A NEW INVARIANT DERIVED FROM LOCAL COHOMOLOGY MODULES

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Communicated by Lucian Beznea

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring of dimension d and M a non-zero finitely generated R-module. In this paper, we introduce a new useful invariant for M, denoted by $\lambda_R(M)$. We study some properties of $\lambda_R(M)$ and, especially in case that $\hat{M}$ is the $\mathfrak{m}$-adic completion of M, we find an interval for $\lambda_{\hat{R}}(\hat{M})$. It is seen that this invariant inherits some of the properties of Krull dimension on exact sequences. Also, as a nice application of this invariant, it is shown that the set $\mathrm{Assh}_R(R)$ has only one element in case that $\lambda_R(R) = 0$ and therefore $(0 : H^d_{\mathfrak{m}}(R))$ is a primary ideal of R when in addition $d \geq 1$.

AMS 2020 Subject Classification: 13D45, 14B15, 13E05.

Key words: associated primes, Krull dimension, local cohomology.

1. INTRODUCTION

Throughout this paper, R denotes a non-trivial commutative Noetherian ring with identity. For an R-module M, the i^{th} local cohomology module of M with respect to an ideal I is defined as

$$H^i_I(M) = \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(R/I^n, M).$$

For each R module L, we denote the set $\{\mathfrak{p} \in \mathrm{Ass}_R(L) : \dim\frac{R}{\mathfrak{p}} = \dim L\}$, by $\mathrm{Assh}_R(L)$. For any ideal $\mathfrak{b}$ of R, we denote $\{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$ and the *radical* of $\mathfrak{b}$, denoted by $\sqrt{\mathfrak{b}}$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. Recall that, for an R-module M, the set $\mathrm{Min}\,\mathrm{Ass}_R(M)$ is defined as

$$\{\mathfrak{p} \in \mathrm{Ass}_R(M) : \nexists \mathfrak{q} \in \mathrm{Ass}_R(M), \mathfrak{q} \subsetneqq \mathfrak{p}\}.$$

Also, the *cohomological dimension* of M *with respect* to I is defined as

$$\mathrm{Cd}(I, M) := \mathrm{Sup}\left\{i \in \mathbb{Z} : H^i_I(M) \neq 0\right\}.$$

It is easy to see that $\mathrm{Cd}(I, M) \leq \dim_R(M)$. For any proper ideal I of R, the *arithmetic rank* of I, denoted by $\mathrm{ara}(I)$, is the least number of elements of R required to generate an ideal which has the same radical as I.

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MATH. REPORTS **25(75)** (2023), *4*, 553–563
doi: 10.59277/mrar.2023.25.75.4.553

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d and M a non-zero finitely generated R-module of dimension n. As a first part of our investigations in this paper, we employ a well-known result on local cohomology modules, see 2.2, to introduce a new invariant for M. We denote the supremum of the set $\{\dim \frac{R}{I} : I$ is an ideal of R and $H^n_I(M) \neq 0\}$, by $\lambda_R(M)$ and study some of its properties, in addition to giving examples. It is seen that $\lambda_R(M) \geq d - n$; however, this inequality can turn into an equality in case R is a Noetherian complete regular local ring, see 2.5 and 2.7. It turns out, in case that $\hat{R}$ is the $\mathfrak{m}$-adic completion of R, then $\lambda_R(M) \leq \lambda_{\hat{R}}(M \otimes_R \hat{R}) \leq 1 + \dim_k \mathfrak{m}/\mathfrak{m}^2 - n$, see 2.8. As another interesting property, it is seen $\lambda_R(M) = \lambda_R(N)$ for every finitely generated R-module N with $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(N)$, see 2.10. In Theorem 2.12, we obtain a useful relation for $\lambda_R(M)$ on exact sequences. Also, in Theorem 2.14, we show that $\lambda_R(M)$ can be obtained in the following way:

$$\lambda_R(M) = \sup\{\lambda_R(\frac{R}{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Assh}_R(M)\}.$$

Our main goal in Section 3 is to find the number of elements in the set $\mathrm{Assh}_R(R)$, which we denote by $|\mathrm{Assh}_R(R)|$. In a nice result, see 3.2, we show that $|\mathrm{Assh}_R(R)| = 1$ whenever $\lambda_R(R) = 0$. In this case, it is seen that the annihilator of d^{th} local cohomology module of R with respect to $\mathfrak{m}$, $(0 :_R H^d_{\mathfrak{m}}(R))$, is a primary ideal. Finally, we try to give some results and related contents to $|\mathrm{Assh}_R(R)|$. For more details about local cohomology modules and any unexplained notation and terminology, we refer the readers to [2] and [6].

