GLOBAL WELL-POSEDNESS OF THE RADIAL SYMMETRY LANDAU-LIFSHITZ-GILBERT EQUATION IN DIMENSIONS 2

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The global solution of the 2-dimensional Landau-Lifshitz-Gilbert (LLG) equation on the sphere $\mathbb{S}^2$ is studied. By the Hasimoto transformation, an equivalent complex-valued equation is deduced under cylindrical symmetric coordinates. Then the global H^2 well-posedness of the Cauchy problem for this complex system with minimal regularity assumptions on the initial data is proved, and the well-posedness of the LLG equation is presented.

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1. INTRODUCTION

This paper discusses the global existence theorem for the Landau-Lifshitz-Gilbert (LLG for short) equation, which comes from the macroscopic ferromagnetic continuum. In a ferromagnet, long-wavelength magnetization S can vary internally, but at each point, its magnitude is equal to the saturation magnetization S_s. The LLG equation, introduced by Landau and Lifshitz [13], predicts the spin of S in response to torque. In 1935, an earlier, but equivalent, LLG equation was written down as:

$$\frac{\partial S}{\partial t} = -\gamma S \times H_{\text{eff}} - \mu S \times (S \times H_{\text{eff}}),$$

where $S = (S_1,\, S_2,\, S_3) \in \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$; H_{eff} denotes the effective field, which is a combination of the external magnetic field, the demagnetizing field, and some quantum mechanical effects. $\times$ denotes the cross product; γ is the electron gyromagnetic ratio; and μ is a phenomenological damping parameter given by (δ is a dimensionless constant called the damping factor)

$$\mu = \delta \frac{\gamma}{S_{\text{s}}}.$$

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Let us consider the simplest situation of an LLG equation in which the effects of anisotropy and external magnetic field have not been included. After time scaling (here we set $\alpha^2+\beta^2=1$ and $\alpha\in[0,\,1]$), this simplest case of the LLG equation takes the form:

$$\frac{\partial S}{\partial t}=\alpha S\times\Delta S-\beta S\times(S\times\Delta S)\,, \tag{1}$$

where the α-term denotes the exchange interaction, and the β-term is the Gilbert damping term.

(1) is a mixture of two well-known partial differential equations, the Schrödinger map equation ($\beta=0$) and the harmonic map heat flow ($\alpha=0$). Because the Schrödinger map equation is the simplest and the most important section of the LLG equation, other papers showing previously known results for (1) will be presented in order. By the difference method, C. Sulem, J. P. Sulem, and C. Bardos [19] established the local well-posedness of (1) for initial data in $H^s(\mathbb{R}^n)$ ($s>n/2+1$ and $n\geq 2$). The equivalent system for the Schrödinger map equation was applied by other authors to prove its well-posedness. Chang, Shatah, and Uhlenbeck [4] established global existence, uniqueness, and regularity in the energy norm for the 1-dimensional Schrödinger map equation. Under radially symmetric coordinates, they also proved similar results [4] for the radially symmetric case assuming low energy in the 2-dimensional case. W. Ding and Y. Wang [5] obtained similar results for any Schrödinger flow from R^n or from a compact Riemannian manifold into a complete Kähler manifold. For general Kähler manifolds, McGahagan [15] presented an approximation scheme for the Schrödinger maps by the wave map and established a well-posedness theorem for it. Much interest and ground-breaking work have arisen within the last decade in the study of the Schrödinger map equation. Small initial conditions will always lead to global solutions in time [2, 12, 3]. In $n\geq 4$ dimensions, Bejenaru, Ionescu, and Kenig [2] proved that the Schrödinger map equation admits a unique solution $S\in(\mathbb{R}:H_O^\infty)$ under a smallness condition near $Q\in\mathbb{S}^2$ as $\|S_0-O\|_{\dot{H}^{n/2}}\ll 1$ ($S_0\in(H_O^\infty)$). Under $n\geq 3$, Ionescu and Kenig [12] obtained the global well-posedness of maps into $\mathbb{S}^2$ with small data in the critical Besov spaces $\dot{B}_O^{n/2}(R^n,\mathbb{S}^2)$. Similarly, global well-posedness with small critical Sobolev norms for $n\geq 2$ was established by Bejenaru, Ionescu, Kenig, and Tataru [3]. However, the Schrödinger map equation with large data is a much more complex problem. Near the collection of families, O_m finite-energy stationary solutions for integer $m\geq 1$, asymptotic stability and blow-up for Schrödinger map equations have been considered by many authors [17, 18, 8].

