ANALYTICAL AND NUMERICAL SOLUTIONS FOR SOME MOTIONS OF VISCOUS FLUIDS WITH EXPONENTIAL DEPENDENCE OF VISCOSITY ON THE PRESSURE

ABDUL RAUF, TAHIR MUSHTAQ QURESHI, and CONSTANTIN FETECAU

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Oscillatory motions of incompressible viscous fluids with exponential dependence of viscosity on the pressure between infinite horizontal parallel plates are analytically and numerically studied. The fluid motion is generated by the lower plate that oscillates in its plane and exact expressions are established for the steady-state solutions. The convergence of starting solutions to the corresponding steady-state solutions is graphically proved. The steady solutions corresponding to the simple Couette flow of the same fluids are obtained as limiting cases of the previous solutions. As expected, the fluid velocity diminishes for increasing values of the pressure-viscosity coefficient and ordinary fluids flow faster. The time required to reach the steady-state is graphically approximated. The spatial profiles of the starting solutions are presented both for oscillatory motions and the simple Couette flow.

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1. **INTRODUCTION**

It is well-known the fact that the fluid viscosity can depend on the pressure and most experiments suggest an exponential variation [2, 3, 21]. At the same time, there exist many theoretical and experimental studies in which the dependence of viscosity on the pressure is intensively studied (see, for instance, [1, 6, 9, 10] and inside references). At a pressure of the order 1000 atm., for instance, the fluid viscosity significantly increases and its effect on the fluid motion cannot be neglected [19]. On the other hand, in most cases, the changes in the fluid density are unimportant in such situations. This is the reason that such liquids are studied as incompressible fluids with pressure-dependent viscosities.

In the existing literature, there are interesting theoretical studies [8, 13] concerning flows of fluids with pressure-dependent viscosities and papers with exact solutions [7, 11] in which the gravity effects have been neglected.

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However, in practice, there are many problems with technical applications in which the influence of gravitation cannot be neglected. For instance, in elastohydrodynamics or in many flows involving geological fluids the gravity effects have a meaningful impact. They can become more pronounced if the pressure varies along the direction in which the gravity acts such as flows through horizontal channels. Kannan and Rajagopal [12] show that the rock glacier significantly varies with depth due to gravity.

In the literature, there are very few exact solutions for unsteady motions of fluids with pressure-dependent viscosity in which the gravity effects to be taken into consideration, although such solutions can be used as tests to verify numerical schemes that are developed to study more complex motion problems.

The first exact solutions for steady or unsteady motions of such fluids between infinite parallel plates or down an inclined plane seem to be those of Rajagopal [15, 16], respectively, Rajagopal and Saccomandi [17] for linear, power-law or exponential dependence of viscosity on pressure. Obtained solutions are different enough from those of classical incompressible viscous fluids and the gravity effects on the motion characteristics are significant. Interesting expressions for steady-state or starting solutions have been already obtained by Prusa [14], respectively, Rajagopal et al. [18] for the modified first and second problems of Stokes of fluids with linear or exponential dependence of viscosity on the pressure. Some uniqueness and qualitative results regarding the nature of solutions are also included in [18].

The aim of this note is to provide new but simpler steady-state solutions for oscillatory motions of incompressible viscous fluids with exponential dependence of viscosity on the pressure between two infinite horizontal parallel plates. The fluid motion is induced by the lower plate that oscillates in its plane.

The obtained solutions satisfy the governing equations and boundary conditions. They are determined in a simple way using a suitable change of the spatial variable and the variables separation method. Solutions of the simple Couette flow of the same fluids are obtained as a limiting case of those corresponding to motions due to cosine oscillations of the boundary.

The convergence of starting solutions (numerical solutions) to the steady-state solutions is graphically proved and the time required to reach the steady-state is graphically approximated. It is also found that ordinary fluids flow faster than the fluids with pressure-dependent viscosity. The spatial profiles of starting solutions are graphically presented both for oscillatory motions and the simple Couette flow.

2. STATEMENT OF THE PROBLEM

Let us consider an incompressible viscous fluid with pressure-dependent viscosity at rest between two infinite horizontal parallel plates at the distance d apart. The constitutive equation of such a fluid, as it results from [15, 16], is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} = -p\mathbf{I} + \eta(p)\mathbf{A}. \tag{1}$$

Here $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ the extra-stress tensor, $\mathbf{A}$ is the first Rivlin-Ericksen tensor, $-p\mathbf{I}$ is the constraint stress due to the requirement of incompressibility and $\eta(p)$ is the pressure-dependent viscosity. In the next equation, we study oscillatory motions of fluids for which

$$\eta(p) = \mu \mathrm{e}^{\alpha(p-p_0)}, \quad \alpha > 0, \tag{2}$$

where μ is the fluid viscosity at the reference pressure p_0 and α is the dimensional pressure-viscosity coefficient [7]. From Eq. (2), as it was experimentally proved, it results that $\eta\,(p) \to \infty$ if $p \to \infty$. Furthermore, $\eta\,(p) \to \mu$ for $\alpha \to 0$ and the constitutive equation (1) reduces to that of ordinary viscous fluids.