2. DEFINITION AND SOME PROPERTIES

We begin this section with the following two preliminary Theorems.

THEOREM 2.1 (Grothendieck's Vanishing Theorem). *Let R be a Noetherian ring, I an ideal of R and M an R-module. Then $H^i_I(M) = 0$ for all $i > \dim_R M$.*

Proof. See [2, Theorem 6.1.2]. □

THEOREM 2.2 (Non-Vanishing Theorem). *Assume that $(R, \mathfrak{m})$ is a local ring and let M be a non-zero finitely generated R-module of dimension n. Then $H^n_{\mathfrak{m}}(M) \neq 0$.*

Proof. See [2, Theorem 6.1.4]. □

The set $\{I : I \text{ is an ideal of } R \text{ and } H_I^n(M) \neq 0\}$ is not empty, as an application of Theorem 2.2. We discuss the set

$$\{\dim\frac{R}{I} : I \text{ is an ideal of } R \text{ and } H_I^n(M) \neq 0\}$$

in the following definition and introduce an invariant, denoted by $\lambda_R(M)$, which is the main object of our investigations.

Definition 2.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a non-zero finitely generated R-module of dimension n. We define the invariant $\lambda_R(M)$ by

$$\lambda_R(M) = \sup\{\dim\frac{R}{I} : I \text{ is an ideal of } R \text{ and } H_I^n(M) \neq 0\}.$$

In particular, when $(R, \mathfrak{m})$ is a local ring of dimension d, we define

$$\lambda_R(R) = \sup\{\dim\frac{R}{I} : I \text{ is an ideal of } R \text{ and } H_I^d(R) \neq 0\}.$$

Note that, in the above set for every ideal I with $H_I^d(R) \neq 0$, it is clear that I automatically is a proper ideal. So by view of Lichtenbaum-Hartshorne Theorem [2, Theorem 8.2.1], $H_I^d(R) \neq 0$ is equivalent to say that there exists a prime ideal β of $\widehat{R}$ such that $\dim\frac{\widehat{R}}{\beta} = d$ and $I\widehat{R} + \beta$ is $\widehat{\mathfrak{m}}$-primary. Therefore, one may write

$$\lambda_R(R) = \sup\{\dim\frac{R}{I} : \dim\frac{\widehat{R}}{\beta} = d, I\widehat{R} + \beta \text{ is } \widehat{\mathfrak{m}}\text{-primary for some } \beta \in \operatorname{Spec}(\widehat{R})\}.$$

This observation shows that $\lambda_R(R)$ is a commutative algebra invariant which may be indicated without referring to local cohomology.

Example 2.4. (i) Let $(R, \mathfrak{m})$ be a Noetherian regular local ring of dimension $2n$ such that $n \geq 2$. Then there exists a regular system of parameters for R such as $x_1, ..., x_{2n}$. Let $\mathfrak{p} = (x_1, ..., x_n)$, $\mathfrak{q} = (x_{n+1}, ..., x_{2n})$, and $\bar{R} = \frac{R}{\mathfrak{p} \cap \mathfrak{q}}$. Clearly $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals and $(\bar{R}, \bar{\mathfrak{m}})$ is a Noetherian local ring. We show that $\lambda_{\bar{R}}(\bar{R}) = n$. Since $\dim(\frac{R}{\mathfrak{p}}) = \dim(\frac{R}{\mathfrak{q}}) = n$, it follows that $\dim\bar{R} \leq n$ because $\dim_R(\frac{R}{\mathfrak{p}} \oplus \frac{R}{\mathfrak{q}}) = n$. Also, $\dim\bar{R} \geq n$ by the epimorphism

$$\bar{R} \to \frac{R}{\mathfrak{p}} \to 0.$$

These lead to $\dim\bar{R} = n$. Now let $\bar{\mathfrak{p}} = \frac{\mathfrak{p}}{\mathfrak{p} \cap \mathfrak{q}}$ and $\bar{\mathfrak{q}} = \frac{\mathfrak{q}}{\mathfrak{p} \cap \mathfrak{q}}$. It follows from the *Mayer-Vietoris Sequence* [2, Theorem 3.2.3], that $H_{\bar{\mathfrak{p}}}^n(\bar{R}) \neq 0$ or

$H^n_{\bar{\mathfrak{q}}}(\bar{R}) \neq 0$ because $\bar{\mathfrak{p}} \cap \bar{\mathfrak{q}} = \bar{0}$ and $\sqrt{\bar{\mathfrak{p}} + \bar{\mathfrak{q}}} = \bar{\mathfrak{m}}$. Since $\lambda_{\bar{R}}(\bar{R}) \leq \dim \bar{R}$, it follows that $\lambda_{\bar{R}}(\bar{R}) = n$.