As background for the LLG equation (1), a few studies presenting previously known results are listed in order. F. Alouges and A. Soyeur [1] established some necessary conditions for the existence of a global weak solution for the LLG equation in $n = 3$. They also proved that, if S satisfies the Neumann boundary conditions, then there are infinitely many weak solutions. If $n = 1$, Guo and Huang [7] established the existence of unique smooth solutions by means of the technique of spatial differences under the periodic boundary condition setting. In dimension $n = 3$, A. Huber [11] proved that there exists a time-periodic solution belonging to $C^1(\mathbb{R}, L^2(\Omega, \mathbb{R}^3)) \cap C(\mathbb{R}, H^2_N(\Omega, \mathbb{R}^3))$, where the bounded domain $\Omega \subset \mathbb{R}^3$ and "$\mathcal{N}$" stands for homogeneous Neumann boundary conditions.

In dimension $n \geq 3$, the LLG equation becomes super-critical with respect to the L^2 norm of $|\nabla S_\epsilon|$ due to $\|\nabla S_\epsilon\|_{L^2_x} \to 0$ as $\epsilon \to 0$. Guo and Hong [6] proved a global existence theorem for solutions of the LLG equation from some n-dimensional manifold M into the $\mathbb{S}^2$ based on links between harmonic maps and the solutions of the LLG equation. For a smallness initial condition on the gradient, the global well-posedness on scaling invariant homogeneous Sobolev space for (1) has been established in $n \geq 3$ by Melcher [16]. According to the work of Melcher [16], there exists a global smooth solution of (1) under the condition $S_0 - S_\infty \in H^1 \cap W^{1,n}$ and small $\|\nabla S_0\|_{L^n}$.

Although one can prove the global existence of weak (or even smooth) solutions of the LLG equation, the smallness condition does not always mean that the global solution exists. In fact, the question of regularity and uniqueness of weak solutions is a delicate problem that depended on the spatial dimension. Much like the Schrödinger map equation, the LLG equation is energy-critical in dimension $n = 2$. This means that the scaling symmetry $S_\epsilon(t, x) = S(t/\epsilon^2, x/\epsilon)$ and the L^2 norm for $|\nabla S_\epsilon|$ are conserved. In dimension 2, the global existence of weak solutions with at most finitely many singularities and the uniqueness among energy non-increasing solutions has been established by [9, 10] based on the Ginzburg-Landau approximations method. In dimension 3, Melcher [14] proved the existence of partially regular weak solutions for the Landau-Lifshitz equation. Hence, the investigation of more flexible and conceptually more adapted approaches would be desirable for obtaining the $n \leq 5$ partially regular and resolving the singularity issue. In this note, we establish a relationship between the LLG equation and the complex Ginzburg-Landau equations. This observation is inspired by recent developments in the context of Schrödinger maps and the LLG equation, as mentioned above. It enables us to prove global existence and regularity in $n = 2$ under a smallness condition. Here, we state our main theorem as follows:

Theorem 1.1. *Assuming $n = 2$ and initial data $S_0 \in H^2_{\mathbb{R}^n}$ for the radial LLG equation*

$$S_t - \alpha S\times\left(S_{1rr}+\frac{1}{r}S_{1r}\right)+\beta S\times\left[S\times\left(S_{1rr}+\frac{1}{r}S_{1r}\right)\right]=0, \tag{2}$$

there exists a global small initial solution ($|\partial_r S_0|$ small enough in L^2 norm):

$$S\in C(\mathbb{R},H^2_{\mathbb{R}^2})\bigcap L^4_{\mathrm{loc}}(\mathbb{R},W^{2,4}_{\mathbb{R}^2}).$$

Remark 1.2. For Theorem 1.1, some comments can be made as follows.

1. If some prerequisites are imposed on the LLG equation, the small initial data solution will become a global one. As seen in the work of Melcher [16], a condition $S_0 - S_\infty \in H^1\cap W^{1,n}$ accompanied with small $\|\nabla S_0\|_{L^n}$ is a compulsory component to deduce the global solvability of the equation. In this work, Theorem 1.1 removes the constraint $S_0 - S_\infty$.

2. Under the normal coordinates, Bejenaru, Ionescu, Kenig, and Tataru [3] have obtained global well-posedness with small data in the critical Sobolev space $\dot{H}^{n/2}$ for the Schrödinger map equation. This indicates that a higher-regularity initial condition is needed in the higher spatial dimensions to prove global well-posedness. Two other papers also confirm this point of view; readers are referred to [12] and [2]. Although these papers all deal with the Schrödinger map rather than the LLG equation, we believe that their results can be extended to the LLG system.

