HOPF CO-BRACE, BRAID EQUATION AND BICROSSED COPRODUCT

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In this paper, we mainly give some equivalent characterisations of Hopf co-braces, show that the full subcategory $\mathcal{HCB}(A)$ of Hopf co-braces is equivalent to the full subcategory $\mathcal{C}(A)$ of bijective 1-cocycles, and prove that the full subcategory $\mathcal{HCB}(A)$ is also equivalent to the category $\mathcal{M}(A)$ of Hopf matched pairs. Moreover, we construct many Hopf co-braces on polynomial Hopf algebras, Long copaired Hopf algebras and Drinfel'd doubles of finite dimensional Hopf algebras. And we also give a sufficient and necessary condition for a given bicrossed coproduct $A \bowtie H$ to be a Hopf co-brace if A or H is a Hopf co-brace.

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1. INTRODUCTION AND PRELIMINARIES

Braces were introduced in [14] by Rump, which are a generalization of Jacobson radical rings, to understand the structure behind non-degenerate involutive set-theoretic solutions of Yang-Baxter equations. They provide a powerful algebraic framework to work with set-theoretic solutions and have also an advantage to discuss braided groups and sets imitating ring theory. Moreover, they have also connections with regular subgroups and orderable groups [4], flat manifolds [15], Hopf-Galois extensions [3]. Through their connection with Yang-Baxter equation and group theory, braces have attracted a lot of attention and obtained a wide range of more influential results, for example [1, 3, 7, 15].

In [2], the authors introduced the concept of Hopf braces and Hopf co-braces. That is, a Hopf brace (or Hopf co-brace) is a kind of special Hopf

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algebra with two different multiplications (or comultiplications) connected with antipode, which is of a new algebraic structure related to the Yang-Baxter equation, and is also a generalization of braces and skew braces. As a basic example of a Hopf brace, we may take the group algebra of a (classical or skew) brace.

The bicrossed product first emerged in group theory, which is constructed from a matched pair of groups, as a natural generalization of the semi-direct product (see [6]). In [1], the author construct Hopf braces by bicrossed product. Naturally, we consider construct Hopf co-braces by bicrossed coproduct, but the difference is that we give sufficient and necessary conditions for constructing Hopf co-braces. We could find that the Drinfel'd double $D(H)$ of a finite dimensional Hopf algebra H was a special type of this bicrossed product in [12]. So we can construct a Hopf co-brace by the dual of Drinfel'd double of a finite dimensional Hopf algebra as an application of bicrossed coproduct.

From [2], we know the finite dual of a cocommutative Hopf brace is a commutative Hopf co-brace. But in the infinite case, it's not necessarily true. In this paper, we proved some results similar to Hopf brace for a infinite dimensional Hopf co-brace. In addition, we added many new examples of Hopf co-braces and constructed Hopf co-braces on polynomial Hopf algebras, Long copaired Hopf algebras.

The paper is organized as follows. In Section 2, we give many examples of Hopf co-braces, and prove that the full subcategory $\mathcal{HCB}(A)$ is equivalent to the full subcategory $\mathcal{C}(A)$ of bijective 1-cocycles. We also obtain a solution of the braid equation by a commutative Hopf co-brace. In Section 3, we mainly study structures of commutative Hopf co-braces, build a correspondence between Hopf co-braces and Hopf matched pairs, and prove that the full subcategory $\mathcal{HCB}(A)$ is equivalent to the category $\mathcal{M}(A)$ of Hopf matched pairs (A, A). In Section 4, we mainly give a sufficient and necessary condition for a given bicrossed coproduct $A \bowtie H$ to be a Hopf co-brace if A or H is a Hopf co-brace, and show that the dual of Drinfel'd double $D(H)$ of a finite dimensional cocommutative Hopf algebra H is a Hopf co-brace.

Throughout this paper, let k be a fixed field, and our considered objects be all meant over k. And we freely use coalgebras and Hopf algebras terminology introduced in [16] and [8].

2. HOPF CO-BRACE AND ITS CATEGORY

In this section, we recall the concept of Hopf co-braces, give many examples of Hopf co-braces on polynomial Hopf algebras and Long copaired Hopf algebras, and mainly prove that the full subcategory $\mathcal{HCB}(A)$ of Hopf cobraces is equivalent to the full subcategory of the category $\mathcal{C}(A)$ of bijective 1-cocycles.