(ii) Let $(R, \mathfrak{m})$ be a Noetherian Complete regular local ring of dimension $d > 0$. Then $\dim \frac{R}{I} = 0$ for every ideal I with $H^d_I(R) \neq 0$. This is clear, because by Lichtenbaum-Hartshorne Theorem, there exists $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\dim \frac{R}{\mathfrak{p}} = d$ and $\dim \frac{R}{I+\mathfrak{p}} = 0$. But $\mathfrak{p} = 0$ since R is integral domain. These conclude that $\lambda_R(R) = 0$.

Lemma 2.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and M a non-zero finitely generated R-module of dimension n. Then $\lambda_R(M) \geq d - n$.*

Proof. Since $\dim \frac{R}{\mathrm{Ann}_R(M)} = n$, it follows that $\mathrm{Ann}_R(M)$ contains a part of system of parameters for R such as $x_1, \ldots, x_{d-n}$. Assume that the elements $y_1, \ldots, y_n \in \mathfrak{m}$ be such that $x_1, \ldots, x_{d-n}, y_1, \ldots, y_n$ is a system of parameters of R. Then

$$\mathfrak{m} = \sqrt{(x_1, ..., x_{d-n}) + (y_1, ..., y_n)} \subseteq \sqrt{\mathrm{Ann}_R(M) + (y_1, ..., y_n)} \subseteq \mathfrak{m}$$

and so $\sqrt{\mathrm{Ann}_R(M) + (y_1, ..., y_n)} = \mathfrak{m}$. Consequently, in view of [2, Theorem 4.2.1], we have the following isomorphisms:

$$H^n_{(y_1,\ldots,y_n)}(M) \cong H^n_{\frac{(y_1,\ldots,y_n)+\mathrm{Ann}\, M}{\mathrm{Ann}\, M}}(M) \cong H^n_{\frac{\mathfrak{m}}{\mathrm{Ann}\, M}}(M) \cong H^n_{\mathfrak{m}}(M).$$

Now, by Theorem 2.2 we get $H^n_{(y_1,\ldots,y_n)}(M) \neq 0$. That is

$$\lambda_R(M) \geq \dim \frac{R}{(y_1, \ldots, y_n)}.$$

□

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring, which is a homomorphic image of a complete regular local ring $(S, \mathfrak{n})$, with $\dim S = d$. Let M be a non-zero finitely generated R-module such that $\dim M = n$. Then $H^n_I(M) = 0$ for all ideals I of R such that $\dim \frac{R}{I} > d - n$.*

Proof. See [5, Proposition 2.12]. □

Theorem 2.7. *Let $(R, \mathfrak{m})$ be a Noetherian complete regular local ring of dimension d and M a non-zero finitely generated R-module of dimension n. Then $\lambda_R(M) = d - n$.*

Proof. Suppose that $\lambda_R(M) = t$. By Definition 2.3, there exists an ideal I of R such that $\dim\frac{R}{I} = t$ and $H^n_I(M) \neq 0$. Thus, $d - n \geq t$ by Theorem 2.6. The assertion follows from Lemma 2.5. □

Let $\hat{R}$ denotes the $\mathfrak{m}$-adic completion of local ring $(R, \mathfrak{m})$. Then, we have the following results.

LEMMA 2.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and M be a non-zero finitely generated R-module of dimension n. Then $\lambda_{\hat{R}}(M \otimes_R \hat{R}) \geq \lambda_R(M)$.*

Proof. Suppose that $\lambda_R(M) = t$. Then there exists an ideal I of R such that $\dim\frac{R}{I} = t$ and $H^n_I(M) \neq 0$. Since $\hat{R}$ is faithfully flat over R, it follows that $H^n_I(M) \otimes_R \hat{R} \neq 0$ and so $H^n_{I\hat{R}}(M \otimes_R \hat{R}) \neq 0$ by [2, Theorem 4.3.2]. Furthermore, one has $\dim\frac{\hat{R}}{I\hat{R}} = \dim\frac{R}{I} = t$. Therefore, $\lambda_{\hat{R}}(M \otimes_R \hat{R}) \geq t$. □

THEOREM 2.9. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a non-zero finitely generated R-module of dimension n. Then*

$$\lambda_R(M) \leq \lambda_{\hat{R}}(M \otimes_R \hat{R}) \leq 1 + \dim_k \mathfrak{m}/\mathfrak{m}^2 - n.$$

