# ALMOST POISED $_2\phi_1$-SERIES EXTENDED WITH TWO INTEGER PARAMETERS

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Almost poised $_2\phi_1$-series extended with two integer parameters are investigated by means of the linearization method. Four analytical formulae are established and fourteen closed formulae are presented as examples.

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## 1. INTRODUCTION AND MOTIVATION

Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and natural numbers with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For two indeterminates $x$ and $q$, define the shifted factorials by $(x;q)_0 = \langle x;q\rangle_0 = 1$ and

$$\left.\begin{aligned}(x;q)_n =&(1-x)(1-qx)\cdots(1-q^{n-1}x)\\ \langle x;q\rangle_n =&(1-x)(1-x/q)\cdots(1-q^{1-n}x)\end{aligned}\right\} \quad \text{for} \quad n\in\mathbb{N}.$$

The rising factorial of negative order can be expressed as

$$(x;q)_{-n} = \frac{1}{(q^{-n}x;q)_n} = q^{\binom{n}{2}}\frac{(-q/x)^n}{(q/x;q)_n} \quad \text{where} \quad n\in\mathbb{N}.$$

The product and fraction of shifted factorials are abbreviated respectively to

$$\begin{aligned}[\,\alpha,\,\beta,\,\cdots,\,\gamma;q\,]_n \quad &= \quad (\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n\,,\\ \left[\begin{matrix}\alpha,\,\beta,\,\cdots,\,\gamma\\ A,\,B,\,\cdots,\,C\end{matrix}\,\middle|\,q\right]_n \quad &= \quad \frac{(\alpha;q)_n\,(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n\,(B;q)_n\cdots(C;q)_n}.\end{aligned}$$

Following Bailey [2] and Gasper–Rahman [8], the basic hypergeometric series (shortly as $q$-series) is defined by

$${}_{1+p}\phi_p\left[\begin{matrix}a_0,a_1,\cdots,a_p\\ b_1,\cdots,b_p\end{matrix}\,\middle|\,q;z\right] = \sum_{n=0}^{\infty}\left[\begin{matrix}a_0,a_1,\cdots,a_p\\ q,\;b_1,\cdots,b_p\end{matrix}\,\middle|\,q\right]_n z^n.$$

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This series is well-poised if the linear condition $qa_0 = a_1b_1 = \cdots = a_pb_p$ is satisfied by its parameters. When this condition is disturbed by integer powers of the base $q$, we say that the series is "almost poised".

Among the numerous identities, the following one (cf. Gasper–Rahman [8, II-20]) for the nonterminating well-poised ${}_6\phi_5$-series plays a fundamental role:

$$ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d \\ \sqrt{a}, & -\sqrt{a}, qa/b, & qa/c, & qa/d \end{matrix} \Big| q; \frac{qa}{bcd}\right] = \left[\begin{matrix} qa, qa/bc, qa/bd, qa/cd \\ qa/b, qa/c, qa/d, qa/bcd \end{matrix} \Big| q\right]_\infty. $$

When $b, d = \pm\sqrt{a}$, the above identity recovers the $q$-Kummer formula established independently by Bailey [3] and Daum [7] (see also Gasper–Rahman [8, II-9]):

$$ {}_2\phi_1\left[\begin{matrix} a,\ c \\ qa/c \end{matrix} \Big| q; -q/c\right] = \frac{(qa;q^2)_\infty}{(qa/c^2;q^2)_\infty}\left[\begin{matrix} qa/c^2, -q \\ qa/c, -q/c \end{matrix} \Big| q\right]_\infty. \tag{1.1} $$

For the $q$-Kummer theorem, we find that there is a positive series counterpart

$$ \begin{aligned} {}_2\phi_1\left[\begin{matrix} a,\ c \\ qa/c \end{matrix} \Big| q; q^{1/2}/c\right] =& \frac{1}{2}\left[\begin{matrix} q^{1/2}, a \\ q^{1/2}/c, qa/c \end{matrix} \Big| q\right]_\infty \\ &\times\left\{\frac{(\sqrt{qa}/c;q^{1/2})_\infty}{(\sqrt{a};q^{1/2})_\infty} + \frac{(-\sqrt{qa}/c;q^{1/2})_\infty}{(-\sqrt{a};q^{1/2})_\infty}\right\} \end{aligned} \tag{1.2} $$

which resembles the following one discovered by Andrews and Askey [1, eq. 3.25]

$$ \begin{aligned} {}_2\phi_1\left[\begin{matrix} a,\ c \\ a/c \end{matrix} \Big| q; q^{1/2}/c\right] =& \frac{1}{2}\left[\begin{matrix} q^{1/2}, a \\ q^{1/2}/c, a/c \end{matrix} \Big| q\right]_\infty \\ &\times\left\{\frac{(\sqrt{a}/c;q^{1/2})_\infty}{(\sqrt{a};q^{1/2})_\infty} + \frac{(-\sqrt{a}/c;q^{1/2})_\infty}{(-\sqrt{a};q^{1/2})_\infty}\right\}. \end{aligned} \tag{1.3} $$