Definition 2.1 ([2]). Let $(H, m, 1)$ be an algebra. A *Hopf co-brace* structure over H consists of the following data:

(1) a Hopf algebra structure $(H, m, 1, \Delta, \varepsilon, S)$,

(2) a Hopf algebra structure $(H, m, 1, \Delta', \epsilon, T)$,

(3) satisfying the following compatibility:

$$h_{1'} \otimes h_{2'1} \otimes h_{2'2} = h_{11'} S(h_2) h_{31'} \otimes h_{12'} \otimes h_{32'} \tag{2.1}$$

for any $h \in H$, where $\Delta(h)$ is denoted by $h_1 \otimes h_2$ and $\Delta'(h)$ denoted by $h_{1'} \otimes h_{2'}$.

Remark 2.2. (1) When H is commutative, Definition 2.1 is the specialization in Vect^{op} of [10, Definition 4.1]. In the general case, it is an immediate dualization of [2, Definition 1.1].

(2) In any given Hopf co-brace $(H, \Delta, \varepsilon; \Delta', \epsilon)$, we obtain $\varepsilon = \epsilon$.

(3) Let (H, Δ, ε) be a Hopf algebra. Then, we easily know that

$$(H, \Delta, \varepsilon; \Delta, \varepsilon)$$

is a Hopf co-brace.

(4) In what follows, we denote the Hopf co-brace in Definition 2.1 by (H, Δ, Δ').

Example 2.3. (1) Let (H, Δ, ε) be a Hopf algebra. Then, $(H, \Delta, \Delta^{coH})$ and $(H, \Delta^{coH}, \Delta)$ are Hopf co-braces.

(2) Let $A = k[g, g^{-1}, x]$ be a Hopf algebra in [11] with the coalgebra structures:

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + 1 \otimes x,$$
$$\varepsilon(g) = 1,\ \varepsilon(x) = 0,$$

and with the antipode: $S(g) = g^{-1},\ S(x) = -x$.

Moreover, $(A, \Delta', \varepsilon, T)$ is a Hopf algebra with the following coalgebra structures:

$$\Delta'(g) = g \otimes g,\ \Delta'(x) = x \otimes 1 + g \otimes x,$$
$$\varepsilon(g) = 1,\ \varepsilon(x) = 0,$$

and with the antipode: $T(g) = g^{-1},\ T(x) = -g^{-1}x$.

When $h = x$ and $h = g$, we can verify that Equation (2.1) holds. Hence (A, Δ, Δ') is a Hopf co-brace.

(3) Let H be a Hopf algebra. If $R = R'_i \otimes R''_i \in H \otimes H$ is a normalized Harrison 2-cocycle in [13], that is, R is satisfied the following conditions:

$$r'_i R'_{i1} \otimes r''_i R'_{i2} \otimes R''_i \quad = \quad R'_i \otimes r'_i R''_{i1} \otimes r''_i R''_{i2},$$

$$\varepsilon(R'_i)R''_i \;=\; 1 \;=\; R'_i\varepsilon(R''_i),$$

where r is a copy of R. Define a comultiplication on H as follows if R is invertible with the inverse $R^{-1} = R'^{-1}_i \otimes R''^{-1}_i$:

$$\Delta_R(h) = R'_i h_1 R'^{-1}_i \otimes R''_i h_2 R''^{-1}_i$$

for any $h \in H$. Then, (H, Δ_R, S^R) is a Hopf algebra with the same counit, where $S^R(x) = R'_i S(R''_i)S(x)S(R'^{-1}_i)R''^{-1}_i$.

Let (H, R) be a Long copaired Hopf algebra in [17], that is, there is an invertible element $R = R'_i \otimes R''_i \in H \otimes H$ such that the following conditions are satisfied:

$$\begin{aligned}
&(LC1)\ R'_i x \otimes R''_i = xR'_i \otimes R''_i, \text{for any } x \in H,\\
&(LC2)\ \varepsilon(R'_i)R''_i = 1,\\
&(LC3)\ R'_{i1} \otimes R'_{i2} \otimes R''_i = R'_i \otimes r'_i \otimes r''_i R''_i,\\
&(LC4)\ R'_i\varepsilon(R''_i) = 1,\\
&(LC5)\ R'_i \otimes R''_{i1} \otimes R''_{i2} = R'_i r'_i \otimes R''_i \otimes r''_i.
\end{aligned}$$

Suppose that (H, R) is a Long copaired Hopf algebra. Then, according to Example 3.1 in [8], we know that R is a normalized Harrison 2-cocycle, so, we obtain a new Hopf algebra (H, Δ_R, S^R), and hence (H, Δ, Δ_R) is a Hopf co-brace.