Proof. It is clear that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \mathfrak{m}\hat{R}/(\mathfrak{m}\hat{R})^2$. Now, by Cohen's structure Theorem [6, Theorem 29.4], $\hat{R}$ is a homomorphic image of a complete regular local ring of dimension $d = 1 + \dim_k \mathfrak{m}/\mathfrak{m}^2$ such as $(S, \mathfrak{n})$. On the other hand, according to the definition there exists an ideal J of $\hat{R}$ such that $\dim\hat{R}/J = \lambda_{\hat{R}}(M \otimes_R \hat{R})$ and $H^n_J(M \otimes_R \hat{R}) \neq 0$. According to Theorem 2.6,

$$\lambda_{\hat{R}}(M \otimes_R \hat{R}) = \dim\hat{R}/J \leq d - n = 1 + \dim\mathfrak{m}/\mathfrak{m}^2 - n.$$

Therefore, the assertion follows from Lemma 2.8. □

LEMMA 2.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M, N be two finitely generated R-modules with* $\operatorname{Supp}_R M = \operatorname{Supp}_R N$. *Then $\lambda_R(M) = \lambda_R(N)$.*

Proof. Let $\dim M = \dim N = n$ and $\lambda_R(M) = t$. Thus, there exists an ideal I of R with $\dim\frac{R}{I} = t$ which leads to $H^n_I(M) \neq 0$. This implies that $\operatorname{Cd}(I, M) \geq n$ and by Theorem 2.1, it follows that $\operatorname{Cd}(I, M) = n$. Now by [4, Theorem 1.2] we find that $\operatorname{Cd}(I, N) = n$. Therefore, $H^n_I(N) \neq 0$ and $\dim\frac{R}{I} = t$. That is $\lambda_R(N) \geq t$. A similar argument as above shows that $\lambda_R(M) \geq \lambda_R(N)$. □

The following example shows that the converse of Lemma 2.10 is not true.

Example 2.11. Let $R = K[[x, y]]$ be the formal power series ring of variables x, y over field K. Then R is a Noetherian complete regular local ring of dimension 2. Now let $M = \dfrac{R}{<x>}$ and $N = \dfrac{R}{<y>}$. Clearly, M and N are two finitely generated R-modules and $\dim M = \dim N = 1$. Therefore, it follows from Theorem 2.7, that $\lambda_R(M) = \lambda_R(N) = 1$. At the same time, $<x> \in \mathrm{Supp}_R(M)$ but $<x> \notin \mathrm{Supp}_R(N)$. Hence, $\mathrm{Supp}_R(M) \neq \mathrm{Supp}_R(N)$

THEOREM 2.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and*

$$0 \to L \to M \to N \to 0$$

be an exact sequence of non-zero finitely generated R-modules. Then

$$\lambda_R(M) \leq \mathrm{Max}\{\lambda_R(L), \lambda_R(N)\}.$$

Proof. Let $\dim L = n_1$, $\dim N = n_2$, $\dim M = n$, and

$$\mathrm{Max}\{\lambda_R(L), \lambda_R(N)\} = t.$$

Suppose that $\lambda_R(M) \geq t + 1$ and look for a contradiction. By Definition 2.3, there exists an ideal I of R such that $\dim \dfrac{R}{I} \geq t + 1$ and $H_I^n(M) \neq 0$. The exact sequence

$$0 \to L \to M \to N \to 0$$

induces a long exact sequence

$$\ldots \to H_I^n(L) \to H_I^n(M) \to H_I^n(N) \to H_I^{n+1}(L) \to \cdots.$$

By Theorem 2.1, we have $H_I^{n+1}(L) = 0$. We distinguish three cases, as follows:

CASE 1: If $n_1 = n_2 = n$, it follows that $H_I^n(L) = 0$ because $\lambda_R(L) \leq t$. Hence, we get $H_I^n(M) \cong H_I^n(N)$. Since $\lambda_R(N) \leq t$ it follows that $H_I^n(N) = 0$ and so $H_I^n(M) = 0$ which is a contradiction.

CASE 2: If $n_1 < n_2$, then $n = n_2$. By Theorem 2.1, $H_I^n(L) = 0$ and therefore $H_I^n(M) \cong H_I^n(N)$. Similar as above, this leads to contradiction because $\lambda_R(N) \leq t$.