Letting $a = q^{-m}$ with $m \in \mathbb{N}_0$ in the above two identities, we derive, under the base replacement $q \to q^2$, their terminating forms

$$ \begin{aligned} {}_2\phi_1\left[\begin{matrix} q^{-2m},\ c \\ q^{2-2m}/c \end{matrix} \Big| q^2; q/c\right] =& \frac{[c,-q;q]_m}{(c;q^2)_m}\cdot q^{-m}, \\ {}_2\phi_1\left[\begin{matrix} q^{-2m},\ c \\ q^{-2m}/c \end{matrix} \Big| q^2; q/c\right] =& \frac{[c,-q;q]_m}{(c;q^2)_m}\cdot\frac{1-q^mc}{1-q^{2m}c}. \end{aligned} $$

When $c \to 0$ further, the last two identities reduce to the same amazing binomial formula (cf. Andrews–Askey [1, eq. 3.26]:

$$ \sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}_{q^2} q^k = (-q;q)_m. $$

Following Andrews–Askey [1], we can provide, for completeness, an elementary proof of (1.2). Recalling the first Heine transformation (Gasper–Rahman [8, III-1])

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, q; z\right] = \left[\begin{matrix} a, bz \\ c, z \end{matrix} \,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix} c/a, \ z \\ bz \end{matrix} \,\middle|\, q; a\right] $$

we can first express the ${}_2\phi_1$-series in question as another ${}_2\phi_1$-series

$$ {}_2\phi_1\left[\begin{matrix} a, \ c \\ qa/c \end{matrix} \,\middle|\, q; q^{1/2}/c\right] = \left[\begin{matrix} q^{1/2}, \ a \\ q^{1/2}/c, qa/c \end{matrix} \,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix} q/c, q^{1/2}/c \\ q^{1/2} \end{matrix} \,\middle|\, q; a\right]. $$

Then the last ${}_2\phi_1$-series can be evaluated, in turn, by means of the $q$-binomial series (Gasper–Rahman [8, II-3])

$$ {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix} \,\middle|\, q; z\right] = \frac{(az; q)_\infty}{(z; q)_\infty} $$

in the following closed form:

$$ \begin{aligned} {}_2\phi_1\left[\begin{matrix} q/c, q^{1/2}/c \\ q^{1/2} \end{matrix} \,\middle|\, q; a\right] &= \frac{1}{2}\left\{ {}_1\phi_0\left[\begin{matrix} q^{1/2}/c \\ - \end{matrix} \,\middle|\, q^{1/2}; \sqrt{a}\right] + {}_1\phi_0\left[\begin{matrix} q^{1/2}/c \\ - \end{matrix} \,\middle|\, q^{1/2}; -\sqrt{a}\right] \right\} \\ &= \frac{1}{2}\left\{ \frac{(\sqrt{qa}/c; q^{1/2})_\infty}{(\sqrt{a}; q^{1/2})_\infty} + \frac{(-\sqrt{qa}/c; q^{1/2})_\infty}{(-\sqrt{a}; q^{1/2})_\infty} \right\}. \end{aligned} $$

Motivated by these two identities (1.2) and (1.3), this paper will examine, for any given pair of integers $\lambda$ and $\rho$, the following general series

$$ \Omega_\lambda^\rho := \Omega_\lambda^\rho(a, c) = {}_2\phi_1\left[\begin{matrix} a, \quad c \\ q^{1+\rho}a/c \end{matrix} \,\middle|\, q; q^{\lambda+1/2}/c\right]. \tag{1.4} $$

By applying the linearization method employed in [5, 6, 4, 9], we shall prove (see Theorem 6) that the $\Omega_\lambda^\rho(a, c)$-series for $\lambda, \rho \in \mathbb{Z}$ is always explicitly evaluable in the $\Omega_0^0(a', c')$-series with the number of terms at most $(1 + |\rho|) \times \big(1 + |\lambda| + |\rho|\big)$.

The rest of the paper will be organized as follows. In the next section, we prove by means of the $q$-binomial theorem two formulae that transform the $\Omega_\lambda^\rho$-series into the $\Omega_\lambda^0$-series. Then the $\Omega_\lambda^0$-series will be explicitly evaluated in Section 3 through the linearization method. Finally, the paper will end up with fourteen further examples as applications.