Indeed, we only need to check that the condition (2.1) is satisfied: for any $h \in H$, where LHB stands for the left-hand side of (2.1) and RHB the right-hand side of (2.1).

$$\begin{aligned}
LHB &= R'_i h_1 R'^{-1}_i \otimes (R''_i h_2 R''^{-1}_i)_1 \otimes (R''_i h_2 R''^{-1}_i)_2\\
&= R'_i h_1 R'^{-1}_i \otimes R''_{i1} h_2 R''^{-1}_{i1} \otimes R''_{i2} h_3 R''^{-1}_{i2}\\
&\overset{(LC5)}{=} R'_i r'_i h_1 R'^{-1}_i \otimes R''_i h_2 R''^{-1}_{i1} \otimes r''_i h_3 R''^{-1}_{i2}\\
&= R'_i r'_i h_1 r'^{-1}_i R'^{-1}_i \otimes R''_i h_2 R''^{-1}_i \otimes r''_i h_3 r''^{-1}_i,
\end{aligned}$$

where $R'^{-1}_i \otimes R''^{-1}_{i1} \otimes R''^{-1}_{i2} = r'^{-1}_i R'^{-1}_i \otimes R''^{-1}_i \otimes r''^{-1}_i$ (see[17, Theorem 3.2]).

$$\begin{aligned}
RHB &= R'_i h_1 R'^{-1}_i S(h_3) r'_i h_4 r'^{-1}_i \otimes R''_i h_2 R''^{-1}_i \otimes r''_i h_5 r''^{-1}_i\\
&\overset{(LC1)}{=} R'_i h_1 R'^{-1}_i S(h_3) h_4 r'_i r'^{-1}_i \otimes R''_i h_2 R''^{-1}_i \otimes r''_i h_5 r''^{-1}_i\\
&= R'_i h_1 R'^{-1}_i r'_i r'^{-1}_i \otimes R''_i h_2 R''^{-1}_i \otimes r''_i h_3 r''^{-1}_i\\
&= R'_i r'_i h_1 r'^{-1}_i R'^{-1}_i \otimes R''_i h_2 R''^{-1}_i \otimes r''_i h_3 r''^{-1}_i,
\end{aligned}$$

where $R'^{-1}_i x \otimes R''^{-1}_i = xR'^{-1}_i \otimes R''^{-1}_i$ (see [17, Theorem 3.2]), for any $x \in H$. So, (2.1) is satisfied, and hence (H, Δ, Δ_R) is a Hopf co-brace.

Let (H, Δ, Δ') and (G, Δ, Δ') be Hopf co-braces. A homomorphism of Hopf co-braces $f : (H, \Delta, \Delta') \to (G, \Delta, \Delta')$ is a linear map f such that f :

$H_\Delta \to G_\Delta$ and $f : H_{\Delta'} \to G_{\Delta'}$ are Hopf algebra homomorphism. It is easy to see that Hopf co-braces form a category.

Fix a Hopf algebra $(H, m, 1, \Delta, \varepsilon, S)$. Let $\mathcal{HCB}(H)$ be the full subcategory of the category of Hopf co-braces with objects (H, Δ, Δ'). This means that the objects of $\mathcal{HCB}(H)$ are Hopf co-braces such that the first Hopf algebra structure is that of H_Δ.

Lemma 2.4. *Let (H, Δ, Δ') be a Hopf co-brace. Then*

$$S(h_1)_{1'}h_2 \otimes S(h_1)_{2'} = S(h_1)h_{21'} \otimes S(h_{22'})$$

for any $h \in H$.

Proof. It is a dualization of [2, Lemma 1.7]. We leave all the details to the reader. □

Lemma 2.5. *Let (H, Δ, Δ') is a Hopf co-brace. Then, (H, Δ) is a left (H, Δ')-comodule coalgebra with*

$$\rho(h) \equiv h_{(-1)} \otimes h_{(0)} = S(h_1)h_{21'} \otimes h_{22'}$$

for any $h \in H$.

