \cap P \cap P^x) \leq Z(P)$, for all $x \in G \backslash N_G(P)$.*

Proof. See [3; Theorem 1]. □

LEMMA 3. (a) *If $H \leq K \leq G$ and H is S-quasinormal in G, then H is S-quasinormal in K.*

(b) *If H is S-quasinormal in G, then H is subnormal in G.*

(c) *Let H be a p-subgroup for some prime p. If H is S-quasinormal in G, then $O^p(G) \leq N_G(H)$, where*

$$O^p(G) = \langle Q \mid Q \text{ is a Sylow } q\text{-subgroup of } G, \text{with } q \neq p \rangle .$$

(d) *Let H be a normal subgroup of G and K an S-quasinormal subgroup of G. Then KH is an S-quasinormal subgroup of G and KH/H is an S-quasinormal subgroup of G/H.*

Proof. (a), (b): See [13].

(c) Let Q be any Sylow q-subgroup of G, with $q \neq p$.

Since H is S-quasinormal in G, it follows that HQ is a subgroup of G. By (a) and (b), H is subnormal in HQ, and since H is a p-subgroup of G, it follows that H is normal in HQ for each Sylow q-subgroup Q of G, with $q \neq p$. Hence $O^p(G) \leq N_G(H)$.

(d) Is clear. □

Lemma 4. *Let $H \lhd G$ such that H is GSE in G. If $H \leq K \leq G$, then H is GSE in K.*

Proof. Since H is GSE in G, it follows that there is a series

$$1 = H_0 < H_1 < ... < H_n = H$$

such that H_i is S-quasinormal in G and $|H_{i+1} : H_i| = prime$ for all $0 \leq i \leq n-1$. By Lemma 3, H_i is S-quasinormal in K for all $0 \leq i \leq n-1$. Hence H is GSE in K. □

Lemma 5. *If K is a supersolvable subgroup of G, then $genz_\infty(G)K$ is supersolvable.*

Proof. See [1; Theorem 2.9]. □

Lemma 6. *Let G be a group. For each Sylow subgroup P of G, fix an ALPE-subgroup $A(P)$ of $\Omega(P)$ having maximal order. Write*

$$A = \langle A(P) \mid P \text{ is a Sylow subgroup of } G \rangle$$

Then G is supersolvable if and only if $A \leq Q_\infty(G)$.

Proof. See [9; Corollary 3.3]. □

Lemma 7. (a) *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If $\Omega(P) \leq genz_\infty(G)$, then G is p-nilpotent.*

(b) *Let P be a normal p-subgroup of G such that G/P is supersolvable. If $\Omega(P) \leq genz_\infty(G)$, then G is supersolvable.*

Proof. (a) see [4; Lemma 3.8] and (b) see [4; Theorem 3.11]. □

3. PROOFS

Proof of Theorem 1. If G is p-nilpotent, then $D(G) \leq O_{p'}(G)$ and $D(G) \cap P = 1$. So (a) implies (b) and (c).

(b) $\Longrightarrow$ (a). Assume that the result is false and let G be a counterexample of minimal order. Let H be any ALPE-subgroup of $A(P)$. By hypothesis, H is an S-quasinormal subgroup of G. Then $HQ \leq G$, for each Sylow q-subgroup Q of G with $q \neq p$. Hence H is a normal subgroup of HQ. Since $A(P)$ is abelian, it follows that $H \lhd A(P)Q$. Then by Lemma 1(a), $A(P)$ is supersolvably embedded in $A(P)Q$ and so $A(P)Q$ is supersolvable. Hence $A(P)Q = A(P) \times Q$. Thus $O^p(G) \leq C_G(A(P))$. Since $G/O^p(G)$ is a p-group, it follows that $D(G) \leq O^p(G)$. Then $A(P) \leq Z(D(G))$ and so $A(P) = \Omega_1(D(G) \cap P)$ by the maximality of $A(P)$. Then by Lemma 2, $D(G)$ is p-nilpotent. Hence $D(G) = (D(G) \cap P)K$, where K is a normal p'-Hall subgroup of $D(G)$. Since K char $D(G)$ and $D(G) \lhd G$, it follows that K is a normal subgroup of G. If $K = 1$, then $D(G) \leq P$ and so $P \lhd G$. By Schur-Zassenhaus theorem, G possesses a p'-Hall subgroup N such that $G/P \cong N$. Clearly, N is nilpotent and $\Omega_1(D(G))N \leq G$. Since the ALPE-subgroups of $A(P) = \Omega_1(D(G))$ are S-quasinormal in G, it follows by Lemma 1(b), that $\Omega_1(D(G))$ is GSE in G. Then by Lemma 4, $\Omega_1(D(G))$ is GSE in $\Omega_1(D(G))N$, i.e., $\Omega_1(D(G)) \leq genz_{\infty}(\Omega_1(D(G)N)$. By Lemma 5, $\Omega_1(D(G))N$ is supersolvable and so $\Omega_1(D(G))N$ is p-nilpotent. Hence $N \leq C_G(\Omega_1(D(G)))$ and since $\Omega_1(D(G)) \leq Z(D(G))$, it follows that $\Omega_1(D(G)) \leq Z(D(G)N)$. Then by Lemma 2, $D(G)N$ is p-nilpotent. Hence N char $D(G)N$ and since $D(G)N \lhd G$, it follows that $N \lhd G$, i.e., G is p-nilpotent; a contradiction. Thus we may assume that $K \neq 1$. Clearly, $D(G/K) = D(G)/K$. By hypothesis and Lemma 3(d), our hypothesis carries over to G/K. Then G/K is p-nilpotent, by the minimality of G. Hence G is p-nilpotent; a final contradiction.