The structure of this paper is as follows. Section 2 presents the deduction of the equivalent complex Ginzburg-Landau type equation from the LLG system. Section 3 establishes some basic estimates of the Ginzburg-Landau semigroup and proves global solvability for the 2-dimensional LLG equation.

2. EQUIVALENT EQUATIONS OF THE LLG EQUATION

This section presents the deduction of an equivalent system under the Hasimoto transformation. This will be used to prove the global existence theorem for the LLG equation in Section 3.

To understand the LLG equation mathematically from another aspect, a change of coordinates, referred to as the Hasimoto transformation, is a fundamental tool. Under this transformation, the LLG equation becomes a single complex system as a nonlinear Ginzburg-Landau equation. The process of deducing this equivalent transformation takes a differential geometry approach. By mapping (2) onto a moving helical space curve in Euclidean space, this new system can be derived. In Euclidean space, the curvature and torsion are defined respectively as:

$$\kappa=(S_r\cdot S_r)^{\frac{1}{2}} \quad \text{and} \quad \tau=\frac{S\cdot(S_r\times S_{rr})}{\kappa^2}. \tag{3}$$

To maintain the parallel characteristic of the two vectors, S is mapped onto the unit tangent vector e_1 to obtain the time evolution equation of e_1 as follows:

$$\frac{\partial}{\partial t}e_1 - \alpha e_1 \times \Delta e_1 + \beta e_1 \times (e_1 \times \Delta e_1) = 0,$$

or equivalently in the form:

$$\frac{\partial}{\partial t}e_1 - \alpha e_1 \times \Delta e_1 - \beta[\Delta e_1 - (e_1 \cdot \Delta e_1)e_1] = 0, \tag{4}$$

where

$$\Delta e_1 = e_{1rr} + \frac{1}{r}e_{1r}.$$

The vectors have a right-hand coordinate relationship, as follows:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_2 \times e_3 \\ e_3 \times e_1 \\ e_1 \times e_2 \end{pmatrix}. \tag{5}$$

The spatial derivatives of the orthogonal basis e_i $(i = 1, 2, 3)$ can be represented by the Frenet formula of the moving frame method as follows:

$$\frac{\partial}{\partial r}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{6}$$

The time derivatives of the space vectors e_i are given by:

$$\frac{\partial}{\partial t}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{7}$$

To deduce more relationships about curvature and torsion, we need to use (6)-(7) and the compatibility condition:

$$\frac{\partial}{\partial t \partial r}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \frac{\partial}{\partial r \partial t}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

to give

$$\kappa_t = -\tau\,\omega_2 - \omega_{3r}, \tag{8}$$

$$\tau_t = \omega_2 \kappa - \omega_{1r} \tag{9}$$

and

$$\omega_1 = \frac{\tau\,\omega_3}{\kappa} - \frac{\omega_{2\,r}}{\kappa}. \tag{10}$$

In addition, we can resolve from (4)-(7) to obtain:

$$\omega_2 = \alpha \left[\kappa_t + \frac{\kappa}{r} \right] + \beta \kappa \tau, \tag{11}$$

and

$$\omega_3 = \alpha \kappa \tau - \beta \left[\kappa_t + \frac{\kappa}{r} \right] \tag{12}$$

respectively.

Combining (11)-(12) with (10) yields:

$$\omega_1 = \alpha \left[\tau^2 - \frac{\kappa_{rr}}{\kappa} + \frac{1}{r^2} - \frac{\kappa_r}{\kappa\, r} \right] - \beta \left[\frac{2\,\kappa_r \tau}{\kappa} + \frac{\tau}{r} + \tau_r \right], \tag{13}$$

Substituting (11)-(12) into (8) gives:

$$\kappa_t = \alpha \left[-2\,\kappa_r \tau - \frac{\tau\,\kappa}{r} - \kappa\,\tau_r \right] - \beta \left[-\kappa\,\tau^2 - \kappa_{rr} - \frac{\kappa}{r^2} + \frac{\kappa_r}{r} \right], \tag{14}$$

Combining (11), (13) and (9) yields:

$$\begin{aligned} \tau_t \;\; = \;\; & -\left(\alpha \left[\tau^2 - \tfrac{\kappa_{rr}}{\kappa} + \tfrac{1}{r^2} - \tfrac{\kappa_r}{\kappa\, r} \right] - \beta \left[\tfrac{2\kappa_r \tau}{\kappa} + \tfrac{\tau}{r} + \tau_r \right] \right)_r \\ & + \left(\alpha \left[\kappa_t + \tfrac{\kappa}{r} \right] + \beta \kappa\, \tau \right) \kappa. \end{aligned} \tag{15}$$