At the moment $t = 0^+$, the lower plate begins to oscillate in its plane with the velocity $V\cos(\omega t)$ or $V\sin(\omega t)$ where V and ω are the amplitude, respectively, the frequency of oscillations. Due to the shear the fluid moves and we are looking for a solution of the form

$$\mathbf{v} = \mathbf{v}(y,t) = v(y,t)\mathbf{i}, \quad p = p(y), \tag{3}$$

where $\mathbf{v}$ is the velocity vector and $\mathbf{i}$ is the unit vector along the x-direction of a suitable system of Cartesian coordinate x, y and z whose upward y-axis is perpendicular to plates.

Introducing Eqs. (3) in (1) and (2) and the obtained results in the motion equations we attain to the next relevant partial or ordinary differential equations

$$\tau(y,t) = \eta(p)\frac{\partial v(y,t)}{\partial y}, \quad \frac{\partial \tau(y,t)}{\partial y} = \rho\frac{\partial v(y,t)}{\partial t}, \quad \frac{dp(y)}{dy} + \rho g = 0, \tag{4}$$

where $\tau\,(y,t) = S_{yx}\,(y,t)$ is the non-trivial shear stress, ρ is the fluid density and g is the gravitational acceleration. The continuity equation is clearly satisfied while Eq. $(4)_3$ involves

$$p = p(y) = \rho g(d-y) + p_0, \tag{5}$$

where $p_0 = p\,(d)$ is the pressure at the upper plate.

Introducing $p(y)$ from Eq. (5) in (2), the obtained result in Eq. $(4)_1$ and eliminating the shear stress $\tau(y,t)$ between the first two equations in Eq. (4), we find the governing equation for velocity field $v(y,t)$, namely

$$\mu e^{\alpha\rho g(d-y)}\frac{\partial^2 v(y,t)}{\partial y^2} - \mu\alpha\rho g e^{\alpha\rho g(d-y)}\frac{\partial v(y,t)}{\partial y} = \rho\frac{\partial v(y,t)}{\partial t}; \quad 0 < y < d,\ t > 0. \tag{6}$$

The non-trivial shear stress corresponding to this motion is given by

$$\tau(y,t) = \mu e^{\alpha\rho g(d-y)}\frac{\partial v(y,t)}{\partial y};\ \ 0 < y < d,\ t > 0\,. \tag{7}$$

Making $\alpha \to 0$ in Eqs. (6) and (7), as expected, the governing equations corresponding to ordinary incompressible viscous fluids performing the same motions are recovered.

The appropriate initial and boundary conditions are

$$v(y,0) = 0, \quad 0 \le y \le d; \qquad v(0,t) = V\cos(\omega t), \quad v(1,t) = 0 \ \text{if}\ t > 0, \tag{8}$$

or

$$v(y,0) = 0, \quad 0 \le y \le d; \qquad v(0,t) = V\sin(\omega t), \quad v(1,t) = 0 \ \text{if}\ t > 0. \tag{9}$$

Introducing the following non-dimensional variables, functions or parameter

$$y^* = \frac{y}{d},\ t^* = \frac{V}{d}t,\ v^* = \frac{v}{V},\ \tau^* = \frac{1}{\rho V^2}\tau,\ \alpha^* = \alpha\rho g d,\ \omega^* = \frac{d}{V}\omega, \tag{10}$$

and omitting the star notation we attain to the next initial and boundary value problems

$$e^{\alpha(1-y)}\frac{\partial^2 v(y,t)}{\partial y^2} - \alpha e^{\alpha(1-y)}\frac{\partial v(y,t)}{\partial y} = \mathrm{Re}\frac{\partial v(y,t)}{\partial t}; \quad 0 < y < 1,\ t > 0, \tag{11}$$

$$v(y,0) = 0,\ 0 \le y \le 1; \qquad v(0,t) = \cos(\omega t), \quad v(1,t) = 0 \ \text{for}\ t > 0, \tag{12}$$

or

$$v(y,0) = 0,\ 0 \le y \le 1; \qquad v(0,t) = \sin(\omega t), \quad v(1,t) = 0 \ \text{for}\ t > 0, \tag{13}$$

where $Re = \frac{Vd}{\nu}$ is the Reynolds number. The dimensionless shear stress is given by

$$\tau(y,t) = \frac{1}{\mathrm{Re}}e^{\alpha(1-y)}\frac{\partial v(y,t)}{\partial y};\ \ 0 < y < 1,\ t > 0. \tag{14}$$