## 2. REDUCTION FORMULAE FROM $\Omega_\lambda^\rho$ TO $\Omega_\lambda^0$

By applying the series rearrangement and the $q$-binomial theorem

$$ (x; q)_m = \sum_{k=0}^{m} q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix} (-x)^k $$

we shall derive, in this section, two transformation formulae that express the $\Omega_\lambda^\rho$-series in terms of the $\Omega_\lambda^0$-series.

## §2.1. $\boldsymbol{\rho \geq 0}$

By inserting the binomial relation in the $\Omega_\lambda^\rho$-series

$$(q^{n-\rho}c;q)_\rho = \sum_{k=0}^{\rho}(-c)^{\rho-k}\begin{bmatrix}\rho\\k\end{bmatrix}q^{\binom{\rho-k}{2}+(n-\rho)(\rho-k)}$$

we can reformulate the following double series

$$\begin{aligned}\Omega_\lambda^\rho(a,c) =&\sum_{n=0}^{\infty}\frac{(a;q)_n(c;q)_n}{(q;q)_n(q^{1+\rho}a/c;q)_n}\Big(\frac{q^{\lambda+\frac{1}{2}}}{c}\Big)^n\sum_{k=0}^{\rho}\frac{(-c)^{\rho-k}}{(q^{n-\rho}c;q)_\rho}\begin{bmatrix}\rho\\k\end{bmatrix}q^{\binom{\rho-k}{2}+(n-\rho)(\rho-k)}\\ =&\sum_{k=0}^{\rho}\frac{(-c)^{\rho-k}}{(q^{-\rho}c;q)_\rho}\begin{bmatrix}\rho\\k\end{bmatrix}q^{\binom{\rho-k}{2}-\rho(\rho-k)}\sum_{n=0}^{\infty}\frac{(a;q)_n(q^{-\rho}c;q)_n}{(q;q)_n(q^{1+\rho}a/c;q)_n}\Big(\frac{q^{\lambda+\rho-k+\frac{1}{2}}}{c}\Big)^n.\end{aligned}$$

Writing the last sum as $\Omega_{\lambda-k}^0(a,q^{-\rho}c)$, we derive the following reduction formula.

Theorem 1 ($\lambda,\ \rho\in\mathbb{Z}$ with $\rho\geq 0$).

$$\Omega_\lambda^\rho(a,c) = \sum_{k=0}^{\rho}q^{\binom{k}{2}}\begin{bmatrix}\rho\\k\end{bmatrix}\frac{(-q/c)^k}{(q/c;q)_\rho}\Omega_{\lambda-k}^0(a,q^{-\rho}c).$$

## §2.2. $\boldsymbol{\rho \leq 0}$

Instead, by putting another binomial relation inside the $\Omega_\lambda^\rho$-series

$$(q^{1+n+\rho}a/c;q)_{-\rho} = \sum_{k=0}^{-\rho}\Big(\frac{-a}{c}\Big)^k\begin{bmatrix}-\rho\\k\end{bmatrix}q^{\binom{k+1}{2}+k\rho+kn}$$

we can analogously manipulate the following double series

$$\begin{aligned}\Omega_\lambda^\rho(a,c) =&\sum_{n=0}^{\infty}\frac{(a;q)_n(c;q)_n}{(q;q)_n(q^{1+\rho}a/c;q)_n}\Big(\frac{q^{\lambda+\frac{1}{2}}}{c}\Big)^n\sum_{k=0}^{-\rho}\Big(\frac{-a}{c}\Big)^k\begin{bmatrix}-\rho\\k\end{bmatrix}\frac{q^{\binom{k+1}{2}+k\rho+kn}}{(q^{1+n+\rho}a/c;q)_{-\rho}}\\ =&\sum_{k=0}^{-\rho}\Big(\frac{-a}{c}\Big)^k\begin{bmatrix}-\rho\\k\end{bmatrix}\frac{q^{\binom{k+1}{2}+k\rho}}{(q^{1+\rho}a/c;q)_{-\rho}}\sum_{n=0}^{\infty}\frac{(a;q)_n(c;q)_n}{(q;q)_n(qa/c;q)_n}\Big(\frac{q^{\lambda+k+\frac{1}{2}}}{c}\Big)^n.\end{aligned}$$

Writing the last sum as $\Omega_{\lambda+k}^0(a,c)$, we derive another reduction formula.