Proof. It is a dualization of [2, Lemma 1.8]. We leave all the details to the reader. □

Remark 2.6. It follows from the Lemma 2.4 that

$$h_{1'} \otimes h_{2'} = h_1 h_{2(-1)} \otimes h_{2(0)},$$
$$h_1 \otimes h_2 = h_{1'}T(h_{2'(-1)}) \otimes h_{2'(0)},$$

for any $h \in H$.

Definition 2.7. A Hopf co-brace (A, Δ, Δ') is said to be commutative if the underlying algebra A is commutative.

Lemma 2.8. *Let (A, Δ, Δ') be a commutative Hopf co-brace. Then the following conclusions hold:*

(1) *A is a left $A_{\Delta'}$-comodule algebra via*

$$\rho(a) \equiv a_{(-1)} \otimes a_{(0)} = S(a_1)a_{21'} \otimes a_{22'},$$

for any $a \in A$.

(2) *A is a right $A_{\Delta'}$-comodule algebra via*

$$\begin{aligned}\varphi(a) &\equiv a_{[0]} \otimes a_{[1]} = T(a_{1'})_{(-1)}a_{2'} \otimes T(a_{1'})_{(0)}a_{3'}\\ &= S(T(a_{1'})_1)T(a_{1'})_{21'}a_{2'} \otimes T(a_{1'})_{22'}a_{3'},\end{aligned}$$

for any $a \in A$.

(3) $(id\otimes S)\rho(a)=\rho(S(a))$, *for all* $a\in A$. *That is,* $a_{(-1)}\otimes S(a_{(0)})=S(a)_{(-1)}\otimes S(a)_{(0)}$.

Proof. (1) It is the second assertion in [10, Proposition 4.3].

(2) It is a special case of [10, Proposition 4.3].

(3) For any $a\in A$, by Lemma 2.4, we have

$$\begin{aligned} S(a)_{(-1)}\otimes S(a)_{(0)} &= a_2S(a_1)_{1'}\otimes S(a_1)_{2'}=S(a_1)_{1'}a_2\otimes S(a_1)_{2'} \\ &= S(a_1)a_{21'}\otimes S(a_{22'})=a_{(-1)}\otimes S(a_{(0)}). \end{aligned}$$

So, (3) is proved. □

PROPOSITION 2.9. *Let* (A,Δ,Δ') *be a commutative Hopf co-brace. Then, the given map*

$$c:A\otimes A\to A\otimes A,\quad c(x\otimes y)=x_{(-1)}y_{[0]}\otimes x_{(0)}y_{[1]},$$

is a solution of the braid equation.

Proof. It is established and proved in the proof of [10, Proposition 4.14]. □

Definition 2.10. Let H and A be Hopf algebras. Assume that A be an H-comodule coalgebra. A *bijective 1-cocycle* is an algebra isomorphism $\pi:A\to H$ such that

$$\pi(a)_1\otimes\pi(a)_2=\pi(a_1)a_{2(-1)}\otimes\pi(a_{2(0)}),$$

for any $a\in A$.

In fact, it is a dualization of [2, Definition 1.10].

Remark 2.11. (1) Any bijective 1-cocycle π satisfies $\varepsilon_H\pi=\varepsilon_A$.

(2) Let $\pi:A\to H$ and $\eta:B\to K$ be two bijective 1-cocycles. A morphism between these bijective 1-cocycles is a pair (f,g) of Hopf algebra maps $f:K\to H$, $g:B\to A$, such that the following conditions are satisfied:

$$\begin{aligned} &\pi g=f\eta, \\ &g(b)_{(-1)}\otimes g(b)_{(0)}=f(b_{(-1)})\otimes g(b_{(0)}), \end{aligned}$$

for any $b\in B$. It is easy to see that bijective 1-cocycles form a category. Fix a Hopf algebra A, we assume that $\mathcal{C}(A)$ is the full subcategory of the category of bijective 1-cocycles with objects $\pi:A\to H$.

THEOREM 2.12. *Let* A *be a Hopf algebra. Then, the full subcategory* $\mathcal{HCB}(A)$ *of Hopf co-braces is equivalent to the full subcategory* $\mathcal{C}(A)$ *of bijective 1-cocycles.*

Proof. It is a dualization of [2, Theorem 1.12]. We leave all the details to the reader. □

3. HOPF CO-BRACE AND HOPF MATCHED PAIR

In this section, we mainly build a correspondence between Hopf co-braces and Hopf matched pairs, and prove that the full subcategory $\mathcal{HCB}(A)$ is equivalent to the category $\mathcal{M}(A)$.