CASE 3: If $n_2 < n_1$, then $n = n_1$ and this time $H_I^n(N) = 0$. Similarly, $\lambda_R(L) \leq t$ leads to $H_I^n(L) = 0$ and so $H_I^n(M) = 0$ which is a contradiction. □

Now we want to describe $\lambda_R(M)$ for an R-module M in another way. But before, we need to present the following auxiliary lemma.

LEMMA 2.13. *Let R be a Noetherian ring and M a non-zero finitely generated R-module. Let $\mathrm{Min\,Ass}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ and put $N := \bigoplus_{i=1}^{k} (\dfrac{R}{\mathfrak{p}_i})$. Then $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(N)$.*

Proof. Clearly, $\mathrm{Supp}_R(N) \subseteq \mathrm{Supp}_R(M)$. Hence, it is enough to show that $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(N)$. Let $\mathfrak{q}$ be an arbitrary element of $\mathrm{Supp}_R(M)$. So there exists $\mathfrak{p} \in \mathrm{Ass}_R(M)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Without loss of generality, we may assume that there exists $1 \leq j \leq k$ such that $\mathfrak{p} = \mathfrak{p}_j \in \{\mathfrak{p}_1, ..., \mathfrak{p}_k\}$. Therefore,

$$N_{\mathfrak{q}} = (\bigoplus_{i=1}^{k} \frac{R}{\mathfrak{p}_i})_{\mathfrak{q}} = (\frac{R_{\mathfrak{q}}}{\mathfrak{p}_1 R_{\mathfrak{q}}}) \bigoplus \cdots \bigoplus (\frac{R_{\mathfrak{q}}}{\mathfrak{p}_k R_{\mathfrak{q}}}).$$

Since $\mathfrak{p}_j \subseteq \mathfrak{q}$ and $\mathfrak{q}R_{\mathfrak{q}} \neq R_{\mathfrak{q}}$, it follows that $\dfrac{R_{\mathfrak{q}}}{\mathfrak{p}_j R_{\mathfrak{q}}} \neq 0$. This leads to $\mathfrak{q} \in \mathrm{Supp}_R(N)$. □

THEOREM 2.14. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a non-zero finitely generated R-module of dimension n. Then*

$$\lambda_R(M) = \sup\{\lambda_R(\frac{R}{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Assh}_R(M)\}.$$

Proof. Let $\sup\{\lambda_R(\frac{R}{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Assh}_R(M)\} = t$. Therefore, there exists $\mathfrak{p} \in \mathrm{Assh}_R(M)$ such that $\lambda_R(\frac{R}{\mathfrak{p}}) = t$ and by view of definition there exists an ideal I of R with $\dim \frac{R}{I} = t$ and $H^n_I(\frac{R}{\mathfrak{p}}) \neq 0$. This concludes $\mathrm{Cd}(I, \frac{R}{\mathfrak{p}}) = n$. Hence, it follows from [4, Theorem 1.2] together with Theorem 2.1 that $\mathrm{Cd}(I, M) = n$. This leads to $H^n_I(M) \neq 0$ and $\lambda_R(M) \geq t$.

Now suppose that $\lambda_R(M) = t$. Then there exists an ideal I of R with $\dim \frac{R}{I} = t$ and $H^n_I(M) \neq 0$. Consequently, we find $\mathrm{Cd}(I, M) = n$. Let $\mathrm{Min\,Ass}_R(M) = \{\mathfrak{p}_1, ..., \mathfrak{p}_k\}$ and put $N := \bigoplus_{i=1}^{k} (\frac{R}{\mathfrak{p}_i})$, $\mathfrak{p}_i \in \mathrm{Min\,Ass}_R(M)$. Hence, $H^n_I(N) \neq 0$ because by view of Lemma 2.13 and [4, Theorem 1.2], $\mathrm{Cd}(I, M) = \mathrm{Cd}(I, N)$. Thus, there exists $\mathfrak{p}_i \in \mathrm{Min\,Ass}_R(M)$ such that $H^n_I(\frac{R}{\mathfrak{p}_i}) \neq 0$. This guarantees that $\dim \frac{R}{\mathfrak{p}_i} = n$. Hence, we have found $\mathfrak{p}_i \in \mathrm{Assh}_R(M)$ with $\lambda_R(\frac{R}{\mathfrak{p}_i}) \geq t$, i.e.,

$$\sup\{\lambda_R(\frac{R}{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Assh}_R(M)\} \geq t.$$

□

At the end of this section, we present the following proposition which is derived from the previous topics.