Theorem 2 ($\lambda,\ \rho \in \mathbb{Z}$ with $\rho \le 0$).

$$\Omega_\lambda^\rho(a,c) = \sum_{k=0}^{-\rho} q^{\binom{k}{2}} \begin{bmatrix} -\rho \\ k \end{bmatrix} \frac{(-q^{1+\rho}a/c)^k}{(q^{1+\rho}a/c;q)_{-\rho}} \Omega_{\lambda+k}^0(a,c).$$

## 3. REDUCTION FORMULAE FROM $\Omega_\lambda^0$ TO $\Omega_0^0$

By making use of the second Heine transformation (Gasper–Rahman [8, III-2])

$${}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, q; z\right] = \left[\begin{matrix} c/b, bz \\ c, z \end{matrix} \,\middle|\, q\right]_\infty {}_2\phi_1\left[\begin{matrix} abz/c,\ b \\ bz \end{matrix} \,\middle|\, q; c/b\right]$$

we can express the $\Omega_\lambda^0(a,c)$-series as

$$\Omega_\lambda^0(a,c) = \left[\begin{matrix} q/c, q^{\lambda+1/2}a/c \\ qa/c, q^{\lambda+1/2}/c \end{matrix} \,\middle|\, q\right]_\infty \times {}_2\phi_1\left[\begin{matrix} a,\ q^{\lambda-1/2}c \\ q^{\lambda+1/2}a/c \end{matrix} \,\middle|\, q; q/c\right]. \tag{3.1}$$

Therefore, in order to evaluate $\Omega_\lambda^0(a,c)$, it suffices to find explicit formulae for the rightmost nonterminating ${}_2\phi_1$-series. For that purpose, we have to invoke the following linearization lemma.

Lemma 3 (Linear representation). *Let $x$ be a variable and $m$ a natural number. Then for three indeterminates $\{u,v,w\}$, the following linear representation formula holds*

$$(wx;q)_m = \sum_{k=0}^{m} (ux;q)_{m-k} \langle vx;q\rangle_k\, \mathcal{E}_m^k(u,v,w), \tag{3.2}$$

*where the connection coefficients $\{\mathcal{E}_m^k(u,v,w)\}$ are independent of $x$ and given by*

$$\mathcal{E}_m^k(u,v,w) = q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix} \frac{(w/u,q)_k (w/v;q)_m}{(w/v;q)_k (u/v;q)_m} \Big(-\frac{u}{v}\Big)^k. \tag{3.3}$$

*Proof.* Recall the following three relations

$$\begin{aligned} \begin{bmatrix} m \\ k \end{bmatrix} &= (-1)^k \frac{(q^{-m};q)_k}{(q;q)_k} q^{mk-\binom{k}{2}}, \\ (ux;q)_{m-k} &= \Big(\frac{-q}{ux}\Big)^k \frac{(ux;q)_m}{(q^{1-m}/ux;q)_k} q^{\binom{k}{2}-mk}, \\ \langle vx;q\rangle_k &= (-vx)^k (1/vx;q)_k q^{-\binom{k}{2}}. \end{aligned}$$

Substituting (3.3) into (3.2), we confirm the lemma by simplifying the finite sum

$$\sum_{k=0}^{m}(ux;q)_{m-k}\langle vx;q\rangle_k \begin{bmatrix} m\\ k\end{bmatrix} q^{\binom{k}{2}}\frac{(w/u,q)_k(w/v;q)_m}{(w/v;q)_k(u/v;q)_m}\Big(-\frac{u}{v}\Big)^k$$

$$=\frac{(ux;q)_m(w/v;q)_m}{(u/v;q)_m}\sum_{k=0}^{m}q^k\left[\begin{matrix} q^{-m},1/vx,w/u,\\ q,q^{1-m}/ux,w/v\end{matrix}\,\Big|\,q;q\right]_k=(wx;q)_m,$$

where the last sum has been evaluated by means of the following $q$-Saalschütz summation formula (cf. [8, II-12])

$${}_3\phi_2\left[\begin{matrix} q^{-m},\ a,\ b\\ c,q^{1-m}ab/c\end{matrix}\,\Big|\,q;q\right]=\left[\begin{matrix} c/a,c/b\\ c,c/ab\end{matrix}\,\Big|\,q\right]_m.$$

□

## §3.1. $\boldsymbol{\lambda\geq 0}$

When $\lambda\geq 0$, specifying in Lemma 3 by

$$\left.\begin{matrix} m\to\lambda\\ x\to q^n\end{matrix}\right\}\quad\text{and}\quad\left\{\begin{matrix} u\to a\\ v\to q^{\lambda-1/2}a/c\\ w\to q^{-1/2}c\end{matrix}\right.$$

we get the equation

$$(q^{n-1/2}c;q)_\lambda=\sum_{k=0}^{\lambda}(q^na;q)_k\langle q^{n+\lambda-1/2}a/c;q\rangle_{\lambda-k}\mathcal{E}_\lambda^{\lambda-k}(a,q^{\lambda-1/2}a/c,q^{-1/2}c).$$