The starting solutions $v_c(y,t)$ and $v_s(y,t)$ corresponding to the previous problems can be presented as sum of steady-state (permanent) and transient components, namely

$$v_c(y,t) = v_{cp}(y,t) + v_{ct}(y,t), \quad v_s(y,t) = v_{sp}(y,t) + v_{st}(y,t). \tag{15}$$

Some time after the motion initiation the behavior of the fluid is characterized by the starting solutions. After this time, when the transients disappear, the fluid moves according to the steady-state solutions $v_{cp}(y,t)$ or $v_{sp}(y,t)$. This time is important for the experimentalists who want to eliminate the transients from their experiments. In order to determine this time, for such motions, at least exact expressions for the steady-state solutions have to be known.

3. EXACT EXPRESSIONS FOR THE STEADY-STATE SOLUTIONS

In order to determine exact expressions for the steady-state solutions $v_{cp}(y,t)$ and $v_{sp}(y,t)$ in the simplest way, we define the complex velocity field

$$u_p(y,t) = v_{cp}(y,t) + iv_{sp}(y,t), \tag{16}$$

where i is the imaginary unit. This complex velocity field $u_p(y,t)$ has to satisfy the partial differential equation

$$\mathrm{e}^{\alpha(1-y)}\frac{\partial^2 u_p(y,t)}{\partial y^2} - \alpha \mathrm{e}^{\alpha(1-y)}\frac{\partial u_p(y,t)}{\partial y} = \mathrm{Re}\frac{\partial u_p(y,t)}{\partial t}; \quad 0 < y < 1, \ t \in R, \tag{17}$$

with the boundary conditions

$$u_p(0,t) = \mathrm{e}^{i\omega t}, \quad u_p(1,t) = 0; \quad t \in R. \tag{18}$$

Making the change of independent variable

$$y = 1 + \ln \sqrt[\alpha]{r} \ \text{ or equivalently } \ r = \mathrm{e}^{\alpha(y-1)}, \tag{19}$$

we find that the unknown function $u_p(r,t)$ has to satisfy the next boundary value problem

$$\alpha^2 r \frac{\partial^2 u_p(r,t)}{\partial r^2} = \mathrm{Re}\frac{\partial u_p(r,t)}{\partial t}; \ \ a < r < 1, \ t \in R, \tag{20}$$

$$u_p(a,t) = \mathrm{e}^{i\omega t}, \quad u_p(1,t) = 0; \quad t \in R. \tag{21}$$

where $a = 1/\mathrm{e}^{\alpha}$.

To solve the previous boundary value problem we are looking for a separable solution

$$u_p(r,t) = V(r)T(t). \tag{22}$$

By substituting $u_p(r,t)$ from Eq. (22) in (20) we find that the functions $V(r)$ and $T(r)$ have to satisfy the ordinary differential equations

$$(23)\qquad r\frac{\mathrm{d}^2V(r)}{\mathrm{d}r^2}-\lambda\frac{\mathrm{Re}}{\alpha^2}V(r)=0,\quad \frac{\mathrm{d}T(t)}{\mathrm{d}t}-\lambda T(t)=0;\qquad a<r<1,\ t\in R.$$

According to the boundary conditions (21), it results that $\lambda=i\omega$ and the function $V(r)$ has to satisfy the following boundary value problem

$$(24)\qquad r\frac{\mathrm{d}^2V(r)}{\mathrm{d}r^2}-\frac{i\omega\mathrm{Re}}{\alpha^2}V(r)=0;\quad V(a)=1,\quad V(1)=0.$$

The solution of the boundary value problem (24) is given by (see, for instance, [22, problem 37, p. 251] for the general solution of the ordinary equation $(23)_1$)

$$(25)\qquad V(r)=\sqrt{\frac{r}{a}}\,\frac{Y_1(b)J_1(b\sqrt{r})-J_1(b)Y_1(b\sqrt{r})}{Y_1(b)J_1(b\sqrt{a})-J_1(b)Y_1(b\sqrt{a})}$$

and the complex velocity field $u_p(r,t)$ is given by

$$(26)\qquad u_p(r,t)=\sqrt{\frac{r}{a}}\,\frac{Y_1(b)J_1(b\sqrt{r})-J_1(b)Y_1(b\sqrt{r})}{Y_1(b)J_1(b\sqrt{a})-J_1(b)Y_1(b\sqrt{a})}\mathrm{e}^{i\omega t};\qquad b=\frac{2}{\alpha}\sqrt{-i\omega\mathrm{Re}}.$$

