EXISTENCE RESULTS FOR PROBLEMS INVOLVING THE $p(x)$-BIHARMONIC OPERATOR

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We study the existence of solutions of classes of nonhomogeneous elliptic problems with Navier boundary condition and involving a fourth order differential operator with variable exponent and power-type nonlinearities. Our arguments to prove these results are based on the direct method of the calculus of variations, estimates of the levels of the associated energy functional and basic properties of the Lebesgue and Sobolev spaces with variable exponent.

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1. INTRODUCTION

The study of problems of elliptic equations and variational problems with $p(x)$-growth condition has attracted more and more attention in recent years. It possesses a solid background in Physics and originates from the study on electrorheological fluids and elastic mechanics. It also has wide applications in different research fields, such as image processing model, stationary thermorheological viscous flows and the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium.

In the study of classes of nonhomogeneous elliptic problems with Dirichlet boundary condition and involving a fourth-order differential operator with variables exponents and power-type nonlinearities, the first result establishes the existence of a nontrivial weak solution in the case of a small perturbation. The proof combines variational methods, including the Ekeland variational principle and the mountain pass theorem of Ambrosetti and Rabinowitz.

The second result is the existence of nontrivial weak solutions for large values of the parameter of very related eigenvalue problem. The direct method of the calculus of variations, estimates of the levels of the associated energy functional and basic properties of the Lebesgue and Sobolev spaces with variable exponent have an important role in the arguments.

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A. Ambrosetti, H. Brezis and G. Cerami [1] studied the qualitative analysis of semilinear elliptic problems involving concave and convex nonlinearities and with Dirichlet boundary condition. They proved several existence, multiplicity and nonexistence results and developed powerful topological and variational methods for the study of such nonlinear problems. In [20] and [15] related existence results are established in the case of elliptic problems with variable exponents and Dirichlet boundary condition. L. Kong [15] proved the existence of a family of eigenvalues in a neighborhood of the origin.

Additional results on higher-order problems or nonlinear partial differential equations with variable exponent can be found in the papers by G. Autuori, F. Colasuonno and P. Pucci [3], Z. Chen [7], F. Colasuonno and P. Pucci [8], A. Kratohvil and I. Necas [16], V. Lubyshev [19], P. Pucci and Q. Zhang [27].

Consider the following nonhomogeneous eigenvalue problem defined by $-\Delta_{p(x)}u = \lambda|u|^{q(x)-2}u$ in a bounded domain of $\mathbb{R}^N$ with a prescribed zero Dirichlet condition on the boundary which is smooth. The $p(x)$-Laplace operator is defined by $\Delta_{p(x)}u = \mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p(x)}-2}\nabla \mathrm{u})$ and the function $p, q \in C^0(\overline{\Omega}, \mathbb{R})$. This nonhomogeneous eigenvalue problem is studied in [20] in the subcritical setting under the basic hypothesis $1 < \min_{\overline{\Omega}} q < \min_{\overline{\Omega}} p < \max_{\overline{\Omega}} q$.

The main result in this direction is the existence of $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ we have at least one nontrivial solution. Since the associated energy functional does not have a mountain pass geometry, see A. Ambrosetti and P. Rabinowitz [2], the proof relies essentially on the Ekeland variational principle, see [11].

The original proof of the mountain pass theorem is based on several powerful deformation techniques developed by R. Palais and S. Smale [24, 23], who developed the main ideas of the Morse theory in the abstract framework of differential topology on infinite-dimensional Riemann manifolds. A simpler proof of the mountain pass theorem is due to H. Brezis and L. Nirenberg [4], who used a pseudo-gradient lemma, a perturbation argument and the Ekeland variational principle. The study initiated in [20, 32] was continued by L. Kong [15] in the framework of the $p(x)$-biharmonic operator $\Delta^2_{p(x)}$, namely $\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$.

For basic definitions and properties concerning the basic function spaces with variable exponent, we refer to the recent monographs of L. Diening, P. Hästo, P. Harjulehto and M. Ruzicka [10] and V. Rădulescu and D. Repovš [32] for related properties of Lebesgue and Sobolev spaces with variable exponents.

In a recent paper [5], the authors studied two classes of nonhomogeneous elliptic problems with Dirichlet boundary condition and involving a fourth order differential operator with variable exponent and power-type nonlinearities.

The first result establishes the existence of a nontrivial weak solution in the case of a small perturbation of the right-hand side and the proof combines variational methods, including the Ekeland variational principle and the mountain pass theorem of Ambrosetti and Rabinowitz. The second result concerns a very related eigenvalue problem involving always a fourth order differential operator with variable exponent and they proved the existence of nontrivial weak solutions. The important steps in their arguments are based on the direct method of the calculus of variations, estimates of the levels of the associated energy functional and basic properties of the Lebesgue and Sobolev spaces with variable exponent.