By inserting this relation in the ${}_2\phi_1$-series displayed in (3.1), we can reformulate the double sum below

$${}_2\phi_1\left[\begin{matrix} a,\ q^{\lambda-1/2}c\\ q^{\lambda+1/2}a/c\end{matrix}\,\Big|\,q;q/c\right]=\sum_{n=0}^{\infty}\Big(\frac{q}{c}\Big)^n\left[\begin{matrix} a,\ q^{\lambda-1/2}c\\ q,q^{\lambda+1/2}a/c\end{matrix}\,\Big|\,q\right]_n$$

$$\times\sum_{k=0}^{\lambda}\frac{(q^na;q)_k\langle q^{n+\lambda-1/2}a/c;q\rangle_{\lambda-k}}{(q^{n-1/2}c;q)_\lambda}\mathcal{E}_\lambda^{\lambda-k}(a,q^{\lambda-1/2}a/c,q^{-1/2}c)$$

$$=\frac{(q^{1/2}a/c;q)_\lambda}{(q^{-1/2}c;q)_\lambda}\sum_{k=0}^{\lambda}\frac{(a;q)_k}{(q^{1/2}a/c;q)_k}\mathcal{E}_\lambda^{\lambda-k}(a,q^{\lambda-1/2}a/c,q^{-1/2}c)$$

$$\times\sum_{n=0}^{\infty}\Big(\frac{q}{c}\Big)^n\left[\begin{matrix} q^ka,\ q^{-1/2}c\\ q,q^{k+1/2}a/c\end{matrix}\,\Big|\,q\right]_n.$$

In view of (3.1), evaluating explicitly the last series

$$(3.4)\qquad {}_2\phi_1\left[\begin{matrix}q^k a,\ q^{-1/2}c\\ q^{k+1/2}a/c\end{matrix}\,\Big|\, q;\frac{q}{c}\right]=\Omega_0(q^k a,c)\times\left[\begin{matrix}q^{1/2}/c, q^{1+k}a/c\\ q/c, q^{k+1/2}a/c\end{matrix}\,\Big|\, q\right]_\infty$$

we establish the following expression

$$\begin{aligned}{}_2\phi_1\left[\begin{matrix}a,\ q^{\lambda-1/2}c\\ q^{\lambda+1/2}a/c\end{matrix}\,\Big|\, q;q/c\right]=&\frac{(q^{1/2}a/c;q)_\lambda}{(q^{-1/2}c;q)_\lambda}\left[\begin{matrix}q^{1/2}/c, qa/c\\ q/c, q^{1/2}a/c\end{matrix}\,\Big|\, q\right]_\infty\\ &\times\sum_{k=0}^{\lambda}\mathcal{E}_\lambda^{\lambda-k}(a,q^{\lambda-1/2}a/c,q^{-1/2}c)\frac{(a;q)_k}{(qa/c;q)_k}\Omega_0(q^k a,c).\end{aligned}$$

Replacing the $\mathcal{E}$-coefficient by (3.3) and then simplifying the result, we find the following theorem.

THEOREM 4 ($\lambda\in\mathbb{Z}$ with $\lambda\geq 0$).

$$\Omega_\lambda^0(a,c)=\frac{(q^{-\frac{1}{2}}c/a;q)_\lambda}{(q^{-\frac{1}{2}}c;q)_\lambda}\sum_{k=0}^{\lambda}q^k\left[\begin{matrix}q^{-\lambda},\ a,\ qa/c^2\\ q, qa/c, q^{\frac{3}{2}-\lambda}a/c\end{matrix}\,\Big|\, q\right]_k\Omega_0^0(q^k a,c).$$

## §3.2. $\boldsymbol{\lambda\leq 0}$

When $\lambda\geq 0$, specifying in Lemma 3 by

$$\left.\begin{matrix}m\to-\lambda\\ x\to q^n\end{matrix}\right\}\quad\text{and}\quad\left\{\begin{matrix}u\to q^{\lambda-1/2}c\\ v\to 1\\ w\to q^{\lambda+1/2}a/c\end{matrix}\right.$$

we get the equation

$$(q^{n+\lambda+1/2}a/c;q)_{-\lambda}=\sum_{k=0}^{-\lambda}(q^{n+\lambda-1/2}c;q)_{-\lambda-k}\langle q^n;q\rangle_k\mathcal{E}_\lambda^k(q^{\lambda-1/2}c,1,q^{\lambda+1/2}a/c).$$