Consequently, coming back to the initial variables, dimensionless steady-state solutions corresponding to our problem are given by

$$(27)\ v_{cp}(y,t)=\sqrt{\mathrm{e}^{\alpha y}}\,\mathrm{Real}\left\{\frac{Y_1(b)J_1\left(b\sqrt{\mathrm{e}^{\alpha(y-1)}}\right)-J_1(b)Y_1\left(b\sqrt{\mathrm{e}^{\alpha(y-1)}}\right)}{Y_1(b)J_1(b\sqrt{a})-J_1(b)Y_1(b\sqrt{a})}\mathrm{e}^{i\omega t}\right\},$$

$$(28)\ v_{sp}(y,t)=\sqrt{\mathrm{e}^{\alpha y}}\,\mathrm{Im}\left\{\frac{Y_1(b)J_1\left(b\sqrt{\mathrm{e}^{\alpha(y-1)}}\right)-J_1(b)Y_1\left(b\sqrt{\mathrm{e}^{\alpha(y-1)}}\right)}{Y_1(b)J_1(b\sqrt{a})-J_1(b)Y_1(b\sqrt{a})}\mathrm{e}^{i\omega t}\right\},$$

where Im denotes the imaginary part of that which follows.

The velocity fields $v_{cp}(y,t)$ and $v_{sp}(y,t)$ given by Eqs. (27) and (28) are independent of the initial condition $(12)_1$ or $(12)_2$ but they satisfy the boundary conditions and the governing equation (11). The corresponding non-dimensional shear stresses, namely

$$(29)\qquad \tau_{cp}(y,t)=\frac{\sqrt{\omega\mathrm{e}^{\alpha}}}{\sqrt{\mathrm{Re}}}\,\mathrm{Real}\left\{\frac{Y_1(b)J_0\left(b\sqrt{\mathrm{e}^{\alpha(y-1)}}\right)-J_1(b)Y_0\left(b\sqrt{\mathrm{e}^{\alpha(y-1)}}\right)}{Y_1(b)J_1(b\sqrt{a})-J_1(b)Y_1(b\sqrt{a})}\ \mathrm{e}^{i(\omega t+3\pi/4)}\right\},$$

$$\tau_{sp}(y,t) =$$

$$(30)\qquad \frac{\sqrt{\omega \mathrm{e}^{\alpha}}}{\sqrt{\mathrm{Re}}}\,\mathrm{Im}\left\{\frac{Y_1(b)J_0\left(b\sqrt{\mathrm{e}^{\alpha(y-1)}}\right)-J_1(b)Y_0\left(b\sqrt{\mathrm{e}^{\alpha(y-1)}}\right)}{Y_1(b)J_1(b\sqrt{a})-J_1(b)Y_1(b\sqrt{a})}\,\mathrm{e}^{i(\omega t+3\pi/4)}\right\},$$

are immediately obtained substituting the expressions of $v_{cp}(y,t)$ and $v_{sp}(y,t)$ in Eg. (14). The dimensionless frictional forces per unit area exerted by the fluid on the upper plate are immediately obtained by substituting y with one in Eqs. (29) and (30).

4. LIMITING CASES

In order to provide the similar solutions for the simple Couette flow of the same fluids with exponential dependence of viscosity on the pressure, as well as to recover some known results from the existing literature or to get some physical insight of results that have been here obtained, we shall consider the following two particular cases:

4.1. Case $\omega \to 0$ (Simple Couette flow)

On the basis of the asymptotic approximations given by Eqs. (A1) from Appendix, it is easy to show that the velocity field $v_{cp}(y,t)$ and the corresponding shear stress $\tau_{cp}(y,t)$ given by Eqs. (27), respectively (29), converge to the solutions

$$(31)\qquad v_{Cp}(y) = \frac{\mathrm{e}^{\alpha y}-\mathrm{e}^{\alpha}}{1-\mathrm{e}^{\alpha}},\quad \text{respectively}\quad \tau_{Cp} = \frac{\alpha \mathrm{e}^{\alpha}}{(1-\mathrm{e}^{\alpha})\mathrm{Re}},$$

if $\omega \to 0$. Their convergence has been also proved by graphical illustrations.

The above expressions of $v_{Cp}(y)$ and τ_{Cp} give the dimensionless velocity field and the adequate shear stress corresponding to the simple Couette flow of fluids with exponential dependence of viscosity on the pressure. The velocity field $v_{Cp}(y)$ can also be obtained solving the boundary value problem (see steady form of Eq. (11) and Eqs. $(12)_2$ and $(12)_3$ with $\omega = 0$)

$$(32)\qquad \frac{\mathrm{d}}{\mathrm{d}y}\left[\mathrm{e}^{\alpha(1-y)}\frac{\mathrm{d}\,v(y)}{dy}\right] = 0;\quad v(0,t) = 1,\quad v(1,t) = 0,$$

while τ_{Cp} is immediately obtained substituting $v_{Cp}(y)$ in Eq. (14). Now, it is important to point out the fact that the steady shear stress τ_{Cp} corresponding to the simple Couette flow of fluids with exponential dependence of viscosity on the pressure is constant on the entire flow field, although the fluid velocity is a function of the spatial variable y.