Proposition 2.15. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension one and $\mathfrak{p}$ a prime ideal in* $\mathrm{Assh}_R(R)$. *Let x be an $R/\mathfrak{p}$-regular element in $\mathfrak{m} \setminus \mathfrak{p}$. Then the following hold.*

(i) $\dfrac{R}{\mathfrak{p}}$ *is a maximal Cohen-Macaulay R-module.*

(ii) $\mathrm{Cd}(< x >, \dfrac{R}{\mathfrak{p}}) = 1$.

(iii) $\dim \dfrac{R}{< x >} \leq \lambda_R(\dfrac{R}{\mathfrak{p}}) \leq \lambda_R(R)$.

(iv) $\mathrm{Cd}(\mathfrak{p}, (\dfrac{R}{\mathfrak{p}})_x) = 0$.

Proof. Since $\mathfrak{p} \in \mathrm{Assh}_R(R)$, it follows that $\mathfrak{p} \neq \mathfrak{m}$. So there exists an element $x \in \mathfrak{m} \setminus \mathfrak{p}$ such that x is a regular element on $R/\mathfrak{p}$ as an R-module.

(i) It follows from the above statement that $1 \leq \mathrm{depth}(\dfrac{R}{\mathfrak{p}}) \leq \dim(\dfrac{R}{\mathfrak{p}}) = \dim R = 1$.

(ii) Since x is a regular sequence on $\dfrac{R}{\mathfrak{p}}$ as an R-module, $\sqrt{Rx + \mathfrak{p}} = \mathfrak{m}$. Therefore,

$$H^1_{Rx}(\frac{R}{\mathfrak{p}}) \cong H^1_{Rx+\mathfrak{p}}(\frac{R}{\mathfrak{p}}) \cong H^1_{\mathfrak{m}}(\frac{R}{\mathfrak{p}}) \neq 0.$$

Now the claim follows from Theorem 2.1.

(iii) $H^1_{Rx}(\dfrac{R}{\mathfrak{p}}) \neq 0$ by part (ii). Thus, $\dim \dfrac{R}{< x >} \leq \lambda_R(\dfrac{R}{\mathfrak{p}})$ and, in view of Theorem 2.14, the assertion is completed.

(iv) The facts that $\Gamma_{\mathfrak{p}}(\dfrac{R}{\mathfrak{p}}) = \dfrac{R}{\mathfrak{p}}$ and $H^i_{Rx+\mathfrak{p}}(\dfrac{R}{\mathfrak{p}}) \cong H^i_{Rx}(\dfrac{R}{\mathfrak{p}})$ for all $i \geq 0$, together with [2, Proposition 8.1.2], induce the following long exact sequence

$$0 \longrightarrow \Gamma_{Rx}(\frac{R}{\mathfrak{p}}) \longrightarrow \frac{R}{\mathfrak{p}} \longrightarrow \Gamma_{\mathfrak{p}}((\frac{R}{\mathfrak{p}})_x) \longrightarrow H^1_{Rx}(\frac{R}{\mathfrak{p}}) \longrightarrow$$

$$H^1_{\mathfrak{p}}(\frac{R}{\mathfrak{p}}) \longrightarrow H^1_{\mathfrak{p}}((\frac{R}{\mathfrak{p}})_x) \longrightarrow H^2_{Rx}(\frac{R}{\mathfrak{p}}) \longrightarrow \ldots$$

It is easy to see that $H^i_{\mathfrak{p}}((\dfrac{R}{\mathfrak{p}})_x) = 0$ for all $i \geq 1$. Moreover, since x is a non-zerodivisor on $\dfrac{R}{\mathfrak{p}}$ as an R-module, it follows from [2, Lemma 2.1.1] that $\Gamma_{Rx}(\dfrac{R}{\mathfrak{p}}) = 0$. Therefore, we have the following exact sequence

$$0 \longrightarrow \frac{R}{\mathfrak{p}} \longrightarrow \Gamma_{\mathfrak{p}}((\frac{R}{\mathfrak{p}})_x) \longrightarrow H^1_{Rx}(\frac{R}{\mathfrak{p}}) \longrightarrow 0.$$

By part (ii), $H^1_{Rx}(\dfrac{R}{\mathfrak{p}}) \neq 0$. This leads to $\mathrm{Cd}(\mathfrak{p}, (\dfrac{R}{\mathfrak{p}})_x) = 0$. □

3. ON THE NUMBER OF PRIMES IN $\mathrm{Assh}_R(R)$

In this section, we are going to discuss the relation between $\lambda_R(R)$ and the number of elements in the set $\mathrm{Assh}_R(R)$. We denote this number by $|\,\mathrm{Assh}_R(R)|$.