By putting this relation inside the ${}_2\phi_1$-series displayed in (3.1), we can manipulate the double sum below

$$\begin{aligned}{}_2\phi_1\left[\begin{matrix}a,\ q^{\lambda-1/2}c\\ q^{\lambda+1/2}a/c\end{matrix}\,\Big|\, q;q/c\right]=&\sum_{n=0}^{\infty}\left(\frac{q}{c}\right)^n\left[\begin{matrix}a,\ q^{\lambda-1/2}c\\ q, q^{\lambda+1/2}a/c\end{matrix}\,\Big|\, q\right]_n\\ &\times\sum_{k=0}^{-\lambda}\frac{\langle q^n;q\rangle_k(q^{n+\lambda-1/2}c;q)_{-\lambda-k}}{(q^{n+\lambda+1/2}a/c;q)_{-\lambda}}\mathcal{E}_\lambda^k(q^{\lambda-1/2}c,1,q^{\lambda+1/2}a/c)\end{aligned}$$

$$= \sum_{k=0}^{-\lambda} \frac{(q^{\lambda-1/2}c;q)_{-\lambda-k}}{(q^{\lambda+1/2}a/c;q)_{-\lambda}} \mathcal{E}_\lambda^k(q^{\lambda-1/2}c,1,q^{\lambda+1/2}a/c)$$

$$\times \sum_{n=k}^{\infty} \left[\begin{matrix} a,\ q^{-k-1/2}c \\ q^{1/2}a/c \end{matrix} \,\middle|\, q\right]_n \frac{(q/c)^n}{(q;q)_{n-k}}$$

$$= \frac{(q^{1/2}a/c;q)_\lambda}{(q^{-1/2}c;q)_\lambda} \sum_{k=0}^{-\lambda} \left(\frac{q}{c}\right)^k \frac{(a;q)_k}{(q^{1/2}a/c;q)_k} \mathcal{E}_\lambda^k(q^{\lambda-1/2}c,1,q^{\lambda+1/2}a/c)$$

$$\times \sum_{n=0}^{\infty} \left[\begin{matrix} q^k a,\ q^{-1/2}c \\ q, q^{k+1/2}a/c \end{matrix} \,\middle|\, q\right]_n \left(\frac{q}{c}\right)^n$$

where the last passage has been justified by the replacement $n \to n+k$ on the summation index $n$. Evaluating the last series by (3.4), we have the following expression

$$ {}_2\phi_1\left[\begin{matrix} a,\ q^{\lambda-1/2}c \\ q^{\lambda+1/2}a/c \end{matrix} \,\middle|\, q; q/c\right] = \frac{(q^{1/2}a/c;q)_\lambda}{(q^{-1/2}c;q)_\lambda} \left[\begin{matrix} qa/c, q^{1/2}/c \\ q/c, q^{1/2}a/c \end{matrix} \,\middle|\, q\right]_\infty$$

$$\times \sum_{k=0}^{-\lambda} \left(\frac{q}{c}\right)^k \mathcal{E}_\lambda^k(q^{\lambda-1/2}c,1,q^{\lambda+1/2}a/c) \frac{(a;q)_k}{(qa/c;q)_k} \Omega_0(q^k a, c).$$

Replacing the $\mathcal{E}$-coefficient by (3.3) and then simplifying the resulting expression, we get another theorem.

Theorem 5 ($\lambda \in \mathbb{Z}$ with $\lambda \le 0$).

$$\Omega_\lambda^0(a,c) = \frac{(q^{\frac12}/c;q)_\lambda}{(q^{\frac12}a/c;q)_\lambda} \sum_{k=0}^{-\lambda} q^{\frac{k}{2}} \left[\begin{matrix} q^\lambda,\ a,\ qa/c^2 \\ q, qa/c, q^{\lambda+\frac12}a/c \end{matrix} \,\middle|\, q\right]_k \Omega_0^0(q^k a, c).$$

## 4. CONCLUSIVE THEOREM AND EXAMPLES

Summing up the results shown in the previous two sections, we can evaluate the $\Omega_\lambda^\rho$-series, for any given pair of integers $\lambda$ and $\rho$, by carrying out the following procedure:

- Step A. If $\rho \neq 0$, we first transform the $\Omega_\lambda^\rho$-series into the $\Omega_{\lambda'}^0$-series by making use of Theorems 1 and 2 and then go to Step B.

- Step B. If $\rho = 0$, we evaluate the $\Omega_\lambda^0$-series by means of Theorems 4 and 5.

Therefore, we have confirmed the following conclusive theorem.

THEOREM 6. *For any given pair of integers* $\lambda$ *and* $\rho$, *the* $\Omega_\lambda^\rho$*-series can be explicitly expressed as a linear combination of the* $\Omega_0^0$*-series with the number of terms at most* $(1+|\rho|)(1+|\lambda|+|\rho|)$.