4.2. Case $\alpha \to 0$ (Oscillatory motions of ordinary viscous fluids)

Using the asymptotic approximations from Eqs. (A2), we can easily prove that for small enough values of the non-dimensional pressure-viscosity coefficient α

$$v_{cp}(y,t) \approx \sqrt[4]{\mathrm{e}^{\alpha y}}\,\mathrm{Real}\left\{\frac{\sin\{b\,[1-\exp\,(\alpha(y-1)/2)]\}}{\sin\{b\,[1-\exp(-\alpha/2)]\}}\;\mathrm{e}^{i\omega t}\right\}, \tag{33}$$

$$v_{sp}(y,t) \approx \sqrt[4]{\mathrm{e}^{\alpha y}}\mathrm{Im}\left\{\frac{\sin\{b\,[1-\exp\,(\alpha(y-1)/2)]\}}{\sin\{b\,[1-\exp(-\alpha/2)]\}}\;\mathrm{e}^{i\omega t}\right\}. \tag{34}$$

Now, using the Maclaurin series expansions of $\exp[\alpha(y-1)/2]$ and $\exp(-\alpha/2)$, the identities (A3) from Appendix and making $\alpha \to 0$ in Eq. (33) and (34), we recover the dimensionless velocity fields [5, Eqs. (43)]

$$\begin{aligned} v_{Vcp}(y,t) &= \mathrm{Real}\left\{\frac{\mathrm{sh}\big[(1-y)\sqrt{i\omega\mathrm{Re}}\big]}{\mathrm{sh}(\sqrt{i\omega\mathrm{Re}})}\;\mathrm{e}^{i\omega t}\right\}, \\ v_{Vsp}(y,t) &= \mathrm{Im}\left\{\frac{\mathrm{sh}\big[(1-y)\sqrt{i\omega\mathrm{Re}}\big]}{\mathrm{sh}(\sqrt{i\omega\mathrm{Re}})}\;\mathrm{e}^{i\omega t}\right\}, \end{aligned} \tag{35}$$

corresponding to incompressible viscous fluids performing the same motions. The adequate shear stresses, namely

$$\begin{aligned} \tau_{Vcp}(y,t) &= -\frac{\sqrt{\omega}}{\sqrt{\mathrm{Re}}}\mathrm{Real}\left\{\frac{\mathrm{ch}\big[(1-y)\sqrt{i\omega\mathrm{Re}}\big]}{\mathrm{sh}(\sqrt{i\omega\mathrm{Re}})}\;\mathrm{e}^{i(\omega t+\pi/4)}\right\}, \\ \tau_{Vsp}(y,t) &= -\frac{\sqrt{\omega}}{\mathrm{Re}}\mathrm{Im}\left\{\frac{\mathrm{ch}\big[(1-y)\sqrt{i\omega\mathrm{Re}}\big]}{\mathrm{sh}(\sqrt{i\omega\mathrm{Re}})}\;\mathrm{e}^{i(\omega t+\pi/4)}\right\}, \end{aligned} \tag{36}$$

can be also obtained as limiting cases of $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$ when $\alpha \to 0$.

It is also worth pointing out the fact that making $\alpha \to 0$ in Eqs. (31) or $\omega \to 0$ in Eqs. $(35)_1$ and $(36)_1$, we recover the well-known dimensionless steady solutions

$$v_{VCp}(y) = 1-y, \qquad \tau_{VCp} = -1/\mathrm{Re}, \tag{37}$$

corresponding to the simple Couette flow of incompressible viscous fluids.

Finally, we would like to end with an important remark regarding the connection between the present velocity fields $v_{Vcp}(y,t)$ and $v_{Vsp}(y,t)$ in dimensional form, namely

$$
(38)\qquad
\begin{aligned}
v_{V\,cp}(y,t) &= V\mathrm{Real}\left\{\frac{\mathrm{sh}\left[(d-y)\sqrt{\frac{i\omega}{\nu}}\right]}{\mathrm{sh}\left(d\sqrt{\frac{i\omega}{\nu}}\right)}\ \mathrm{e}^{i\omega t}\right\},\\
v_{V\,sp}(y,t) &= V\mathrm{Im}\left\{\frac{\mathrm{sh}\left[(d-y)\sqrt{\frac{i\omega}{\nu}}\right]}{\mathrm{sh}\left(d\sqrt{\frac{i\omega}{\nu}}\right)}\ \mathrm{e}^{i\omega t}\right\},
\end{aligned}
$$

and the similar solutions of the Stokes' second problem [20, Eqs. (48) and (49) with $M=0$]

$$
(39)\qquad
\begin{aligned}
v_{S\,cp}(y,t) &= V\exp\left(-y\sqrt{\tfrac{\omega}{2\nu}}\right)\cos\left(\omega t-y\sqrt{\tfrac{\omega}{2\nu}}\right),\\
v_{S\,sp}(y,t) &= V\exp\left(-y\sqrt{\tfrac{\omega}{2\nu}}\right)\sin\left(\omega t-y\sqrt{\tfrac{\omega}{2\nu}}\right).
\end{aligned}
$$