In what follows, we give the dualization of the classical definition of matched pair of Hopf algebras, and it is also the third case considered in [5, Corollary 2.17], when the involved monoidal category is Vect^{op}.

Definition 3.1. Let A and H be Hopf algebras. A *Hopf matched pair* is a pair (A, H) with two coactions

$$H \xrightarrow{\varphi} H \otimes A \xleftarrow{\rho} A$$

such that (A, ρ) is a left H-comodule algebra, (H, φ) a right A-comodule algebra, and the following compatibilities hold:

$(HM1)$ $a_{(-1)}\varepsilon_A(a_{(0)}) = \varepsilon_A(a)1_H, \ \ \varepsilon_H(h_{[0]})h_{[1]} = \varepsilon_H(h)1_A,$

$(HM2)$ $a_{(-1)} \otimes a_{(0)1} \otimes a_{(0)2} = a_{1(-1)}a_{2(-1)[0]} \otimes a_{1(0)}a_{2(-1)[1]} \otimes a_{2(0)},$

$(HM3)$ $h_{[0]1} \otimes h_{[0]2} \otimes h_{[1]} = h_{1[0]} \otimes h_{1[1](-1)}h_{2[0]} \otimes h_{1[1](0)}h_{2[1]},$

$(HM4)$ $h_{[0]}a_{(-1)} \otimes h_{[1]}a_{(0)} = a_{(-1)}h_{[0]} \otimes a_{(0)}h_{[1]},$

for any $a \in A, h \in H$, where $\rho(a)$ is denoted by $a_{(-1)} \otimes a_{(0)}$ and $\varphi(h)$ denoted by $h_{[0]} \otimes h_{[1]}$.

Example 3.2. (1) Let $A = k[g, g^{-1}, x]$ be a Hopf algebra as in Example 2.3, and let $H = k[X, a^{\pm}, b^{\pm}]$ a Hopf algebra in [11] with the following structures:

$$\begin{aligned} \Delta(a) &= a \otimes a, \ \Delta(b) = b \otimes b, \ \Delta(X) = X \otimes ab + ab \otimes X \\ \varepsilon(a) &= \varepsilon(b) = 1, \ \varepsilon(X) = 0, \\ S(a) &= a^{-1}, \ S(b) = b^{-1}, \ S(X) = -a^{-2}b^{-2}X. \end{aligned}$$

Then, it is easy to get a Hopf matched pair (A, H, ρ, φ) with coactions as follows:

$$\begin{aligned} \rho(g) &= 1 \otimes g, \ \rho(x) = a \otimes x, \\ \varphi(X) &= X \otimes g, \ \varphi(a) = a \otimes 1, \ \varphi(b) = b \otimes 1. \end{aligned}$$

(2) Let H and A be Hopf algebras. An invertible element $R = R'_i \otimes R''_i$ in $H \otimes A$ is called a weak R-matrix of H and A in [9] if the following conditions are satisfied:

$(WM1)$ $(\Delta \otimes id)(R) = R'_i \otimes r'_i \otimes R''_i r''_i,$

$(WM2)$ $(id \otimes \Delta)(R) = R'_i r'_i \otimes r''_i \otimes R''_i,$

where $r = r'_i \otimes r''_i$ is a copy of R.

Then, by Lemma 1.3 in [9], (H, A, ρ, φ) is a Hopf matched pair with the coactions as follows:

$$\rho : H \to A \otimes H, \rho(h) = \tau(R)(1 \otimes h)\tau(R^{-1}),$$
$$\varphi : A \to A \otimes H, \varphi(a) = \tau(R)(a \otimes 1)\tau(R^{-1}),$$

where τ is the twisted map and R^{-1} the inverse of R.