Consider the fourth order nonlinear elliptic equation with variable exponent and Navier boundary condition

$$(1)\qquad \begin{cases} \Delta^2_{p(x)}u + a(x)|u|^{p(x)-2}u = \lambda w(x) f(u), & x \in \Omega \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases}$$

where $a(x)$ and $w(x)$ are nonnegative potentials and the nonlinear term f behaves like

$$f(u) = |u|^{\gamma(x)-2}u - |u|^{\beta(x)-2}u,$$

where $\gamma,\ \beta > 1$ are continuous functions and under the basic hypothesis

$$(2)\qquad \gamma(x) < \beta(x) < p(x), \quad \text{for all } x \in \overline{\Omega}.$$

The main result in [15] asserts that there exists $\lambda^* > 0$ such that problem (1) has at least one nontrivial solution for any $\lambda \in (0, \lambda^*)$.

In the present paper, we establish several existence results for problems related to (1) but under some basic assumptions different from (2).

In [5] the authors considered the following nonlinear problem

$$(3)\qquad \begin{cases} \Delta^2_{p(x)}u + a|u|^{p(x)-2}u = \lambda(|u|^{\gamma(x)-2} - |u|^{\beta(x)-2})u, & x \in \Omega \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases}$$

where λ is a positive parameter and $a \geq 0$. Under two different assumptions, they showed that problem (3) has at least one nontrivial solution provided that the positive parameter λ is small enough.

In a different situation they considered a problem very close to (3). They considered the following eigenvalue nonlinear problem

$$(4)\qquad \begin{cases} \Delta^2_{p(x)}u + a|u|^{p(x)-2}u = \lambda|u|^{\gamma(x)-2}u - |u|^{\beta(x)-2}u, & x \in \Omega \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases}$$

In this case, they established a sufficient condition for the existence of nontrivial solutions provided that the parameter λ is large enough.

The variable exponent Lebesgue and Sobolev spaces are generalizations of the classical Lebesgue and Sobolev spaces, replacing the constant exponent p with an exponent function $p(\cdot)$. These spaces have been the subject of constant interest since the beginning of the 20th century both as function spaces with intrinsic interest and for their applications to problems arising in nonlinear partial differential equations and the calculus of variations.

In this paper, we will use some basic definitions and properties concerning the basic function spaces with variable exponent summarized in [5]. We refer to the recent monographs of D. Cruz-Uribe and A. Fiorenza [9], L. Diening, P. Hästo, P. Harjulehto and M. Ruzicka [10] and V. Rădulescu and D. Repovš [32] for further developments and related properties of Lebesgue and Sobolev spaces with variable exponents and their history.

In the present paper, we consider the fourth-order nonlinear elliptic equation with variable exponent and Navier boundary condition

$$\begin{cases} \Delta^2_{p_1(x)}u + \Delta^2_{p_2(x)}u + a|u|^{p_0(x)-2}u = \lambda(|u|^{\gamma(x)-2} - |u|^{\beta(x)-2})u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

$$\begin{cases} \Delta^2_{p_1(x)}u + \Delta^2_{p_2(x)}u + a|u|^{p_0(x)-2}u = \lambda|u|^{\gamma(x)-2}u - |u|^{\beta(x)-2}u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where a is a nonnegative constant potential, λ is a positive parameter and $\gamma, \beta > 1$ are continuous functions and for all $i \in \{0,1,2\}$, $p_i \in C^0(\overline{\Omega}, \mathbb{R})$. Set $p(x) = \min\limits_{i=0,1,2} p_i(x)$ and $q(x) = \max\limits_{i=0,1,2} p_i(x)$.

We study the problems (5) and (6) under one of the following hypotheses

$$1 < \min\{p(x), \beta(x)\} < \max\{q(x), \beta(x)\} < \gamma(x) < \min_{i=0,1,2} p_i^*(x) \quad \text{for all } x \in \overline{\Omega}, \tag{7}$$

$$1 < \gamma(x) < \min\{p(x), \beta(x)\} < \max\{q(x), \beta(x)\} < \min_{i=0,1,2} p_i^*(x) \quad \text{for all } x \in \overline{\Omega}. \tag{8}$$

Where, for $i \in \{0,1,2\}$, $p_i^*(x)$ denote the critical Sobolev exponents, namely

$$p_i^*(x) = \begin{cases} \dfrac{Np_i(x)}{N - 2p_i(x)} & \text{if } 2p_i(x) < N \\ +\infty & \text{if } 2p_i(x) \geq N. \end{cases} \tag{9}$$

In a four-dimensional manifold, this type of equations and similar ones arise from the problem of prescribing the so-called Q-curvature. More precisely, given (M, g) a four-dimensional Riemannian manifold, the problem consists in finding a conformal metric $\tilde{g}$ for which the corresponding Q-curvature $Q_{\tilde{g}}$ is a-priori prescribed.