Using the complex Hasimoto transformation, it can be assumed that:

$$\Phi = \frac{\kappa}{2} \exp \left[i \int_0^r \tau(t, \tilde{r}) \mathrm{d}\tilde{r} \right], \tag{16}$$

Calculating the time derivative of (16) leads to:

$$\Phi_t = \frac{1}{2} \left[\kappa_t \mathrm{e}^{i \int_0^r \tau \mathrm{d}\tilde{r}} + i \kappa \mathrm{e}^{i \int_0^r \tau \mathrm{d}\tilde{r}} \int_0^r \tau_t \mathrm{d}\tilde{r} \right], \tag{17}$$

where

$$R(t) = \left(\int \tau_t \mathrm{d}\tilde{r} \right)_{r=0}.$$

By (14) and (15), (17) can be transformed into the nonlinear Ginzburg-Landau equation as follows:

$$\begin{aligned} & i\Phi_t - (i\beta - \alpha) \left(\Phi_{rr} + \tfrac{\Phi_r}{r} - \tfrac{\Phi}{r^2} \right) + 4\,\beta\,\Phi \int_0^r \Im \left(\overline{\Phi} \Phi_r \right) \mathrm{d}r \\ & + \alpha \left(2\,|\Phi|^2\,\Phi + 4\Phi \int_0^r \tfrac{|\Phi|^2}{r} \mathrm{d}r \right) - \Phi R(t) = 0. \end{aligned} \tag{18}$$

$R(t)$ is a real-valued function in (18). It can be used to rearrange (18) to be a single expression. More clearly, the transformation

$$Q = \Phi \exp \left[-i \int_0^t R(t') \mathrm{d}\tilde{t'} \right], \tag{19}$$

can be used to re-write (18) as:

$$iQ_t - (i\beta - \alpha)\left(Q_{rr} + \tfrac{Q_r}{r} - \tfrac{Q}{r^2}\right) + 4\,\beta\,Q\int_0^r \Im\left(\overline{Q}Q_r\right)\,\mathrm{d}r \tag{20}$$
$$+\alpha\left(2\,|Q|^2\,Q + 4Q\int_0^r \tfrac{|Q|^2}{r}\,\mathrm{d}r\right) = 0.$$

The functions $(t,r) \mapsto \Phi$ (or $(t,r) \mapsto Q$) are said to be the Hasimoto transformation. Under these transforms, two equations can be gathered in a complex form. It is difficult to go back directly from solutions of the complex system to the LLG equation by the Hasimoto transformation. Nonetheless, some estimates of S can be obtained by the complex variable Q (or Φ).

3. GLOBAL EXISTENCE OF THE SOLUTION

Equation (20) is a complex equation. The kernel of the n-dimensional linear Ginzburg–Landau equation

$$iQ_t - (i\beta - \alpha)\,\Delta Q = 0$$

is a mixture of the Poisson kernel and the heat kernel as follows:

$$\mathbb{K}_\alpha(r,t) = \frac{e^{-\frac{|x|^2}{4(\beta+i\alpha)t}}}{\sqrt{2\pi(\beta+i\alpha)t}},$$

which leads to a semigroup

$$(U_\alpha(t)\varphi)(x) = \int_{R^n} \mathbb{K}_\alpha(x-x',t)\varphi(x')\mathrm{d}x'. \tag{21}$$

In another setting, (21) can also be written as $U_\alpha(t)\varphi = e^{(\beta+i\alpha)t\Delta}\varphi$. Much like the Schrödinger semigroup $e^{it\Delta}\varphi$, the Ginzburg-Landau semigroup has a certain smoothing effect on the Schrödinger derivatives. It is well known that for $2 \le p \le \infty$ and p' with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\left\|e^{(\beta+i\alpha)t\Delta}\varphi\right\|_{L^p_x} \le \left\|e^{i\alpha t\Delta}\varphi\right\|_{L^p_x} \le_C |t|^{-\frac{n}{2}(1-\frac{2}{p})}\|\varphi\|_{L^{p'}_x}, \tag{22}$$

which leads by dispersive estimate to:

$$\left\|e^{it\Delta}\varphi\right\|_{L^\infty_x} \le_C |t|^{-n/2}\|\varphi\|_{L^1_x}$$

and complex interpolation from

$$\left\|e^{i\alpha t\Delta}\varphi\right\|_{L^2_x} = \|\varphi\|_{L^2_x},$$

denotes

$$\mathcal{A}\varphi(t) = \int_0^t U_\alpha(t-t')\varphi(t')\,\mathrm{d}t'.$$

From the estimate (22), we obtain

$$\|\mathcal{A}\varphi(t)\|_{L^p_x} \leq_C \int_0^t |t-t'|^{-\frac{n}{2}(1-\frac{2}{p})} \|\varphi(t')\|_{L^{p'}_x} \, \mathrm{d}t'. \tag{23}$$