As expected, graphical representations (which are not included here) show that for values of the distance d (greater than 1.9) the diagrams of $v_{V\,cp}(y,t)$ and $v_{V\,sp}(y,t)$ against t are almost identical to those of $v_{S\,cp}(y,t)$, respectively $v_{S\,sp}(y,t)$ for $y\in[0,1.4]$. Consequently, the velocity fields corresponding to motions between infinite horizontal parallel plates induced by the lower plate that oscillates in its plane converge to the solutions of the second problem of Stokes on the most part of the flow range if the distance d between plates is greater or equal to 1.9.

5. NUMERICAL RESULTS AND CONCLUSIONS

The main purpose of this note is to provide exact but simple expressions for the non-dimensional steady-state components $v_{cp}(y,t)$ and $v_{sp}(y,t)$ of the starting solutions $v_c(y,t)$, respectively $v_s(y,t)$, corresponding to two oscillatory motions of the incompressible viscous fluids with exponential dependence of viscosity on the pressure between infinite horizontal parallel plates. The corresponding steady-state shear stresses $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$ are also determined and the frictional forces per unit area exerted by the fluid on the upper plate can be immediately obtained. The fluid motion is induced by the lower plate that oscillates in its plane. The solutions that have been obtained can be useful for those who want to eliminate the transients from their experiments. They are independent of the initial condition but satisfy the boundary conditions and governing equations. As expected, if the dimensionless pressure-viscosity coefficient $\alpha\to 0$, the obtained expressions reduce to the well-known solutions corresponding to ordinary incompressible viscous fluids subject to the same movements.

On the other hand, if the oscillations' frequency $\omega \to 0$, the velocity field $v_{cp}(y,t)$ and the associate shear stress $\tau_{cp}(y,t)$ given by Eqs. (27), respectively (29), converge to the dimensionless steady solutions $v_{Cp}(y,t)$, respectively $\tau_{Cp}(y,t)$ corresponding to the simple Couette flow of the incompressible viscous fluids with exponential dependence of viscosity on the pressure. Furthermore, making $\alpha \to 0$ in Eqs. (31) or $\omega \to 0$ in Eqs. $(35)_1$ and $(36)_1$, the dimensionless steady solutions (37) corresponding to the steady simple Couette flow of incompressible viscous fluids are recovered.

In the following, in order to bring to light some physical aspects of results that have been here established, as well as to confirm their correctness, graphical representations were depicted in Figs. 1-5. The convergence of dimensionless starting solutions $v_c(y,t)$ and $v_s(y,t)$ (numerical solutions) to their steady-state components $v_{cp}(y,t)$, respectively $v_{sp}(y,t)$ is graphically proved in Figs. 1.

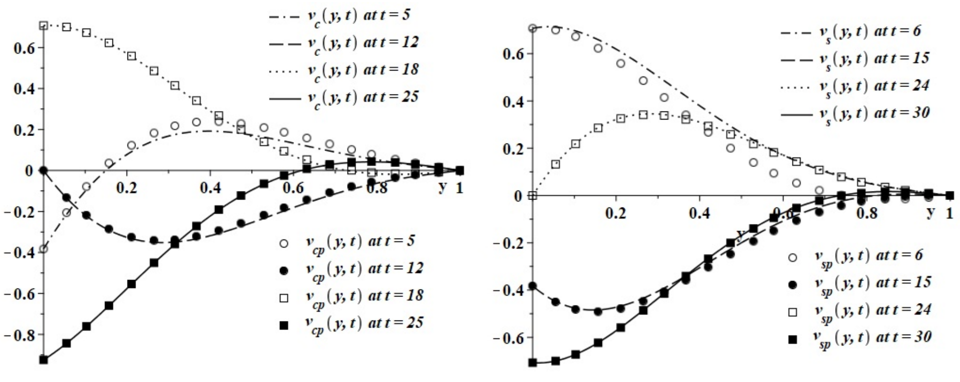

Figure 1 – Convergence of starting solutions $v_c(y,t)$ and $v_s(y,t)$ (numerical solutions) to their steady-state components $v_{cp}(y,t)$, respectively $v_{sp}(y,t)$, for $\alpha = 0.95$, $\omega = \pi/8$ and Re $= 100$.