LEMMA 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension zero. Then $\lambda_R(R) = 0$ and $|\mathrm{Assh}_R(R)| = 1$.*

Proof. For every ideal I of R, $\dim\frac{R}{I} = 0$. This implies that $\lambda_R(R) = 0$. Also, it is clear that $\mathrm{Assh}_R(R) = \{\mathfrak{m}\}$. □

THEOREM 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and $\lambda_R(R) = 0$. Then $|\mathrm{Assh}_R(R)| = 1$.*

Proof. When R is an integral domain, there is nothing to prove. Moreover, if $d = 0$ the assertion follows from Lemma 3.1. Therefore, we may assume that R is not an integral domain and that $d \geq 1$. Clearly, the set $\mathrm{Assh}_R(R)$ is a non-empty set. Suppose that $\mathrm{Assh}_R(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$, where $k \geq 2$ and $\mathfrak{p}_i \neq \mathfrak{p}_j$ for each $i \neq j$. Then $\bigcap_{i=2}^{k} \mathfrak{p}_i \nsubseteq \mathfrak{p}_1$ and so there exists an element $x_1 \in \bigcap_{i=2}^{k} \mathfrak{p}_i \backslash \mathfrak{p}_1$. Because x_1 is a regular element on $\frac{R}{\mathfrak{p}_1}$ as an R-module, we can extend it to a system of parameters such as $x_1, \dots, x_d$ for the R-module $\frac{R}{\mathfrak{p}_1}$. Therefore, in case that $d = 1$, we have $\sqrt{< x_1 > + \mathfrak{p}_1} = \mathfrak{m}$. Now by [2, Theorem 4.2.1] and Theorem 2.2, we can write

$$H^1_{\mathfrak{p}_1}(\frac{R}{\mathfrak{p}_2}) \cong H^1_{\frac{< x_1 > + \mathfrak{p}_1}{< x_1 >}}(\frac{R}{\mathfrak{p}_2}) \cong H^1_{\frac{\mathfrak{m}}{< x_1 >}}(\frac{R}{\mathfrak{p}_2}) \cong H^1_{\mathfrak{m}}(\frac{R}{\mathfrak{p}_2}) \neq 0.$$

On the other hand, the exact sequence

$$0 \to \mathfrak{p}_2 \to R \to \frac{R}{\mathfrak{p}_2} \to 0, \qquad (*)$$

induces a long exact sequence

$$... \to H^1_{\mathfrak{p}_1}(\mathfrak{p}_2) \to H^1_{\mathfrak{p}_1}(R) \to H^1_{\mathfrak{p}_1}(\frac{R}{\mathfrak{p}_2}) \to H^2_{\mathfrak{p}_1}(\mathfrak{p}_2) \to ....$$

Since $\dim_R(\mathfrak{p}_2) \leq 1$, it follows that $H^2_{\mathfrak{p}_1}(\mathfrak{p}_2) = 0$ by Theorem 2.1, and so $H^1_{\mathfrak{p}_1}(R) \neq 0$. This shows that $\lambda_R(R) \geq \dim\frac{R}{\mathfrak{p}_1} = 1$ which is a contradiction.

In case that $d > 1$, in view of [7, Lemma 2.6], there exists a prime ideal $\frac{\mathfrak{q}}{\mathfrak{p}_1} \in \operatorname{Min}\left(\frac{(x_2, \ldots, x_d) + \mathfrak{p}_1}{\mathfrak{p}_1}\right)$ such that $\dim \frac{R}{\mathfrak{q}} = 1$ and $\operatorname{height}(\frac{\mathfrak{q}}{\mathfrak{p}_1}) = d - 1$. The latter yields $x_1 \notin \mathfrak{q}$. This follows that x_1 is a regular element on $\frac{R}{\mathfrak{q}}$ as an R-module and consequently, $\sqrt{< x_1 > + \mathfrak{q}} = \mathfrak{m}$. Therefore,

$$H^d_{\mathfrak{q}}(\frac{R}{\mathfrak{p}_2}) \cong H^d_{\frac{< x_1 > + \mathfrak{q}}{< x_1 >}}(\frac{R}{\mathfrak{p}_2}) \cong H^d_{\frac{\mathfrak{m}}{< x_1 >}}(\frac{R}{\mathfrak{p}_2}) \cong H^d_{\mathfrak{m}}(\frac{R}{\mathfrak{p}_2}) \neq 0.$$

By a similar argument as in the case $d = 1$, one can find from the exact sequence $(*)$ that $H^d_{\mathfrak{q}}(R) \neq 0$. That is $\lambda_R(R) \geq \dim \frac{R}{\mathfrak{q}} = 1$ which is a contradiction. □