The right-hand side of (23) is a convolution $H * G$, where

$$G(t) = \|\varphi(t)\|_{L^{p'}_x} \quad \text{and} \quad H(t) = C|t|^{-\frac{n}{2}(1-\frac{2}{p})}.$$

If we set

$$\frac{1}{\gamma} = \frac{n}{2}(1-\frac{2}{p}), \quad \gamma > 1,$$

by the weak Young inequality,

$$\|H * G\|_{L^q_t} \leq_C \|H\|_{L^\gamma_t} \|G\|_{L^{q'}_t}. \tag{24}$$

The mixed space-time Lebesgue spaces are defined as the set of all functions φ with

$$\|\varphi\|_{L^q_t L^p_x} = \left(\int \|\varphi(t')\|^q_{L^p_x} \, \mathrm{d}t' \right)^{\frac{1}{q}},$$

where $(p,\ q)$ are strict Strichartz pairs that satisfy $2 < q \leq \infty$, $2 \leq p \leq \infty$ and

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2}. \tag{25}$$

(24) indicates a control of the norm

$$\|Q\|_{L^q_t L^p_x} \leq_C \|\varphi\|_{L^{q'}_t L^{p'}_x}$$

for the Ginzberg-Landau type equation

$$iQ_t - (i\beta - \alpha)\,\Delta Q = \varphi. \tag{26}$$

Lemma 3.1. *Let $(q,\ p)$ be strict Strichartz pairs (see (25)). Then there exists a constant $C > 0$ such that the dispersive inequality for the linear Ginzburg-Landau propagator can be expressed as:*

$$\left\| e^{(\beta+i\alpha)t\Delta} \varphi \right\|_{L^p_x} \leq_C |t|^{-\frac{n}{2}(1-\frac{2}{p})} \|\varphi\|_{L^{p'}_x}$$

and the Hardy-Littlewood-Sobolev inequality can be expressed as follows:

$$\|\mathcal{A}\varphi(t)\|_{L^q_t L^p_x} \leq_C \|\varphi\|_{L^{q'}_t L^{p'}_x}.$$

To prove the regularity of the radial equation (20) in n-space dimensions, an estimate for Q is needed. Obviously, Lemma 3.1 indicates the control of the norm of Q that comes from (20). We claim that the variation of constants formula defines a function Q in (26) that satisfies the Strichartz estimate.

Lemma 3.2. *Let $(q,\ p)$ be strict Strichartz pairs. In n-space dimensions, Q is the solution to the linear equation*

$$iQ_t - (i\beta - \alpha)\,(\Delta Q) = g, \qquad Q(0) = Q_0. \tag{27}$$

Then Q satisfies the following space-time Strichartz estimate:

$$\|Q\|_{L_t^\infty L_x^2} + \|Q\|_{L_t^q L_x^p} \leq_C \|Q_0\|_{L_x^2} + \|g\|_{L_t^{q'} L_x^{p'}}\,. \tag{28}$$

More clearly, (28) is a combination of Lemma 3.1 and the Duhamel formula of (27), as follows:

$$Q = U_\alpha(t)Q_0 + \int_0^t U_\alpha(t-t')g(t')\,\mathrm{d}t'.$$

Before we proceed to present the theorem, we would like to make a few remarks concerning Lemmas 3.1 and 3.2.

Remark 3.3. In Lemmas 3.1 and 3.2, the dispersive inequality, Hardy – Littlewood – Sobolev inequality, and space-time Strichartz estimate for the linear Ginzburg-Landau semigroup apply to the extreme case $\beta = 0$. Furthermore, the following theorems of each section remain true for the Schrödinger map equation.

Lemma 3.2 (or Lemma 3.1) will be used to deduce an estimate of the nonlinear terms of the complex equation (20). On 2-dimensional space, we will prove a global result Q as follows.