The first result of this paper establishes the existence of a nontrivial weak solution in the case of a small perturbation. The proof of this result combines variational methods, including the Ekeland variational principle and the Ambrosetti-Rabinowitz mountain pass theorem. The second existence result concerns a closely related eigenvalue problem involving always a fourth order differential operator with variable exponent.

2. DEFINITIONS AND RELATED PROPERTIES OF WEAK SOLUTIONS

2.1. Function spaces with variable exponent

Consider the set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For all $p \in C_+(\overline{\Omega})$, we define

$$p^+ = \sup_{x\in\Omega} p(x) \qquad \text{and} \qquad p^- = \inf_{x\in\Omega} p(x).$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \left\{u;\ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, \mathrm{d}x < \infty\right\}.$$

This vector space is a Banach space if it is endowed with the *Luxemburg norm*, which is defined by

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} \mathrm{d}x \le 1\right\}.$$

The function space $L^{p(x)}(\Omega)$ is reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions with compact support are dense in $L^{p(x)}(\Omega)$ if $p^+ < \infty$.

Let $L^{q(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$, i.e $1/p(x) + 1/q(x) = 1$. If $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ then the following Hölder-type inequality holds:

$$\left|\int_\Omega uv\, \mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)}|v|_{q(x)}\,. \tag{10}$$

The inclusion between Lebesgue spaces also generalizes the classical framework, namely if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1 \le p_2$ in Ω then the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ holds true.

If k is a positive integer number and $p \in C_+(\overline{\Omega})$, we define the variable exponent Sobolev space by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega):\ D^\alpha u \in L^{p(x)}(\Omega),\ \text{for all } |\alpha| \le k\}.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^{N} \alpha_i$ and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N}} \,.$$

On $W^{k,p(x)}(\Omega)$ we consider the following norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)}.$$

Then $W^{k,p(x)}(\Omega)$ is a reflexive and separable Banach space (see [17, Theorem 3.1]). Let $W_0^{k,p(x)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Consider the function space E defined by

$$E = W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega).$$

Then E is a separable and reflexive Banach space if it is equipped with the norm

$$\|u\|_E = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}.$$

Cf. Kong [15, p. 251], the norms $\|u\|_E$ and $|\Delta u|_{p(x)}$ are equivalent.

If a is a positive number, define for all $u \in E$

$$\|u\|_a = \inf \left\{ \lambda > 0;\ \int_\Omega \left(\left| \frac{\Delta u}{\lambda} \right|^{p(x)} + a \left| \frac{u}{\lambda} \right|^{p(x)} \right) \mathrm{d}x \le 1 \right\} .$$

Then $\|u\|_a$ is well-defined and it is a norm which is equivalent with the norms $\|u\|_E$ and $|\Delta u|_{p(x)}$ in E.

Let $\varrho_a : E \to \mathbb{R}$ be the *modular function* defined by

$$\varrho_a(u) = \int_\Omega \left(|\Delta u|^{p(x)} + a|u|^{p(x)} \right) \mathrm{d}x. \tag{11}$$

If (u_n), $u \in E$ then the following properties are true:

$$\|u\|_a > 1 \ \Rightarrow\ \|u\|_a^{p^-} \le \varrho_a(u) \le \|u\|_a^{p^+}, \tag{12}$$

$$\|u\|_a < 1 \ \Rightarrow\ \|u\|_a^{p^+} \le \varrho_a(u) \le \|u\|_a^{p^-}, \tag{13}$$

$$\|u_n - u\|_E \to 0 \quad \Leftrightarrow \quad \varrho_a(u_n - u) \to 0. \tag{14}$$

The proofs of relations (12), (13) and (14) can be found in [6, Proposition 1]. Let $p^*(x)$ denote the *critical Sobolev exponent*, defined as in (9).

We point out that if $p, q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding $E \hookrightarrow L^{q(x)}(\Omega)$ is compact, see Kong [15, Proposition 1.3].

2.2. **Definitions and main tools**

Consider the function space E defined by

$$E = W_0^{1,q(x)}(\Omega) \cap W^{2,q(x)}(\Omega),$$

For $(u, v) \in E^2$, denote by

$$L(u,v) := \int_\Omega |\Delta u|^{p_1(x)-2}\Delta u \Delta v \mathrm{d}x + \int_\Omega |\Delta u|^{p_2(x)-2}\Delta u \Delta v \mathrm{d}x + a\int_\Omega |u|^{p_0(x)-2} uv \mathrm{d}x.$$

Definition 2.1. We say that λ is an *eigenvalue* of problem (5) if there exists $u \in E \setminus \{0\}$ such that for all $v \in E$

$$L(u,v) = \lambda \int_\Omega (|u|^{\gamma(x)-2}uv - |u|^{\beta(x)-2}uv)\mathrm{d}x.$$

Remark 2.2. If λ is an eigenvalue of problem (5), the corresponding function $u \in E \setminus \{0\}$ is a weak solution of problem (5).