(c) $\Longrightarrow$ (a). Since $\Omega_1(D(G) \cap P) \leq Z(P)$, it follows that $\Omega_1(D(G) \cap P)$ is elementary abelian. So $A(P) = \Omega_1(D(G) \cap P)$ and since the ALPE-subgroups of $A(P)$ are S-quasinormal in $N_G(P)$, it follows that every subgroup of $\Omega_1(D(G) \cap P)$ is S-quasinormal in $N_G(P)$. Clearly, $D(N_G(P)) \leq D(G)$. So $\Omega_1(D(N_G(P)) \cap P) \leq \Omega_1(D(G) \cap P)$. Then every subgroup of $\Omega_1(D(N_G(P) \cap P)$ is S-quasinormal in $N_G(P)$. Hence $N_G(P)$ is p-nilpotent, by the statement (b) $\Longrightarrow$ (a). Therefore G is p-nilpotent, by Lemma 2. □

Proof of Theorem 2. (a) Suppose that the ALPE-subgroups of $A(P)$ are S-quasinormal in G. Theorem 1 implies that G is r-nilpotent, where r is the smallest prime dividing the order of G. Then $G = RK$, where R is a Sylow r-subgroup of G and K is a normal r'-Hall subgroup of G. Since $G/K \cong R$

is nilpotent, it follows that $D(G) \leq K$. Hence $D(G)$ is a group of odd order. Let H be any ALPE-subgroup of $A(P)$. By hypothesis H is S-quasinormal in G. Then by Lemma 3(c), $O^p(G) \leq N_G(H)$, where $A(P)$ is a p-group. Since $D(G) \leq O^p(G)$, it follows that $H \lhd D(G)$. So the ALPE-subgroups of $A(P)$ are normal in $D(G)$. Then by Lemma 1(a), $A(P) \leq Q_\infty(D(G))$, for each Sylow subgroup P of G. Hence by Lemma 6, $D(G)$ is supersolvable. Thus $D(G) \cap P \lhd D(G)$, where $D(G) \cap P$ is a Sylow p-subgroup of $D(G)$ and p is the largest prime dividing the order of $D(G)$. Since $D(G) \cap P$ char $D(G)$ and $D(G) \lhd G$, it follows that $D(G) \cap P \lhd G$. Clearly, $D(G/D(G) \cap P) = D(G)/D(G) \cap P$. By hypothesis and Lemma 3(d), our hypothesis carries over to $G/D(G) \cap P$. Then $G/D(G) \cap P$ is supersolvable, by the induction on the order of G. Since $\Omega_1(D(G) \cap P)$ char $D(G) \cap P$ and $D(G) \cap P \lhd G$, it follows that $\Omega_1(D(G) \cap P) \lhd G$. By hypothesis and Lemma 1(b), $\Omega_1(D(G) \cap P) \leq genz_\infty(G)$. Hence G is supersolvable, by Lemma 7(b).