PROPOSITION 3.3. *Let (A, Δ, Δ') be a commutative Hopf co-brace. Then, $(A_{\Delta'}, A_{\Delta'})$ is a Hopf matched pair with coactions as follows:*

$$\begin{aligned} \rho(a) &\equiv a_{(-1)} \otimes a_{(0)} = S(a_1)a_{21'} \otimes a_{22'}, \\ \varphi(a) &\equiv a_{[0]} \otimes a_{[1]} = T(a_{1'})_{(-1)}a_{2'} \otimes T(a_{1'})_{(0)}a_{3'} \\ &= S(T(a_{1'})_1)T(a_{1'})_{21'}a_{2'} \otimes T(a_{1'})_{22'}a_{3'}, \end{aligned}$$

for any $a \in A$.

Proof. It is a dualization of [2, Proposition 3.1]. We leave all the details to the reader. □

PROPOSITION 3.4. *Let (A, Δ') be a commutative Hopf algebra with antipode T. Assume that (A, A) is a Hopf matched pair with coactions ρ and φ, such that*

$$a_{1'} \otimes a_{2'} = a_{1'(-1)}a_{2'[0]} \otimes a_{1'(0)}a_{2'[1]} \tag{3.1}$$

holds. Then, (A, Δ, Δ') is a commutative Hopf co-brace with

$$\Delta(a) \equiv a_1 \otimes a_2 = a_{1'}T(a_{2'(-1)}) \otimes a_{2'(0)},$$

$$S(a) = a_{(-1)}T(a_{(0)}),$$

for all $a \in A$.

Proof. It is a dualization of [2, Proposition 3.2]. We leave all the details to the reader. □

Let (A, Δ) be a commutative Hopf algebra with antipode S. Let $\mathcal{M}(A)$ be the category with objects Hopf matched pairs (A, A) such that the condition (3.1) is satisfied, and all morphisms Hopf algebra homomorphisms $f : A \to A$ such that $\rho f(a) = (f \otimes f)\rho(a)$, $\varphi f(a) = (f \otimes f)\varphi(a)$, for all $a \in A$.

THEOREM 3.5. *Let (A, Δ) be a commutative Hopf algebra with antipode S. Then, the full subcategory $\mathcal{HCB}(A)$ of Hopf co-braces is equivalent to the category $\mathcal{M}(A)$ of Hopf matched pairs.*

Proof. We have two functors as follows:

$$\begin{array}{rcl} F:\mathcal{HCB}(A) & \to & \mathcal{M}(A), \quad F((A,\Delta,\Delta')) = (A,A), \\ F(f) & = & f, \end{array}$$

where (A,A) is the Hopf matched pair as in Proposition 3.3.

$$\begin{array}{rcl} G:\mathcal{M}(A) & \to & \mathcal{CB}(A), \quad G((A,A)) = (A,\Delta,\Delta'), \\ G(f) & = & f, \end{array}$$

where (A,Δ,Δ') is a Hopf co-brace as in Proposition 3.4.

By a direct calculation, we can show that $\mathcal{HCB}(A)$ is equivalent to $\mathcal{M}(A)$.
□

4. HOPF CO-BRACE ON BICROSSED COPRODUCT

In this section, we mainly construct Hopf co-braces on bicrossed coproducts.

Assume that (A,H,ρ,φ) is a Hopf matched pair in Definition 3.1, and give $A\otimes H$ the tensor algebra structure. Define a comultiplication on $A\otimes H$ as follows: for all $a\in A, h\in H$,

$$\widetilde{\Delta}_{A\otimes H}(a\otimes h) = (a_1\otimes a_{2(-1)}h_{1[0]})\otimes(a_{2(0)}h_{1[1]}\otimes h_2).$$

Then, by [9], $A\otimes H$ is a Hopf algebra, whose antipode is given by

$$\widetilde{S}(a\otimes h) = S_A(h_{[1]})S_A(a_{(0)})\otimes S_H(h_{[0]})S_H(a_{(-1)}).$$

In what follows, we call the Hopf algebra a bicrossed coproduct of A and H, and denote it by $A\bowtie H$, whose comultiplication is denoted by $\widetilde{\Delta}$.