Definition 2.3. We say that λ is an *eigenvalue* of problem (6) if there exists $u \in E \setminus \{0\}$ such that for all $v \in E$

$$L(u,v) = \lambda \int_\Omega |u|^{\gamma(x)-2}uv \mathrm{d}x - \int_\Omega |u|^{\beta(x)-2}uv \mathrm{d}x.$$

Remark 2.4. If λ is an eigenvalue of problem (6), the corresponding function $u \in E \setminus \{0\}$ is a weak solution of problem (6).

The energy functional associated to problem (5) is defined as follows:

$$\mathcal{E}_\lambda(u) = \sum_{i=1}^{2} \left(\int_\Omega \frac{1}{p_i(x)} |\Delta u|^{p_i(x)} + \frac{1}{2p_0(x)} a|u|^{p_0(x)}\mathrm{d}x \right) - \lambda \int_\Omega \left[\frac{|u|^{\gamma(x)}}{\gamma(x)} - \frac{|u|^{\beta(x)}}{\beta(x)} \right] \mathrm{d}x,$$

for all $u \in E$.

The energy functional associated to problem (6) is defined as follows:

$$\mathcal{J}_\lambda(u) = \sum_{i=1}^{2} \left(\int_\Omega \frac{1}{p_i(x)} |\Delta u|^{p_i(x)} + \frac{1}{2p_0(x)} a|u|^{p_0(x)}\mathrm{d}x \right) - \lambda \int_\Omega \frac{|u|^{\gamma(x)}}{\gamma(x)}\mathrm{d}x + \int_\Omega \frac{|u|^{\beta(x)}}{\beta(x)}\mathrm{d}x,$$

for all $u \in E$.

3. THE MAIN RESULTS AND RELATED PROPERTIES

We study problem (5) first under the hypothesis (8) and then under the hypothesis (7).

Hypothesis (8) implies that $\mathcal{E}_\lambda$ is well-defined, of class C^1, and for all $v \in E$

$$\langle \mathcal{E}'_\lambda(u), v\rangle = \sum_{i=1}^{2} \int_\Omega \left(|\Delta u|^{p_i(x)-2} \Delta u \Delta v + \frac{1}{2} a |u|^{p_0(x)-2} uv \right) \mathrm{d}x$$
$$- \lambda \int_\Omega \left(|u|^{\gamma(x)-2} - |u|^{\beta(x)-2} \right) uv \mathrm{d}x.$$

The first result of this paper is the following.

Theorem 3.1. *Assume that one of the hypotheses* (7) *or* (8) *is satisfied. Then there exists a positive number λ^*, such that for all $\lambda \in (0, \lambda^*)$, problem* (5) *has at least one nontrivial weak solution.*

Next we are concerned with the study of problem (6).

Theorem 3.2. *Assume that the hypothesis* (8) *is satisfied. Then there exists a positive number λ^{**} such that for all $\lambda \in (\lambda^{**}, \infty)$ problem* (6) *has at least one nontrivial weak solution.*

We point out that under hypothesis (8) the minimization problem related to the equation (5) does not have a mountain pass geometry. More precisely, $\mathcal{E}_\lambda$ satisfies only one of the geometric hypotheses of the mountain pass theorem, namely the existence of a mountain between two prescribed villages. However, the second geometric assumption of the mountain pass theorem is not fulfilled because this valley is close to the first village and not across the chain of mountains, as requested by the mountain pass theorem. For this reason, the existence of the solution follows with different arguments and only for small perturbations in terms of λ.

We remark that Theorem 3.1 is established in a very close case in [5] which uses a property related to Theorem 2.1 in [15]. However, this result is based on the assumption (8), which is more general than the corresponding hypothesis (2.1) in [15].

The proofs of Theorems 3.1 and 3.2 are similar to those of Theorems 3.1 and 3.2 in [5] in which the authors use some ideas developed in [20, 31, 32] in the framework of $p(x)$-Laplace operators and extended in [15] to biharmonic operators with variable exponent.

3.1. **Geometric properties**

We are first concerned with the proof of Theorem 3.1 if the hypothesis (8) is fulfilled.

We have $\mathcal{E}_\lambda(0) = 0$. We first establish the following auxiliary property.