(b) Suppose that $\Omega_1(D(G) \cap P) \leq Z(P)$ and the ALPE-subgroups of $A(P)$ are S-quasinormal in $N_G(P)$. Theorem 1 implies that G is r-nilpotent, where r is the smallest prime dividing the order of G. Then $G = RK$, where R is a Sylow r-subgroup of G and K is a normal r'-Hall subgroup of G. Since $G/K \cong R$ is nilpotent, it follows that $D(G) \leq K$. Hence $D(G)$ is a group of odd order. Since $\Omega_1(D(G) \cap P) \leq Z(P)$, it follows that $\Omega_1(D(G) \cap P)$ is elementary abelian and so $A(P) = \Omega_1(D(G) \cap P)$. Then each subgroup of $\Omega_1(D(G) \cap P)$ is S-quasinormal in $N_G(P)$, for each Sylow subgroup P of G. Clearly, $D(K) \leq D(G)$. Then by Lemma 3(a), each subgroup of $\Omega_1(D(K) \cap P)$ is S-quasinormal in $N_G(P) \cap K = N_K(P)$, for each Sylow subgroup P of K. Hence K is supersolvable by the induction on the order of G. Then Q char K, where Q is a Sylow q-subgroup of K and q is the largest prime dividing the order of K, and since $K \lhd G$, it follows that $Q \lhd G$. Now consider the factor group G/Q. Put $D(G/Q) = L/Q$. Since $G/L \cong (G/Q)/(L/Q)$ is nilpotent, it follows that $D(G)Q \leq L$ and since $(G/Q)/(D(G)Q/Q) \cong G/D(G)Q$ is nilpotent, it follows that $L \leq D(G)Q$. Hence $L = D(G)Q$, i.e., $D(G/Q) = D(G)Q/Q$. For each Sylow subgroup P of G, if $(|P|, q) = 1$, then $|P \cap Q| = 1$ and $|D(G)P \cap Q| = |D(G) \cap Q|$. So

$$|(D(G) \cap Q)(P \cap Q)| = \frac{|D(G) \cap Q|\,|P \cap Q|}{|D(G) \cap Q \cap P|} = |D(G) \cap Q| = |D(G)P \cap Q|.$$

Also, if $(|P|, q) = q$, then $P = Q$ as $Q \lhd G$ and so

$$|(D(G) \cap Q)(P \cap Q)| = |Q| = |D(G)P \cap Q|.$$

Hence, for each Sylow subgroup P of G, $|(D(G) \cap Q)(P \cap Q)| = |D(G)P \cap Q|$ and since $(D(G) \cap Q)(P \cap Q) \leq D(G)P \cap Q$, it follows that

$$(D(G) \cap Q)(P \cap Q) = D(G)P \cap Q.$$

Then by [8; Lemma 1.2, p. 2], $D(G)Q \cap PQ = (D(G) \cap P)Q$. Hence

$$\Omega_1(D(G/Q) \cap PQ/Q) = \Omega_1(D(G)Q/Q \cap PQ/Q) = \Omega_1((D(G)Q \cap PQ)/Q)$$
$$= \Omega_1((D(G) \cap P)Q/Q) = \Omega_1(D(G) \cap P)Q/Q \leq Z(P)Q/Q \leq Z(PQ/Q).$$

Clearly, $A(P)Q/Q$ is an ALPE-subgroup of $\Omega_1(D(G/Q) \cap PQ/Q)$ of maximal order. By hypothesis and Lemma 3(d), the ALPE-subgroups of $A(P)Q/Q$ are S-quasinormal in $N_G(P)Q/Q = N_{G/Q}(PQ/Q)$. Then our hypothesis carries over to G/Q. Hence G/Q is supersolvable by the induction on the order of G. So $G/D(G) \cap Q$ is supersolvable.

Since each subgroup H of $A(Q) = \Omega_1(D(G) \cap Q)$ is S-quasinormal in $N_G(Q) = G$, it follows by Lemma 3(c), that $O^q(G) \leq N_G(H)$ and since $\Omega_1(D(G) \cap Q) \leq Z(Q)$, it follows that $H \lhd G$. Then by Lemma 1(a), $\Omega_1(D(G) \cap Q) \leq Q_\infty(G)$ and so $\Omega_1(D(G) \cap Q) \leq genz_\infty(G)$. Hence G is supersolvable, by Lemma 7(b). □

Proof of Theorem 3. By Theorem 2, K is supersolvable. Then P char K, where P is a Sylow p-subgroup of K and p is the largest prime dividing the order of K, and since $K \lhd G$, it follows that $P \lhd G$. Since G/K is supersolvable, it follows that $G/D(G) \cap K$ is supersolvable. Then