PROPOSITION 4.1. *Let (A,Δ,Δ') be a Hopf co-brace, and H a commutative cocommutative Hopf algebra. If $(A_{\Delta'},H,\rho,\varphi)$ is a Hopf matched pair, and the map ρ also makes A_Δ into a left H-comodule coalgebra. Then, $(A\otimes H,\widehat{\Delta},\widetilde{\Delta})$ is a Hopf co-brace, if and only if*

$$h_{[0]}\otimes h_{[1]1}\otimes h_{[1]2} = h_{1[0]}S(h_2)h_{3[0]}\otimes h_{1[1]}\otimes h_{3[1]}, \tag{4.1}$$

for all $h\in H$, where $\widehat{\Delta}$ denotes the comultiplication of the usual tensor coalgebra of $A_\Delta\otimes H$, i.e. for all $a\in A, h\in H$,

$$\widehat{\Delta}_{A_\Delta\otimes H}(a\otimes h) = (a_1\otimes h_1)\otimes(a_2\otimes h_2)$$

and $\widetilde{\Delta}$ is given by

$$\widetilde{\Delta}_{A_{\Delta'}\otimes H}(a\otimes h) = (a_{1'}\otimes a_{2'(-1)}h_{1[0]})\otimes(a_{2'(0)}h_{1[1]}\otimes h_2).$$

Proof. It is a dualization of [1, Theorem 2.1]. We leave all the details to the reader. □

Remark 4.2. (1) Let (A, Δ, Δ') be a Hopf co-brace, and H a commutative cocommutative Hopf algebra. Suppose that the right $A_{\Delta'}$-comodule action of H is trivial. Then, by Definition 3.1, $(A_{\Delta'}, H, \rho, \varphi)$ is a Hopf matched pair if $(A_{\Delta'}, \rho)$ is a left H-comodule bialgebra.

It is obvious that the condition (4.1) holds. So, according to Proposition 4.1, $(A \otimes H, \widehat{\Delta}, \widetilde{\Delta})$ is a Hopf co-brace if (A, ρ) is a left H-comudule coalgebra, where the comultiplication $\widetilde{\Delta}$ is given by

$$\widetilde{\Delta}(a \otimes h) = a_{1'} \otimes a_{2'(-1)}h_1 \otimes a_{2'(0)} \otimes h_2.$$

In this case, the comultiplication $\widetilde{\Delta}$ of the bicrossed coproduct $A_{\Delta'} \bowtie H$ is actually the comultiplication of the usual smash coproduct on $A_{\Delta'} \otimes H$.

(2) Suppose that A is a Hopf algebra with comultiplication Δ. Then, (A, Δ, Δ) is a Hopf co-brace. If H is a commutative cocommutative Hopf algebra, and (A, ρ) is a left H-comudule bialgebra. Then, by the above remark, we know that the smash coproduct $(A \times H, \widehat{\Delta}, \widetilde{\Delta})$ is a Hopf co-brace.

Proposition 4.3. *Let A be a Hopf algebra, and (H, Δ, Δ') a commutative Hopf co-brace. If $(A, H_{\Delta}, \rho, \varphi)$ is a Hopf matched pair, and (A, ρ') a left $H_{\Delta'}$-comodule bialgebra (whose comodule structure is given by $\rho'(a) = a_{(-1)'} \otimes a_{(0)'}$ for $a \in A$). Then, $(A \otimes H, \widetilde{\Delta}, \bar{\Delta})$ is a Hopf co-brace, if and only if for all $a \in A, h \in H$,*

$$a_{(-1)'} \otimes a_{(0)'(-1)} \otimes a_{(0)'(0)} = a_{(-1)11'}S(a_{(-1)2})a_{(0)(-1)'} \otimes a_{(-1)12'} \otimes a_{(0)(0)'}, \tag{4.2}$$

$$h_{1'} \otimes h_{2'[0]} \otimes h_{2'[1]} = h_{1[0]11'}S(h_{1[0]2})h_{1[1](-1)'}h_2 \otimes h_{1[0]12'} \otimes h_{1[1](0)'}, \tag{4.3}$$

where $\bar{\Delta}$ denotes the comultiplication of smash coproduct on $A \otimes H_{\Delta'}$, that is, for all $a \in A, h \in H$,

$$\begin{aligned}\bar{\Delta}(a \otimes h) &\equiv (a \otimes h)_{\bar{1}} \otimes (a \otimes h)_{\bar{2}} \\ &= a_1 \otimes a_{2(-1)'}h_{1'} \otimes a_{2(0)'} \otimes h_{2'},\end{aligned}$$

and $\widetilde{\Delta}$ is given by

$$\widetilde{\Delta}_{A \otimes H_{\Delta}} = (a_1 \otimes a_{2(-1)}h_{1[0]}) \otimes (a_{2(0)}h_{1[1]} \otimes h_2).$$

Proof. It is a dualization of [1, Theorem 2.5]. We leave all the details to the reader. □

Remark 4.4. (1) Assume that H is a commutative Hopf algebra. Then, by Example 2.3, we know that $(H, \Delta, \Delta' = \Delta^{coH})$ is a commutative Hopf co-brace.