Lemma 3.3. There exists a positive number λ^* such that for all $\lambda \in (0, \lambda^*)$ there are positive numbers r and η such that $\mathcal{E}_\lambda(u) \geq r$ for all $u \in E$ with $\|u\| = \eta$.

Proof. We observe that

$$\mathcal{E}_\lambda(u) \geq \sum_{i=1}^{2} \int_\Omega \left(\frac{1}{p_i^+} |\Delta u|^{p_i(x)} + \frac{1}{2p_0^+} a|u|^{p_0(x)} \right) \mathrm{d}x - \frac{\lambda}{\gamma^-} \int_\Omega |u|^{\gamma(x)} \mathrm{d}x + \frac{\lambda}{\beta^+} \int_\Omega |u|^{\beta(x)} \mathrm{d}x.$$

Then,

$$\begin{aligned} \mathcal{E}_\lambda(u) &\geq \frac{1}{p^+} \varrho_a(u) - \frac{\lambda}{\gamma^-} \int_\Omega |u|^{\gamma(x)} \mathrm{d}x + \frac{\lambda}{\beta^+} \int_\Omega |u|^{\beta(x)} \mathrm{d}x \\ &\geq \frac{1}{q^+} \varrho_a(u) - \frac{\lambda}{\gamma^-} \int_\Omega |u|^{\gamma(x)} \mathrm{d}x. \end{aligned}$$

Where $\varrho_a(u) = \int_\Omega (|\Delta u|^{p(x)} + a|u|^{p(x)}) \mathrm{d}x$. Fix $\eta \in (0, 1)$ and assume that $\|u\|_E = \eta$. We obtain

$$\mathcal{E}_\lambda(u) \geq \frac{1}{q^+} \|u\|_E^{p^+} - \frac{\lambda}{\gamma^-} \int_\Omega |u|^{\gamma(x)} \mathrm{d}x.$$

Since the embedding $E \hookrightarrow L^{\gamma(x)}(\Omega)$ is continuous, there exists $C_1 > 0$ such that for all $u \in E$

$$\mathcal{E}_\lambda(u) \geq \frac{1}{q^+} \|u\|_E^{p^+} - \lambda C_1 \|u\|_E^{\gamma^-} = \frac{\eta^{p^+}}{q^+} - \lambda C_1 \eta^{\gamma^-}.$$

Now, taking λ^* sufficiently small, we deduce that for all $\lambda \in (0, \lambda^*)$ there exists $r > 0$ such that $\mathcal{E}_\lambda(u) \geq r$ for all $u \in E$ with $\|u\|_E = \eta$.

The proof of Lemma 3.3 is now complete. $\square$

Next, we establish the existence of a valley near the origin.

Lemma 3.4. There exist $v \in E$ and $t_0 > 0$ such that $\mathcal{E}_\lambda(tv) < 0$ for all $t \in (0, t_0)$.

Proof. Fix $v \in E \setminus \{0\}$ such that $v \geq 0$. For all $t \in (0,1)$, we have

$$\mathcal{E}_\lambda(tv) = \int_\Omega \sum_{i=1}^{2} \left(\tfrac{t^{p_i(x)}}{p_i(x)} |\Delta v|^{p_i(x)} + \tfrac{1}{2p_0(x)} a v^{p_0(x)} \right) \mathrm{d}x \quad -\lambda \int_\Omega \frac{t^{\gamma(x)}}{\gamma(x)} v^{\gamma(x)} \mathrm{d}x + \lambda \int_\Omega \frac{t^{\beta(x)}}{\beta(x)} v^{\beta(x)} \mathrm{d}x,$$

then

$$\begin{aligned}\mathcal{E}_\lambda(tv) &\leq \frac{t^{p^-}}{p^-} \varrho_a(v) - \lambda \frac{t^{\gamma^+}}{\gamma^+} \int_\Omega v^{\gamma(x)} \mathrm{d}x + \lambda \frac{t^{\beta^-}}{\beta^-} \int_\Omega v^{\beta(x)} \mathrm{d}x \\ &= C_1 t^{p^-} + C_2 t^{\beta^-} - C_3 t^{\gamma^+},\end{aligned}$$

where C_1, C_2, C_3 are positive numbers.

Using hypothesis (8), we deduce that $\mathcal{E}_\lambda(tv) < 0$, provided that $t > 0$ is sufficiently small. The proof of Lemma 3.4 is complete. □

As a conclusion of these results, we point out that hypothesis (8) implies that problem (5) satisfies the existence of a mountain between two prescribed villages.

3.2. **A compactness property**

Since the right-hand side of equation (5) does not satisfy the Ambrosetti-Rabinowitz condition, we cannot deduce that $\mathcal{E}_\lambda$ satisfies the Palais-Smale condition, that is, any Palais-Smale sequence is relatively compact. However, we prove in what follows that there is a bounded Palais-Smale sequence that contains a strongly convergent subsequence.