The afore-described procedure is realized by appropriately devised *Mathematica* commands, that are executed to produce several closed formulae of the $\Omega_\lambda^\rho$-series for small integers $\lambda$ and $\rho$. In order to avoid complicated expressions, we shall write the identities under the replacements $q \to q^2$ and $a \to a^2$. As a showcase, the first two examples in (1.2) and (1.3) can be reformulated as

$$(\Omega_0^0) \quad {}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^2a^2/c \end{matrix}\,\middle|\, q^2; q/c\right] = \left[\begin{matrix} q, a^2 \\ q/c, q^2a^2/c \end{matrix}\,\middle|\, q^2\right]_\infty \left\{\frac{(qa/c;q)_\infty}{2(a;q)_\infty} + \frac{(-qa/c;q)_\infty}{2(-a;q)_\infty}\right\},$$

$$(\Omega_0^{-1}) \quad {}_2\phi_1\left[\begin{matrix} a^2,\ c \\ a^2/c \end{matrix}\,\middle|\, q^2;\ q/c\right] = \left[\begin{matrix} q, a^2 \\ q/c,\ \ a^2/c \end{matrix}\,\middle|\, q^2\right]_\infty \left\{\frac{(a/c;q)_\infty}{2(a;q)_\infty} + \frac{(-a/c;q)_\infty}{2(-a;q)_\infty}\right\}.$$

Fourteen remarkable examples are recorded as further applications. They do not seem to have appeared previously in the mathematical literature.

*Example* 1 ($\lambda = 1$ and $\rho = 0$).

$${}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^2a^2/c \end{matrix}\,\middle|\, q^2; q^3/c\right] = \left[\begin{matrix} q, a^2 \\ q/c, q^2a^2/c \end{matrix}\,\middle|\, q^2\right]_\infty \times \Big\{\theta(a)+\theta(-a)\Big\}$$

$$\text{where} \quad \theta(a) = \frac{(qa/c;q)_\infty}{2a(a;q)_\infty}.$$

*Example* 2 ($\lambda = 1$ and $\rho = 1$).

$${}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^4a^2/c \end{matrix}\,\middle|\, q^2; q^3/c\right] = \left[\begin{matrix} q, a^2 \\ q^3/c, q^4a^2/c \end{matrix}\,\middle|\, q^2\right]_\infty \times \Big\{\theta(a)+\theta(-a)\Big\}$$

$$\text{where} \quad \theta(a) = \frac{(q^2a/c;q)_\infty}{2a(1-q^2/c)(a;q)_\infty}.$$

*Example* 3 ($\lambda = 0$ and $\rho = 1$).

$${}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^4a^2/c \end{matrix}\,\middle|\, q^2; q/c\right] = \left[\begin{matrix} q, a^2 \\ q/c, q^4a^2/c \end{matrix}\,\middle|\, q^2\right]_\infty \times \Big\{\theta(a)+\theta(-a)\Big\}$$

$$\text{where} \quad \theta(a) = \frac{(q^2a/c;q)_\infty}{2(a;q)_\infty} \times \frac{q^2-c+qa-q^2a}{q^2-c}.$$

*Example* 4 ($\lambda = 1$ and $\rho = -1$).

$${}_2\phi_1\left[\begin{matrix} a^2,\ c \\ a^2/c \end{matrix}\,\middle|\, q^2; q^3/c\right] = \left[\begin{matrix} q, a^2 \\ q/c, a^2/c \end{matrix}\,\middle|\, q^2\right]_\infty \times \Big\{\theta(a)+\theta(-a)\Big\}$$

$$\text{where} \quad \theta(a) = \frac{(a/c;q)_\infty}{2(a;q)_\infty} \times \frac{q+a-qa-qc}{a(1-qc)}.$$

*Example* 5 ($\lambda = -1$ and $\rho = -1$).

$$ {}_2\phi_1\left[\begin{matrix} a^2,\ c \\ a/c \end{matrix}\,\Big|\, q^2; q^{-1}/c\right] = \left[\begin{matrix} q, a \\ q/c, a^2/c \end{matrix}\,\Big|\, q^2\right]_\infty \times \Big\{\theta(a)+\theta(-a)\Big\} $$
$$ \text{where}\quad \theta(a) = \frac{(a/c;q)_\infty}{2(a;q)_\infty} \times \frac{a+c-ac-qc}{1-qc}. $$

*Example* 6 ($\lambda = 0$ and $\rho = -2$).

$$ {}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^{-2}a/c \end{matrix}\,\Big|\, q^2; q/c\right] = \left[\begin{matrix} q, a \\ q/c, q^{-2}a^2/c \end{matrix}\,\Big|\, q^2\right]_\infty \times \Big\{\theta(a)+\theta(-a)\Big\} $$
$$ \text{where}\quad \theta(a) = \frac{(q^{-1}a/c;q)_\infty}{2(a;q)_\infty} \times \frac{q+a-qa-q^2c}{q(1-qc)}. $$

*Example* 7 ($\lambda = 1$ and $\rho = 2$).

$$ {}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^6a/c \end{matrix}\,\Big|\, q^2; q^3/c\right] = \left[\begin{matrix} q, a \\ q^3/c, q^6a^2/c \end{matrix}\,\Big|\, q^2\right]_\infty \times \Big\{\theta(a)+\theta(-a)\Big\} $$
$$ \text{where}\quad \theta(a) = \frac{(q^3a/c;q)_\infty}{2(a;q)_\infty} \times \frac{c-q^3-q^2a+q^3a}{ac(q^2-c)(q^4-c)}. $$