Suppose that (A, H, ρ, φ) is a Hopf matched pair, and the coaction of left $H_{\Delta'}$-comodule bialgebra on A is trivial. Then, Eq.(4.2) and Eq.(4.3) are satisfied. So, according to Proposition 4.3, the bicrossed coproduct $(A \bowtie H, \widetilde{\Delta}, \widehat{\Delta})$ is a Hopf co-brace, where $\bar{\Delta} = \widehat{\Delta}$ since the coaction of left $H_{\Delta'}$-comodule on A is trivial.

(2) Let A and H be two commutative Hopf algebras, and (A, ρ') a left $H_{\Delta^{coH}}$-comodule bialgebra. Suppose that the coaction of the left H-comodule algebra on A is trivial. Then, it is easy to see Eq.(4.2) holds, and Eq.(4.3) amounts to

$$h_{1[0]} \otimes h_2 \otimes h_{1[1]} = h_{1[0]} \otimes h_{1[1](-1)'} h_2 \otimes h_{1[1](0)'} \tag{4.4}$$

for all $h \in H$.

By Definition 3.1, (A, H, ρ, φ) is a Hopf matched pair if (H, φ) a right A-comodule bialgebra. It is obvious that $(H, \Delta, \Delta' = \Delta^{coH})$ is a Hopf co-brace. So, according to Proposition 4.3, $(A \otimes H, \widetilde{\Delta}, \bar{\Delta})$ is Hopf co-brace, whose comultiplication $\widetilde{\Delta}$ is given by

$$\widetilde{\Delta}(a \otimes h) = a_1 \otimes h_{1[0]} \otimes a_2 h_{1[1]} \otimes h_2.$$

Note that the comultiplication $\widetilde{\Delta}$ of the bicrossed coproduct $A \bowtie H$ is actually the comultiplication of the usual smash coproduct on $A \otimes H$.

Example 4.5. (1) Let H be a finite dimensional cocommutative Hopf algebra with antipode S, h_i a basis of H and h_i^* the corresponding dual basis of H^*, and let

$$R = h_i \otimes h_i^* \in H^{op} \otimes H^*.$$

Then, by [9], R is a weak R-matrix of $H^{op} \otimes H^*$ with the inverse $R^{-1} = S^{-1}(h_i) \otimes h_i^*$. So, by Example 3.2 (2), we know that $(H^{op}, H^*, \rho, \varphi)$ is a Hopf matched pair, and hence one can form the bicrossed coproduct $H^{op} \bowtie H^*$, whose comultiplication of $H^{op} \bowtie H^*$ is given by

$$\widetilde{\Delta}(x \otimes f) = x_1 \otimes h_i^* f_1 h_j^* \otimes S^{-1}(h_j) x_2 h_i \otimes f_2$$

for all $x \in H^{op}, f \in H^*$.

Therefore, according to Remark 4.4(1), the dual $(D(H)^*, \widetilde{\Delta}, \widehat{\Delta})$ of Drinfel'd double $D(H)$ is a Hopf co-brace.

(2) Let $H_4 = k\{1, g, x, gx\}$ be Sweedler's 4-Hopf algebra with char$k \neq 2$. As an algebra, H is generated by g and x with relations

$$g^2 = 1, \quad x^2 = 0, \quad xg = -gx.$$

The coalgebra structure and antipode are determined by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x,$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g, \quad S(x) = gx.$$

Let $A = kZ_2$, where Z_2 is written multiplicatively as $\{1, a\}$, and

$$R = \frac{1}{2}(1 \otimes 1 + 1 \otimes a + g \otimes 1 - g \otimes a) \in H \otimes A.$$

Then, by [9], one can easily see that R is a weak R-matrix of $H \otimes A$ with $R^{-1} = R$. So, by Lemma 1.3 in [9], we have the bicrossed coproduct $H_4 \bowtie kZ_2$, and according to Remark 4.4(1), we know that $(H_4 \bowtie kZ_2, \widetilde{\Delta}, \widehat{\Delta})$ is a Hopf co-brace.

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