Returning to Lemma 3.3, we have

$$\inf_{u \in \partial B} \mathcal{E}_\lambda(u) \geq r > 0, \tag{15}$$

where

$$B := \{u \in E;\ \|u\|_a < \eta\}.$$

By Lemma 3.4, there exists $v \in E$ such that

$$\mathcal{E}_\lambda(tv) < 0 \quad \text{for all } t > 0 \text{ small enough.} \tag{16}$$

Set

$$m := \inf_{u \in \overline{B}} \mathcal{E}_\lambda(u).$$

Then m is finite and using relation (16), we deduce that $m < 0$. By (15) it follows that

$$\inf_{u \in \partial B} \mathcal{E}_\lambda(u) - \inf_{u \in \overline{B}} \mathcal{E}_\lambda(u) > 0.$$

Fix $\varepsilon > 0$ such that

$$\varepsilon < \inf_{u\in\partial B} \mathcal{E}_\lambda(u) - \inf_{u\in\overline{B}} \mathcal{E}_\lambda(u).$$

The functional $\mathcal{E}_\lambda$ restricted to the complete metric space $\overline{B}$ satisfies the hypotheses of the Ekeland variational principle. A straightforward computation (see [32]) shows that there exists a bounded sequence $(u_n) \subset B$ such that

$$\mathcal{E}_\lambda(u_n) \to m \quad \text{and} \quad \|\mathcal{E}_\lambda'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{17}$$

So, up to a subsequence, we can assume that

$$u_n \rightharpoonup u_0 \text{ in } E, \quad u_n \to u_0 \text{ in } L^{\gamma(x)}(\Omega), \text{ and } u_n \to u_0 \text{ in } L^{\beta(x)}(\Omega).$$

We claim that, in fact,

$$u_n \to u_0 \quad \text{in } E.$$

Using the second information in relation (17), we deduce that for all $\varphi \in E$

$$\begin{aligned}&\sum_{i=1}^{2}\int_\Omega \left(|\Delta u_n|^{p_i(x)-2}\Delta u_n \Delta(u_n-u_0) + \frac{1}{2}a|u_n|^{p_0(x)-2}u_n(u_n-u_0)\right)\mathrm{d}x - \\ &\lambda\int_\Omega \left(|u_n|^{\gamma(x)-2} - |u_n|^{\beta(x)-2}\right)u_n(u_n-u_0)\mathrm{d}x \to 0 \qquad \text{as } n\to\infty.\end{aligned} \tag{18}$$

By [15, Lemma 2.1(b)], the operator $\mathcal{E}_\lambda' : E \to E^*$ is an operator of type (S_+). Thus we obtain that $u_n \to u_0$ in E, which is our claim. So, by (17),

$$\mathcal{E}_\lambda(u_0) = m < 0 \quad \text{and} \quad \mathcal{E}_\lambda'(u_0) = 0.$$

We conclude that u_0 is a nontrivial weak solution of problem (5). Thus any $\lambda \in (0, \lambda^*)$ is an eigenvalue of problem (5). The proof of Theorem 3.1 is now complete, provided that hypothesis (8) is fulfilled.

We are now concerned with the related property if condition (7) is satisfied. We first observe that under this new hypothesis, the conclusion of Lemma 3.3 remains true. Next, since condition (7) implies that the dominating term in the right-hand side of problem (5) is $|u|^{\gamma(x)-2}u$, we prove in what follows the existence of a valley across the chain of mountains.

Lemma 3.5. There exist $v \in E$ and $t_0 > 0$ such that $\mathcal{E}_\lambda(tv) < 0$ for all $t > t_0$.

Proof. Fix $v \in E \setminus \{0\}$ such that $v \geq 0$. We have, for all $t > 1$,

$$\mathcal{E}_\lambda(tv) = \sum_{i=1}^{2}\int_\Omega \frac{t^{p_i(x)}}{p_i(x)}\left(|\Delta v|^{p_i(x)} + \frac{1}{2p_0(x)}av^{p_0(x)}\right)\mathrm{d}x - \lambda\int_\Omega \frac{t^{\gamma(x)}}{\gamma(x)}v^{\gamma(x)}\mathrm{d}x$$
$$+ \lambda\int_\Omega \frac{t^{\beta(x)}}{\beta(x)}v^{\beta(x)}\mathrm{d}x,$$

then

$$\begin{aligned}\mathcal{E}_\lambda(tv) &\le \frac{t^{q^+}}{p^-}\varrho_a(v) - \lambda\frac{t^{\gamma^-}}{\gamma^+}\int_\Omega v^{\gamma(x)}\mathrm{d}x + \lambda\frac{t^{\beta^+}}{\beta^-}\int_\Omega v^{\beta(x)}\mathrm{d}x\\ &\le C_4 t^{p^-} + C_5 t^{\beta^+} - C_6 t^{\gamma^-},\end{aligned}$$

where C_4, C_5, C_6 are positive numbers.