The boundary condition on the upper plate is clearly satisfied. Differences between the corresponding solutions hastily diminish in both cases and the non-dimensional time to reach the steady-state $t_s = 30$ (the time after which the profiles of starting and steady-state solutions are almost identical) for the motion induced by sine oscillations of the lower plate is larger than $t_c = 25$ of the motion due to cosine oscillations of the wall. This is obvious [4], since at $t = 0$ the plate velocity is zero.

In Figures 2 and 3, the convergence of the dimensionless velocity $v_C(y,t)$ (numerical solutions) corresponding to the simple Couette flow to its steady component $v_{Cp}(y)$ is proved for two different values of the Reynolds number Re, respectively, the non-dimensional pressure-viscosity coefficient α. As expected, in all cases the fluid velocity increases in time and smoothly decreases from the maximum value one on the moving plate to the zero asymptotic value on the stationary plate. In addition, the velocity values are higher or lower for distinct values of the time t on the entire flow domain. The required time to reach the steady state is an increasing function with regard to the Reynolds number Re and diminishes for increasing values of α. Consequently, the steady state is rather obtained for motions of fluids with pressure-dependent viscosity as compared to ordinary fluids.

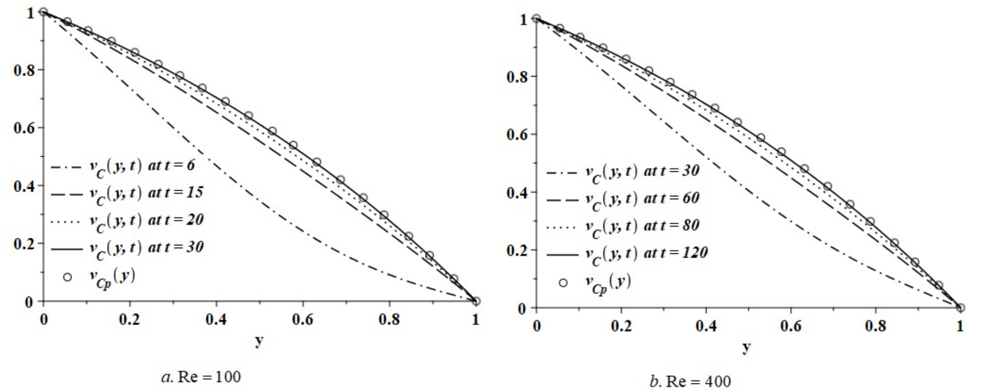

Figure 2 – Convergence of the starting solution $v_C(y,t)$ (numerical solution) to its steady component $v_{Cp}(y)$ given by Eq. $(31)_1$ for $\alpha = 0.95$.

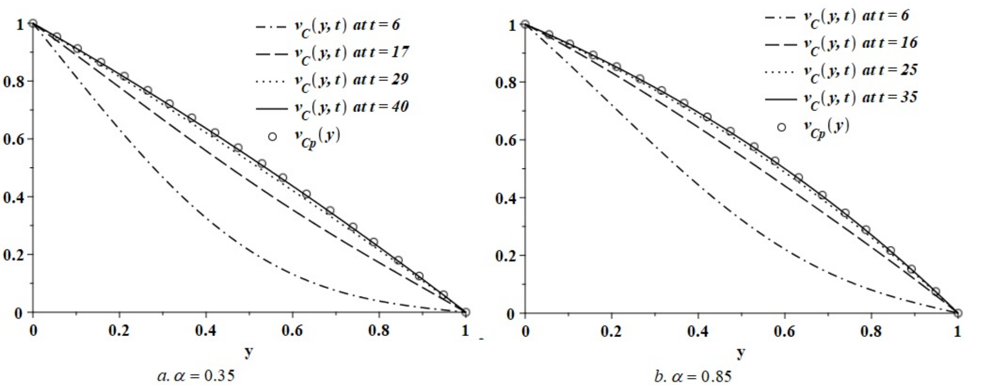

Figure 3 – Convergence of the starting solution $v_C(y,t)$ (numerical solution) to its steady component $v_{Cp}(y)$ given by Eq. $(31)_1$ for Re = 100.

For comparison, the spatial profiles of the dimensionless starting velocity fields $v_c(y,t)$ and $v_s(y,t)$ (numerical solutions) are together presented in Figs. 4 for Re $= 100$, $\omega = \pi/8$ and $\alpha = 0.95$. In both cases, the initial and boundary conditions are satisfied and the phase difference between the two motions and their oscillatory behaviour can be easy observed. Figs. 5a and 5b provide the spatial profiles of the starting velocity field $v_C(y,t)$ corresponding to the simple Couette flow at two different values of the Reynolds number Re. All imposed initial and boundary conditions are clearly satisfied and the fluid velocity slowly decreases for increasing values of Re. This is possible since the fluid viscosity diminishes for increasing values of Re and the fluid flows faster.