Theorem 3.4. *Assuming initial data $S_0 \in H^2_{\mathbb{R}^2}$ (or $Q_0 \in H^1_{\mathbb{R}^2}$) and $\|\partial_r S_0\|_{L^2} \ll 1$ for the radial LLG Equation:*

$$S_t - \alpha S\times\left(S_{1rr} + \frac{1}{r}S_{1r}\right) + \beta S\times\left[S\times\left(S_{1rr} + \frac{1}{r}S_{1r}\right)\right] = 0,$$

which can be equivalently written as

$$iQ_t - (i\beta - \alpha)\left(Q_{rr} + \tfrac{Q_r}{r} - \tfrac{Q}{r^2}\right) + 4\,\beta\,Q\int_0^r \Im\left(\overline{Q}Q_r\right)\,\mathrm{d}r$$

$$+\alpha\left(2\,|Q|^2\,Q + 4\,Q\int_0^r \tfrac{|Q|^2}{r}\,\mathrm{d}r\right) = 0,$$

there exists a global small initial data solution

$$Q \in C(\mathbb{R}, H^1_{\mathbb{R}^2}) \bigcap L^4_{\mathrm{loc}}(\mathbb{R}, W^{1,4}_{\mathbb{R}^2}).$$

Proof. Under the 2-dimensional space, we use the transformation $W = e^{i\theta}Q$ (or $Q = e^{-i\theta}W$) to re-write (20) as follows:

$$iW_t - (i\beta - \alpha)\,\Delta W = G, \tag{29}$$

where

$$\Delta W = W_{rr} + \frac{W_r}{r} + \frac{W_{\theta\theta}}{r^2}$$

and

$$G = -4\,\beta\, W \int_0^r \Im\left(\overline{W} W_r\right)\, \mathrm{d}r - \alpha\left(2\,|W|^2\, W + 4\, W \int_0^r \frac{|W|^2}{r}\, \mathrm{d}r\right).$$

An $L^2_{\mathbb{R}^2}$ estimate of Q is needed. Let Q be a suitable smooth solution of (29). Multiplying this by $\overline{W}$, integrating, and taking the image part yields:

$$\frac{1}{2}\|W(t)\|^2_{L^2_{\mathbb{R}^2}} - \frac{1}{2}\|W_0\|^2_{L^2_{\mathbb{R}^2}} + \beta \int_{\mathbb{R}^2} \|\nabla W(\tau)\|^2_{L^2_{\mathbb{R}^2}}\, \mathrm{d}\tau = 0. \tag{30}$$

This also indicates a control of the norm as follows:

$$\|W(t)\|^2_{L^2_{\mathbb{R}^2}} \leq \|W_0\|^2_{L^2_{\mathbb{R}^2}}$$

or

$$\|Q(t)\|^2_{L^2_{\mathbb{R}^2}} \leq \|Q_0\|^2_{L^2_{\mathbb{R}^2}}. \tag{31}$$

Under the constraint $n = 2$, from Lemma 3.2,

$$\|W\|_{L^\infty L^2_{\mathbb{R}^2}} + \|W\|_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq_C \|W_0\|_{L^2_{\mathbb{R}^2}} + \|G\|_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}}. \tag{32}$$

Combining (32) with $||W||_{L_{\mathbb{R}^2}} = ||Q||_{L_{\mathbb{R}^2}}$, we obtain:

$$\begin{aligned} ||W||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq_C \; & ||Q_0||_{L^2_{\mathbb{R}^2}} + ||Q||^3_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \\ & + ||Q \textstyle\int_0^r \frac{|Q|^2}{r}\, \mathrm{d}r||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} + ||Q \int_0^r \Im\left(\overline{W} W_r\right)\, \mathrm{d}r||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}}. \end{aligned} \tag{33}$$

The $L^4_{\mathbb{R}\times\mathbb{R}^2}$ norm estimate can then be deduced for the last two terms in (33). First, by (31), the Young inequality, and the Hardy inequality:

$$\begin{aligned} ||Q \textstyle\int_0^r \frac{|Q|^2}{r}\, \mathrm{d}r||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} & \leq ||Q||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} || \textstyle\int_0^r \frac{|Q|^2}{r}\, \mathrm{d}r||_{L^2_{\mathbb{R}\times\mathbb{R}^2}} \\ & \leq_C ||Q||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} |||Q|^2||_{L^2_{\mathbb{R}\times\mathbb{R}^2}} \\ & \leq_C ||Q||^3_{L^4_{\mathbb{R}\times\mathbb{R}^2}}. \end{aligned} \tag{34}$$