Using hypothesis (7), we have $\gamma^- > \max\{p^-, \beta^+\}$. It follows that $\mathcal{E}_\lambda(tv) < 0$, provided that $t > 0$ is sufficiently large and the proof is complete. □

3.3. The Palais-Smale condition

The energy functional $\mathcal{E}_\lambda : E \to \mathbb{R}$ is said to satisfy the Palais-Smale condition on E if any sequence $(u_n) \subset E$ such that

$$\mathcal{E}_\lambda(u_n) = O(1) \text{ and } \|\mathcal{E}'_\lambda(u_n)\|_{E^*} = o(1) \text{ as } n \to \infty, \tag{19}$$

is relatively compact.

Let $(u_n) \subset E$ be a sequence such that (19) is fulfilled. We claim that

$$(u_n) \text{ is bounded in } E. \tag{20}$$

Arguing by contradiction, we suppose that the sequence (u_n) is unbounded in E. Without loss of generality, we can assume that $\|u_n\|_a > 1$ for all $n \ge 1$. Using relation (19), we have

$$\begin{aligned}O(1) + o(\|u_n\|) &= \mathcal{E}_\lambda(u_n) - \frac{1}{\gamma^-}\langle \mathcal{E}'_\lambda(u_n), u_n\rangle\\ &= \sum_{i=1}^{2}\int_\Omega \left(\frac{1}{p_i(x)}|\Delta u_n|^{p_i(x)} + \frac{1}{2p_0(x)}a|u_n|^{p_0(x)}\right)\mathrm{d}x\\ &- \lambda\int_\Omega \left[\frac{|u_n|^{\gamma(x)}}{\gamma(x)} - \frac{|u_n|^{\beta(x)}}{\beta(x)}\right]\mathrm{d}x\\ &- \frac{1}{\gamma^-}\sum_{i=1}^{2}\int_\Omega \left(|\Delta u_n|^{p_i(x)} + \frac{1}{2}a|u_n|^{p_0(x)}\right)\mathrm{d}x\\ &- \frac{\lambda}{\gamma^-}\int_\Omega \left(|u_n|^{\gamma(x)} - |u_n|^{\beta(x)}\right)\mathrm{d}x.\end{aligned}$$

We deduce that

$$\begin{aligned}O(1)+o(\|u_n\|) &= \mathcal{E}_\lambda(u_n) - \frac{1}{\gamma^-}\langle \mathcal{E}'_\lambda(u_n), u_n\rangle\\ &\ge \left(\frac{1}{q^+} - \frac{1}{\gamma^-}\right)\int_\Omega \left(|\Delta u_n|^{p^-} + a|u_n|^{p^-}\right)\mathrm{d}x\end{aligned}$$

$$+\lambda\int_\Omega\left(\frac{1}{\beta(x)}-\frac{1}{\gamma^-}\right)|u_n|^{\beta(x)}\mathrm{d}x+\lambda\int_\Omega\left(\frac{1}{\gamma^-}-\frac{1}{\gamma(x)}\right)|u_n|^{\gamma(x)}\mathrm{d}x.$$

Using now the hypothesis (7), we conclude that

$$O(1)+o(\|u_n\|)\geq\left(\frac{1}{q^+}-\frac{1}{\gamma^-}\right)\|u_n\|_a^{p^-}\quad\text{as } n\to\infty.$$

Since $\gamma^->q^+$ it follows that

$$\|u_n\|_a=O(1)\quad\text{as } n\to\infty.$$

This shows that our claim (20) is true. So, up to a subsequence, we can assume that

$$u_n\rightharpoonup u_0 \text{ in } E,\quad u_n\to u_0 \text{ in } L^{\gamma(x)}(\Omega),\ \text{and } u_n\to u_0 \text{ in } L^{\beta(x)}(\Omega).$$

We show in what follows that

$$u_n\to u_0\quad\text{in } E.$$

Using the second information in relation (19), we deduce that for all $\varphi\in E$

$$\sum_{i=1}^{2}\int_\Omega\left[|\Delta u_n|^{p_i(x)-2}\Delta u_n\Delta\varphi+\frac{1}{2}a|u_n|^{p_0(x)-2}u_n\varphi\right]\mathrm{d}x-$$
$$\lambda\int_\Omega\left(|u_n|^{\gamma(x)-2}-|u_n|^{\beta(x)-2}\right)u_n\varphi\mathrm{d}x\to 0$$

as $n\to\infty$.