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and $\lambda_R(R) = 0$. Then $(0 :_R H^d_{\mathfrak{m}}(R))$ is a primary ideal of R.*

Proof. It follows from Theorem 3.2 that $|\operatorname{Assh}_R(R)| = 1$. Let $\operatorname{Assh}_R(R) = \{\mathfrak{p}\}$. Therefore, in view of [1, Theorem 2.8] and [2, Lemma 7.3.1], we find that $\operatorname{Ass}_R(\frac{R}{(0 : H^d_{\mathfrak{m}}(R))}) = \{\mathfrak{p}\}$. Now the proof is completed by [6, Theorem 6.6]. □

Theorem 3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $x_1, \ldots, x_d$ be a system of parameters for R and $\bar{R} = \frac{R}{< x_1, ..., x_d >}$. Then the following conditions hold;*
(i) $\lambda_{\bar{R}}(\bar{R}) = 0$,
(ii) $\lambda_R(\bar{R}) = d$.

Proof. (i) It is clear by Theorem 3.1.

(ii) Since $\dim_R \bar{R} = 0$, we have $\lambda_R(\bar{R}) = \sup\{\dim \frac{R}{I} : \Gamma_I(\bar{R}) \neq 0\}$. Thus, $\lambda_R(\bar{R}) = d$ because $\Gamma_0(\bar{R}) \neq 0$. □

Lemma 3.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and I be an ideal of R with $\operatorname{Cd}(I, R) = d$. Then $\operatorname{ara}(J) = d$ for every ideal J of R such that $I \subseteq J \subseteq \mathfrak{m}$.*

Proof. By [3, Theorem 2.5], we have the following epimorphism

$$H^d_J(R) \to H^d_I(R) \to 0.$$

Since $H^d_I(R) \neq 0$ it follows that $H^d_J(R) \neq 0$, which implies $\operatorname{Cd}(J, R) = d$. Now, in view of [2, Definition 3.3.4], we have $\operatorname{ara}(J) \geq d$. Therefore, the proof is completed by [7, Corollary 2.8]. □

Theorem 3.6. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* d*. Suppose that* $\operatorname{ara}(\mathfrak{p}) < d$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$*. Then* $\lambda_R(R) = 0$*.*

Proof. Let $\lambda_R(R) = t \geq 1$. Then there exists an ideal I of R with $H_I^d(R) \neq 0$ and $\dim \frac{R}{I} = t$. These yield that $\operatorname{Cd}(I, R) = d$ and that there exists an ideal $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$ such that $\dim \frac{R}{\mathfrak{p}} = t$. Now, in view of Lemma 3.5, $\operatorname{ara}(\mathfrak{p}) = d$ which is a contradiction. Therefore, $\lambda_R(R) = 0$. □

Corollary 3.7. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* d*. Let* $\operatorname{ara}(\mathfrak{p}) < d$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$*. Then* $|\operatorname{Assh}_R(R)| = 1$

Proof. Follows from Theorem 3.2 and Theorem 3.6. □

Acknowledgments. The authors are deeply grateful to the reviewer for a very careful reading of the manuscript and useful suggestions in improving the quality of the paper.

REFERENCES

[1] K. Bahmanpour, J. A'zami, and G. Ghasemi, *On the annihilators of local cohomology modules*. J. Algebra **363** (2012), 8–13.

[2] M.P. Brodmann and R.Y. Sharp, *Local Cohomology. An Algebraic Introduction with Geometric Applications*. Cambridge Stud. Adv. Math. **60**, Cambridge Univ. Press, Cambridge, 1998.

[3] L. Chu, *Top local cohomology modules with respect to a pair of ideals*. Proc. Amer. Math. Soc. **139** (2010), 777–782.

[4] K. Divaani-Aazar, R. Naghipour, and M. Tousi, *Cohomological dimension of certain algebraic varieties*. Proc. Amer. Math. Soc. **130** (2002), 3537–3544.

[5] G. Ghasemi, K. Bahmanpour, and J. A'zami, *On the cofiniteness of Artinian local cohomology modules*. J. Algebra Appl. **15** (2016), *4*, 1650070.

[6] H. Matsumura, *Commutative Ring Theory*. Cambridge Stud. Adv. Math. bf 8, Cambridge Univ. Press, Cambridge, 1986.

[7] A.A. Mehrvarz, K. Bahmanpour, and R. Naghipour, *Arithmetic rank, cohomological dimension and filter regular sequences*. J. Algebra Appl. **8** (2009), 855–862.

Received November 26, 2020

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