*Example* 8 ($\lambda = -1$ and $\rho = -2$).

$$ {}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^{-2}a/c \end{matrix}\,\Big|\, q^2; q^{-1}/c\right] = \left[\begin{matrix} q, a \\ q/c, q^{-2}a^2/c \end{matrix}\,\Big|\, q^2\right]_\infty \times \Big\{\theta(a)+\theta(-a)\Big\} $$
$$ \text{where}\quad \theta(a) = \frac{(q^{-1}a/c;q)_\infty}{2(a;q)_\infty} \times \frac{a+qc-qac-q^2c}{q(1-qc)}. $$

*Example* 9 ($\lambda = 2$ and $\rho = 1$).

$$ {}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^4a/c \end{matrix}\,\Big|\, q^2; q^5/c\right] = \left[\begin{matrix} q, a \\ q/c, q^4a^2/c \end{matrix}\,\Big|\, q^2\right]_\infty \times \Big\{\theta(a)+\theta(-a)\Big\} $$
$$ \text{where}\quad \theta(a) = \frac{(q^2a/c;q)_\infty}{2(a;q)_\infty} \times \frac{qc-c+ac-q^2a}{qa^3c(1-q^2/c)}. $$

*Example* 10 ($\lambda = 2$ and $\rho = 2$).

$$ {}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^6a/c \end{matrix}\,\Big|\, q^2; q^5/c\right] = \left[\begin{matrix} q, a \\ q^3/c, q^6a^2/c \end{matrix}\,\Big|\, q^2\right]_\infty \times \Big\{\theta(a)+\theta(-a)\Big\} $$
$$ \text{where}\quad \theta(a) = \frac{(q^3a/c;q)_\infty}{2(a;q)_\infty} \times \frac{c(ac+qc-c-q^3a)}{qa^3(q^2-c)(q^4-c)}. $$

*Example* 11 ($\lambda = 2$ and $\rho = 0$).

$$
{}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^2a^2/c \end{matrix} \,\middle|\, q^2; q^5/c\right] = \left[\begin{matrix} q, a^2 \\ q/c, q^2a^2/c \end{matrix} \,\middle|\, q^2\right]_\infty \times \Big\{\theta(a) + \theta(-a)\Big\}
$$

$$
\text{where}\quad \theta(a) = \frac{(qa/c;q)_\infty}{2(a;q)_\infty} \times \frac{a^2 - qa^2 - ac + qa + c - qc}{a^3(1-qc)}.
$$

*Example* 12 ($\lambda = 2$ and $\rho = 3$).

$$
{}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^8a^2/c \end{matrix} \,\middle|\, q^2; q^5/c\right] = \left[\begin{matrix} q, a^2 \\ q^5/c, q^8a^2/c \end{matrix} \,\middle|\, q^2\right]_\infty \times \Big\{\theta(a) + \theta(-a)\Big\}
$$

$$
\text{where}\quad \theta(a) = \frac{(q^4a/c;q)_\infty}{2(a;q)_\infty} \times \frac{ac + qc - c - q^3a^2 - q^4a + q^4a^2}{qa^3c(1-q^2/c)(1-q^4/c)(1-q^6/c)}.
$$

*Example* 13 ($\lambda = -1$ and $\rho = 0$).

$$
{}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^2a^2/c \end{matrix} \,\middle|\, q^2; q^{-1}/c\right] = \left[\begin{matrix} q, a^2 \\ q/c, q^2a^2/c \end{matrix} \,\middle|\, q^2\right]_\infty \times \Big\{\theta(a) + \theta(-a)\Big\}
$$

$$
\text{where}\quad \theta(a) = \frac{(qa/c;q)_\infty}{2(a;q)_\infty} \times \frac{a^2 - qa^2 - ac + qa + c - qc}{1-qc}.
$$

*Example* 14 ($\lambda = -1$ and $\rho = -3$).

$$
{}_2\phi_1\left[\begin{matrix} a^2,\ c \\ q^{-4}a^2/c \end{matrix} \,\middle|\, q^2; q^{-1}/c\right] = \left[\begin{matrix} q, a^2 \\ q/c, q^{-4}a^2/c \end{matrix} \,\middle|\, q^2\right]_\infty \times \Big\{\theta(a) + \theta(-a)\Big\}
$$

$$
\text{where}\quad \theta(a) = \frac{(q^{-1}a/c;q)_\infty}{2(a;q)_\infty} \times \frac{a^2 + qa - qa^2 + q^3c - q^4c - q^3ac}{q^3(1-qc)}.
$$

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