Second, by (31) and the Young inequality, the norm for the last term of

(33) can be estimated as:

$$
\begin{aligned}
||Q \textstyle\int_0^r \Im\left(\overline{W}W_r\right)\,\mathrm{d}r||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} &\leq ||Q \textstyle\int_0^r \Im\left(\overline{W}W_r\right)\,\mathrm{d}r||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} \\
&\leq_C ||Q \textstyle\int_0^r \Re\left(\overline{W}W_r\right)\,\mathrm{d}r||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} \\
&\leq_C ||Q|W|^2||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} \\
&\leq_C ||Q||^3_{L^4_{\mathbb{R}\times\mathbb{R}^2}}.
\end{aligned} \tag{35}
$$

According to (35), the structure of $Q\int_0^r \Im\left(\overline{W}W_r\right)\,\mathrm{d}r$ is similar to the cubic nonlinearity $|W|^2\,W$. According to (34) and (35),

$$
||Q\int_0^r \frac{|Q|^2}{r}\,\mathrm{d}r||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} + ||Q\int_0^r \Im\left(\overline{W}W_r\right)\,\mathrm{d}r||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} \leq_C ||Q||^3_{L^4_{\mathbb{R}\times\mathbb{R}^2}}. \tag{36}
$$

Combining (33) and (36), the norm of Q satisfies:

$$
||Q||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq_C ||Q_0||_{L^2_{\mathbb{R}^2}} + ||Q||^3_{L^4_{\mathbb{R}\times\mathbb{R}^2}}. \tag{37}
$$

If $||Q_0||_{L^2_{\mathbb{R}^2}}$ is small enough, combining it with the iteration argument (using different initial times T_j $(j = 0, 1, 2...)$, $T_0 = 0$ to make sure that C is small enough in each iteration step), (37) indicates a global estimate (for $Q \in L^4_{\mathbb{R}\times\mathbb{R}^2}$):

$$
||Q||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq_C ||Q_0||_{L^2_{\mathbb{R}^2}}
$$

or

$$
||\partial_r S||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq_C ||\partial_r S_0||_{L^2_{\mathbb{R}^2}}.
$$

To achieve higher-order estimates (on H^2) and deduce the well-posedness of the classical solutions, one can differentiate (with respect to $x_i, i = 1, 2$) (29) and define $V = W_{x_i}$ to obtain:

$$
iV_t - (i\beta - \alpha)\,\Delta V = \widetilde{G}, \tag{38}
$$

where

$$
|\widetilde{G}| \leq_C \frac{|Q|^3}{r} + Q|Q|^2 + |V|\left(|Q|^2 + \int_0^r \Im\left(\overline{Q}Q_r\right)\,\mathrm{d}r + \int_0^r \frac{|Q|^2}{r'}\mathrm{d}r'\right).
$$

Similarly, the Duhamel formula of (38) gives the solution as follows:

$$
V = U_\alpha(t)V_0 + \int_0^t U_\alpha(t-t')\widetilde{G}(t')\,\mathrm{d}t',
$$

where V_0 denotes the initial condition of V.

According to Lemma 3.2, we obtain (similar to (32)):

$$\|V\|_{L^\infty L^2_{\mathbb{R}^2}} + \|V\|_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq_C \|V_0\|_{L^2_{\mathbb{R}^2}} + \left\|\widetilde{G}\right\|_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}}. \tag{39}$$

(39) will be applied to estimate the $\widetilde{G}$ term of (38). Because $Q(0,t)=0$, the space norm of Q/r can be controlled by

$$||\frac{Q}{r}||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq_C ||\partial_r Q||_{L^4_{\mathbb{R}\times\mathbb{R}^2}}.$$

Hence, the space norm of $\frac{|Q|^3}{r} + Q|Q|^2$ can be bounded by

$$\begin{aligned} ||\tfrac{|Q|^3}{r} + Q|Q|^2||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} &\leq ||Q||^3_{L^4_{\mathbb{R}\times\mathbb{R}^2}} + ||\tfrac{|Q|^3}{r}||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} \\ &\leq_C ||Q||^3_{L^4_{\mathbb{R}\times\mathbb{R}^2}} + ||Q||^2_{L^4_{\mathbb{R}\times\mathbb{R}^2}} ||\tfrac{Q}{r}||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \\ &\leq_C ||Q||^3_{L^4_{\mathbb{R}\times\mathbb{R}^2}} + ||Q||^2_{L^4_{\mathbb{R}\times\mathbb{R}^2}} ||V||_{L^4_{\mathbb{R}\times\mathbb{R}^2}}. \end{aligned} \tag{40}$$