$$(G/D(G) \cap P)/(D(G) \cap K/D(G) \cap P) \cong G/D(G) \cap K$$

is supersolvable. By hypothesis and Lemma 3(d), our hypothesis carries over to $G/D(G) \cap P$. Then $G/D(G) \cap P$ is supersolvable by the induction on the order of G. Since $\Omega_1(D(G) \cap P)$ char $D(G) \cap P$, it follows that $\Omega_1(D(G) \cap P) \lhd G$. By the hypothesis (a) and (b), the ALPE-subgroups of $A(P)$ are S-quasinormal in $G = N_G(P)$. Then by Lemma 1(b), $\Omega_1(D(G) \cap P) \leq genz_\infty(G)$. If $p > 2$, then G is supersolvable, by Lemma 7(b). Thus, we may assume that $p = 2$. So $K = P$ is a 2-subgroup of G. Since G/P is supersolvable, it follows that G/P is 2-nilpotent. Then G/P possesses a normal $2'$-Hall subgroup LP/P, where L is a $2'$-Hall subgroup of G and so $LP \lhd G$. Since $\Omega_1(D(G) \cap P) \lhd G$, it follows that $\Omega_1(D(G) \cap P)L \leq G$. Since $\Omega_1(D(G) \cap P) \leq genz_\infty(G)$, it follows by Lemma 4, that $\Omega_1(D(G) \cap P) \leq genz_\infty(\Omega_1(D(G) \cap P)L)$. Then by Lemma 7(a), $\Omega_1(D(G) \cap P)L = \Omega_1(D(G) \cap P) \times L$. Then $L \leq C_G(\Omega_1(D(G) \cap P))$. So each subgroup of $\Omega_1(D(G) \cap P)$ is S-quasinormal in LP.

Since $\Omega_1(D(LP) \cap P) \leq \Omega_1(D(G) \cap P)$ and each subgroup of $\Omega_1(D(G) \cap P)$ is S-quasinormal in LP, it follows by Theorem 1, that LP is 2-nilpotent. Then L char LP and since $LP \lhd G$, it follows that $L \lhd G$. Hence $D(G) \leq L$. Therefore $G \cong G/D(G) \cap P$ is supersolvable. □

Acknowledgments. This project was funded by the Deanship of Scientific Research (DSR), King Abdualaziz University, Jeddah, under grant No. (D- 207-130-1441). The authors, therefore, gratefully acknowledge DSR technical and financial support.

REFERENCES

[1] R.K. Agrawal, *Generalized center and hypercenter of a finite group.* Proc. Amer. Math. Soc. **54** (1976), 13-21.

[2] M.M. AL-Shomrani and M. Ezzat Mohamed, *Finite groups in which some abelian subgroups are quasinormal.* JP J. Algebra, Number Theory Appl. **19** (2010), *2*, 175-183.

[3] M. Asaad, *On p-nilpotence of finite groups.* J. Algebra **277** (2004), 157-164.

[4] M. Asaad and M. Ezzat Mohamed, *On generalized hypercenter of a finite group.* Comm. Algebra **29** (2001), *5*, 2239-2248.

[5] M. Asaad, A.A. Heliel, and M. Ezzat Mohamed, *Finite groups with some subgroups of prime power order S-quasinormally embedded.* Comm. Algebra **32** (2004), *5*, 2019-2027.

[6] M. Asaad, M. Ramadan, and M. Ezzat Mohamed, *Finite groups with normal subgroups of prime power orders.* Pure Math. Appl. (PU.M.A.), **5** (1994), *3*, 251-256.

[7] J. Buckley, *Finite groups whose minimal subgroups are normal.* Math. Z. **116** (1970), 15-17.

[8] K. Doerk and T. Hawkes, *Finite Soluble Groups.* De Gruyter Expositions in Mathematics **4**, Walter De Gruyter, Berlin-New York, 1992.

[9] M. Ezzat Mohamed, *Finite groups whose $\mathfrak{F}$-hypercenter contains certain subgroups of prime power order.* Algebra Colloq. **16** (2009), *4*, 587-592.

[10] M. Ezzat Mohamed, M.I. Elashiry, and M.M. Al-Shomrani, *The influence of F_{hq}-supplemented subgroups on the structure of a finite group.* Math. Reports (Bucur.), **21(71)** (2019), *2*, 135-144.

[11] M. Ezzat Mohamed, M.M. Al-Shomrani, and M.I. Elashiry, *On F_{hq}-supplemented subgroups of a finite group.* Period. Math. Hungar. **76** (2018), *2*, 265-270.

[12] B. Huppert, *Endliche Gruppen.* Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen **134**, Berlin-Heidelberg-New York, Springer-Verlag, 1967.

[13] O.H. Kegel, *Sylow-Gruppen und Subnormlteiler endlicher Gruppen.* Math. Z. **78** (1962), 205-221.

[14] N. Itô, *Uber eine zur Frattini-gruppe duale Bildung.* Nagoya Math. J. **9** (1955), 123-127.

[15] M. Weinstein (Ed.), *Between Nilpotent and Solvable.* Polygonal Publishing House, 1982.

Received May 11, 2020

M. M. Al-Shomrani
King Abdulaziz University
Faculty of Science, Department of Mathematics
Jeddah 21589, Saudi Arabia
malshamrani@kau.edu.sa

M. Ezzat Mohamed
Fayoum University
Faculty of Science, Department of Mathematics
Fayoum, Egypt
mohaezzat@yahoo.com