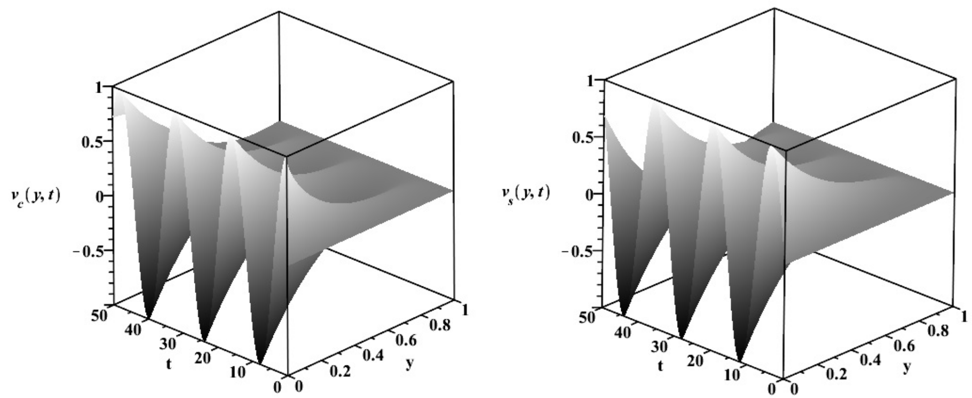

Figure 4 – Spatial profiles of the velocity fields $v_c(y,t)$ and $v_s(y,t)$ (numerical solutions) for $\alpha = 0.95$, $\omega = \pi/8$ and Re $= 100$.

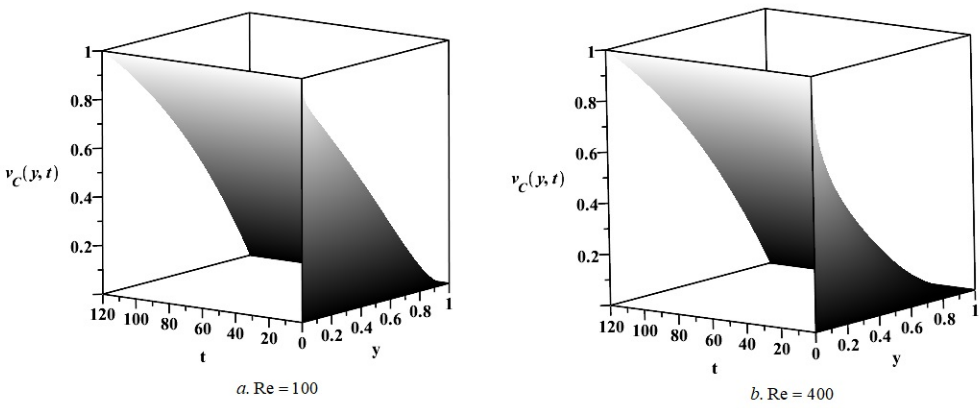

Figure 5 – Spatial profiles of the velocity field $v_C(y,t)$ (numerical solutions) corresponding to the simple Couette flow for $\alpha = 0.95$.

The main results that have been obtained in this note are as follows:

– Exact and simple expressions are established for the steady-state solutions of some oscillatory motions of fluids with exponential dependence of viscosity on the pressure between parallel plates.

– The steady solutions corresponding to the simple Couette flow of the same fluids as well as some known results from the existing literature are obtained as limiting cases of previous expressions.

– The steady shear stress corresponding to the simple Couette flow of such fluids is constant on the entire flow domain although the fluid velocity is a function of the spatial variable y.

– The convergence of starting solutions to their steady components has been graphically proved. Spatial profiles of starting solutions (numerical solutions) are also presented.

– Required time to reach the stationary state has been graphically determined for all motions. It is found that ordinary fluids flow faster in comparison with fluids with pressure-dependent viscosity.

6. APPENDIX

(A1)
$$J_0(z) \approx 1, \quad J_1(z) \approx \frac{z}{2},$$
$$Y_0(z) \approx \frac{2}{\pi}\left[\ln\left(\frac{z}{2}\right) + \gamma\right], \quad Y_1(z) \approx -\frac{2\pi}{z} \text{ for } z << 1,$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

(A2)
$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left[z - \frac{(2\nu+1)\pi}{4}\right],$$
$$Y_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left[z - \frac{(2\nu+1)\pi}{4}\right] \text{ for } |z| >> 1,$$

(A3)
$$\sin(iz) = i\mathrm{sh}(z), \quad \cos(iz) = \mathrm{ch}(z).$$

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Abdul Rauf
Air University Islamabad (Multan Campus)
Department of Mathematics
Pakistan
attari_ab092@gmail.com

Tahir Mushtaq Qureshi
COMSATS University Islamabad (Vehari Campus)
Department of Mathematics
Pakistan
tahmush@hotmail.com

Constantin Fetecau
Academy of Romanian Scientists
Bucharest, Romania
c_fetecau@yahoo.com