$|V||Q|^2$ can be estimated by

$$|||V||Q|^2||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} \leq_C ||V||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} ||Q||^2_{L^4_{\mathbb{R}\times\mathbb{R}^2}}. \tag{41}$$

Combining the Young and Hardy inequalities, the space norm of $|V|\int_0^r \frac{|Q|^2}{r'}\mathrm{d}r'$ can be bounded by:

$$\begin{aligned} &|||V|\int_0^r \frac{|Q|^2}{r'}\mathrm{d}r'||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} \\ \leq_C &||V||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} ||\int_0^r \frac{|Q|^2}{r'}\mathrm{d}r'||_{L^2_{\mathbb{R}\times\mathbb{R}^2}} \\ \leq_C &||V||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} ||Q||^2_{L^4_{\mathbb{R}\times\mathbb{R}^2}}. \end{aligned} \tag{42}$$

By (31) and the Young inequality, our estimates for the nonlinearity $|V||Q|^2$ will suffice to show that:

$$\begin{aligned} |||V|\textstyle\int_0^r \Im\left(\overline{Q}Q_r\right)\,\mathrm{d}r||_{L^{4/3}_{\mathbb{R}\times\mathbb{R}^2}} &\leq ||V||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} ||\textstyle\int_0^r \Im\left(\overline{Q}Q_r\right)\,\mathrm{d}r||_{L^2_{\mathbb{R}\times\mathbb{R}^2}} \\ &\leq_C ||V||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} |||Q|^2||_{L^2_{\mathbb{R}\times\mathbb{R}^2}} \\ &\leq_C ||V||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} ||Q||^2_{L^4_{\mathbb{R}\times\mathbb{R}^2}}. \end{aligned} \tag{43}$$

According to the estimates (39)-(43),

(44) $\|V\|_{L^\infty L^2_{\mathbb{R}^2}} + \|V\|_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq_C \|V_0\|_{L^2_{\mathbb{R}^2}} + ||Q||^3_{L^4_{\mathbb{R}\times\mathbb{R}^2}} + ||Q||^2_{L^4_{\mathbb{R}\times\mathbb{R}^2}} ||V||_{L^4_{\mathbb{R}\times\mathbb{R}^2}}.$

Combining (32) with (44) yields:

(45) $$\|V\|_{L^\infty L^2_{\mathbb{R}^2}} + \|V\|_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq C(Q_0, V_0, T),$$

where $C(Q_0, V_0, T)$ is a bounded function depending on Q_0, V_0, and T.

In particular, we have estimates on the norm $H^1_{\mathbb{R}^2}$ for some initial data ($Q_0 \in L^2_{\mathbb{R}^2}$ (or $Q \in L^4_{\mathbb{R}^2}$) and $V_0 \in L^2_{\mathbb{R}^2}$). The above is also enough to establish the regularity claimed in Theorem 3.4: if Q_0 small enough, we obtain a global solution and prove Theorem 3.4. □

As is already known, if $||Q_0||_{L^2_{\mathbb{R}^2}}$ is small enough, combining it with the iteration argument (37) provides a global estimate (for $Q \in L^4_{\mathbb{R}^2}$):

$$||Q||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq_C ||Q_0||_{L^2_{\mathbb{R}^2}}.$$

Note that under radial coordinates, the relationship of the Q-norm and the S-norm is:

$$\|Q\|^4_{L^4_{\mathbb{R}^2}} = \int_{\mathbb{R}^2} |Q|^4 \mathrm{d}x_1 \mathrm{d}x_2 = \int_0^{+\infty} \left(\frac{1}{2}\kappa\right)^4 r\mathrm{d}r = 2^{-4} \|\partial_r S\|^4_{L^4_{\mathbb{R}^2}},$$

and therefore

$$||\partial_r S||_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq_C ||\partial_r S_0||_{L^2_{\mathbb{R}^2}}.$$

At the same time, (45) also indicates that:

$$\|Q_x\|_{L^\infty L^2_{\mathbb{R}^2}} + \|Q_x\|_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq C(Q_0, V_0, T),$$

or

$$\left\|\partial_r^2 S\right\|_{L^\infty L^2_{\mathbb{R}^2}} + \left\|\partial_r^2 S\right\|_{L^4_{\mathbb{R}\times\mathbb{R}^2}} \leq C(Q_0, V_0, T).$$

According to the above relationship of Q and S, Theorem 1.1 has already been proved by Theorem 3.4.

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