With the same arguments as in the first case and since $\mathcal{E}'_\lambda: E\to E^*$ is an operator of type (S_+), we conclude that $u_n\to u_0$ in E, which shows that Palais-Smale condition is satisfied. At this stage it is enough to apply the mountain pass theorem in order to obtain a nontrivial weak solution of problem (5) for each $\lambda>0$, provided that the condition (7) is satisfied.

The proof of Theorem 3.1 is complete.

3.4. **Proof of Theorem 3.2.**

We first show that $\mathcal{J}_\lambda$ is coercive, namely $\mathcal{J}_\lambda(u)\to+\infty$ as $\|u\|_a\to\infty$. Indeed, for all $u\in E$ with $\|u\|_a>1$, we have

$$\begin{aligned}\mathcal{J}_\lambda(u)&\geq\frac{1}{q^+}\int_\Omega\left(|\Delta u|^{p^-}+a|u|^{p^-}\right)\mathrm{d}x-\frac{\lambda}{\gamma^-}\int_\Omega|u|^{\gamma(x)}\mathrm{d}x+\frac{1}{\beta^+}\int_\Omega|u|^{\beta(x)}\mathrm{d}x\\&\geq\frac{1}{q^+}\|u\|_a^{p^-}-\frac{\lambda}{\gamma^-}\int_\Omega|u|^{\gamma(x)}\mathrm{d}x\\&\geq\frac{1}{q^+}\|u\|_a^{p^-}-\frac{c\lambda}{\gamma^-}\|u\|_a^{\gamma^+},\end{aligned}$$

where c is the best constant of the continuous embedding $E \hookrightarrow L^{\gamma(x)}(\Omega)$. By hypothesis (8) we have $p^- > \gamma^+$, which infers that the energy functional $\mathcal{J}_\lambda$ is coercive.

Let (v_n) be a minimizing sequence of the functional $\mathcal{J}_\lambda$ in E. Since $\mathcal{J}_\lambda$ is coercive, we deduce that (v_n) is a bounded sequence. So, up to a subsequence, we can assume that

$$v_n \rightharpoonup v_0 \text{ in } E, \quad v_n \to v_0 \text{ in } L^{\gamma(x)}(\Omega), \text{ and } v_n \to v_0 \text{ in } L^{\beta(x)}(\Omega).$$

Using now the lower semicontinuity of $\mathcal{J}_\lambda$ (see [15, Lemma 2.1 (a)]), we deduce that v_0 is a global minimizer of $\mathcal{J}_\lambda$ on E. It remains to prove that $v_0 \neq 0$. We have $\mathcal{J}_\lambda(0) = 0$. Thus it is enough to show that

$$\inf\{\mathcal{J}_\lambda(v);\ v \in E\} < 0 \quad \text{for } \lambda \text{ large enough.}$$

Indeed, let us consider the following constrained minimization problem

$$\begin{aligned}\lambda^{**} := \inf\Bigg\{ \sum_{i=1}^{2} \int_\Omega \left(\frac{1}{p_i(x)} |\Delta w|^{p_i(x)} + \frac{1}{2p_0(x)} a |w|^{p_0(x)} \right) \mathrm{d}x \\ + \int_\Omega \frac{|w|^{\beta(x)}}{\beta(x)} \mathrm{d}x;\ w \in E' \Bigg\},\end{aligned} \tag{21}$$

where

$$E' = \left\{ w \in E \text{ and } \int_\Omega \frac{|w|^{\gamma(x)}}{\gamma(x)} \mathrm{d}x = 1 \right\}. \tag{22}$$

If $(w_n) \subset E'$ is an arbitrary minimizing sequence of problem (21) then (w_n) is bounded. Thus, up to a subsequence, (w_n) converges weakly in E and strongly in $L^{\gamma(x)}(\Omega)$ and $L^{\beta(x)}(\Omega)$ to some w_0 satisfying

$$\int_\Omega \frac{|w_0|^{\gamma(x)}}{\gamma(x)} \mathrm{d}x = 1$$

and

$$\lambda^{**} = \sum_{i=1}^{2} \int_\Omega \left(\frac{1}{p_i(x)} |\Delta w_0|^{p_i(x)} + \frac{1}{2p_0(x)} a |w_0|^{p_0(x)} \right) \mathrm{d}x + \int_\Omega \frac{|w_0|^{\beta(x)}}{\beta(x)} \mathrm{d}x > 0.$$

We conclude that

$$\mathcal{J}_\lambda(w_0) = \lambda^{**} - \lambda < 0 \quad \text{for all } \lambda > \lambda^{**},$$

hence w_0 is a nontrivial weak solution of problem (6). The proof of Theorem 3.2 is complete.

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