SUBALGEBRA INDEPENDENCE

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Subobject independence as morphism co-possibility has recently been defined in [2] and studied in the context of algebraic quantum field theory. This notion of independence is handy when it comes to systems coming from physics, but when directly applied to classical algebras, subobject independence is not entirely satisfactory. The sole purpose of this note is to introduce the notion of subalgebra independence, which is a slight variation of subobject independence, yet this modification enables us to connect subalgebra independence to more traditional notions of independence. Apart from drawing connections between subalgebra independence and coproducts and congruences, we mainly illustrate the notion by discussing examples.

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1. INTRODUCTION

Specifying notions of independence of subsystems of a larger system is crucial in the axiomatic approach to algebraic quantum field theory. It turns out that such notions of independence can be specified in a number of nonequivalent ways, Summers [8] gives a review of the rich hierarchy of independence notions; for a non-technical review of subsystem independence concepts that include more recent developments as well, see [9]. Generalizing earlier attempts, a purely categorial formulation of independence of subobjects as morphism co-possibility has been introduced and studied in the recent papers [5, 6] and [2]. Two subobjects of an object are defined to be independent if any two morphisms on the two subobjects are jointly implementable by a single morphism on the larger object. More precisely, let us recall the definition from [2]. Suppose M is a class of monomorphisms and H is another class of morphisms of a category.

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Definition 1.1. M-morphisms $f_A : A \to X$ and $f_B : B \to X$ are called H-independent if for any two H-morphisms $\alpha : A \to A$ and $\beta : B \to B$ there is an H-morphism $\gamma : X \to X$ such that the diagram below commutes.

$$\begin{array}{ccccc} A & \xrightarrow{f_A} & X & \xleftarrow{f_B} & B \\ \downarrow{\scriptstyle \alpha} & & \vdots{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{f_A} & X & \xleftarrow{f_B} & B \end{array}$$

The objects A and B can be regarded as M-subobjects of X, and it is intuitively clear why H-independence of M-subobjects A and B is an independence condition: fixing the morphism α_A on object A does not interfere with fixing any morphism α_B on object B, and vice versa. That is to say, morphisms can be independently chosen on these objects seen as subobjects of object X.

In algebraic quantum field theory, independence given by the definition above is specified in the context of the category of special C^*-algebras taken with the class of operations (completely positive, unit preserving, linear maps) between C^*-algebras. Considerations from physics ensure injectivity of the "large system" X and therefore, extending morphisms from the subobjects to the larger object in which independence is defined as always possible.

Although the definitions employed in [2] are rather general, they become too restrictive when injectivity is not guaranteed. To reiterate: the main concern is that independence of A and B should not depend on whether morphisms can be extended to the entire X, but rather one should care for extensions to the subobject "generated by" A and B only. In other words, in a concrete category of structures, independence of A and B should depend only on how elements that can be term-defined from A and B relate to each other and not on elements that have "nothing to do" with A and B. Algebraically, term-definable elements are exactly the elements of the substructure generated by A and B. Defining the notion of a generated subobject in category theoretic terms is not unproblematic and we do not take the trouble here to deal with such issues. Instead, we focus almost exclusively on concrete algebras or categories of algebras. We introduce a slight modification to Definition 1.1 which makes it more useful among algebras. We illustrate this "usefulness" by examples where subalgebra independence coincide with well-known traditional notions of independence:

- Subset independence is disjointness.
- Subspace independence is linear independence.

- Boolean subalgebra independence is logical independence.
- Abelian subgroup independence is the traditional notion of group independence.[1]

Finally, we mention a related concept that we call congruence independence.

2. SUBALGEBRA INDEPENDENCE

Let us fix an algebraic (or more generally, a first order) similarity type. When we speak about algebras or structures, then we understand these algebras (structures) to have the same similarity type. We use the convention that algebras are denoted by Fraktur letters $\mathfrak{A}$ and the universe of the algebra $\mathfrak{A}$ is denoted by the same but capital letter A. For subalgebras $\mathfrak{A}, \mathfrak{B}$ of $\mathfrak{X}$ we write $\mathfrak{A} \vee \mathfrak{B}$ for the subalgebra of $\mathfrak{X}$ generated by $A \cup B$.

Definition 2.1 (Subalgebra-independence). Let $\mathfrak{X}$ be an algebra and $\mathfrak{A}, \mathfrak{B}$ be subalgebras of $\mathfrak{X}$. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *subalgebra-independent in* $\mathfrak{X}$ if for any homomorphisms $\alpha : \mathfrak{A} \to \mathfrak{A}$ and $\beta : \mathfrak{B} \to \mathfrak{B}$ there is a homomorphism $\gamma : \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A} \vee \mathfrak{B}$ such that the diagram below commutes.

$$\begin{array}{ccccc}
\mathfrak{A} & \xrightarrow{\subseteq} & \mathfrak{A} \vee \mathfrak{B} & \xleftarrow{\supseteq} & \mathfrak{B} \\
\downarrow \alpha & & \vdots\, \gamma & & \downarrow \beta \\
\mathfrak{A} & \xrightarrow{\subseteq} & \mathfrak{A} \vee \mathfrak{B} & \xleftarrow{\supseteq} & \mathfrak{B}
\end{array}$$

The homomorphism γ is called the joint extension of α and β (to $\mathfrak{A} \vee \mathfrak{B}$). We write $\mathfrak{A} \downarrow_{\mathfrak{X}} \mathfrak{B}$ when $\mathfrak{A}$ and $\mathfrak{B}$ are subalgebra-independent in $\mathfrak{X}$, and we might omit the subscript $\mathfrak{X}$ when it is clear from the context.

When the algebras in question have particular names, e.g. groups, fields, etc., then we specify the independence as "subgroup-independence", "subfield-independence" etc.

Comparing subalgebra independence with Definition 1.1, it is clear that the inclusion mappings take the role of M-morphisms and H is the class of all homomorphisms between algebras. The main difference, however, is that in subalgebra independence we extend the mappings α and β to the substructure generated by $A \cup B$ only. We also note that H could be chosen differently, e.g. it could be the class of automorphisms, leading to variations of the notion of independence. We do not discuss such variations in this paper.

[1] However, the case of non-Abelian groups is very different.

Before discussing the examples, let us state some useful propositions. First, it is an immediate consequence of the definition of subalgebra independence that the joint extension of α and β is always unique (if exists):

Proposition 2.2. *If the joint extension* $\gamma : \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A} \vee \mathfrak{B}$ *of* $\alpha : \mathfrak{A} \to \mathfrak{A}$ *and* $\beta : \mathfrak{B} \to \mathfrak{B}$ *exists, then it is unique and is given by*

$$\gamma\big(t^{\mathfrak{A}\vee\mathfrak{B}}(\vec{a},\vec{b})\big) \;=\; t^{\mathfrak{A}\vee\mathfrak{B}}\big(\alpha(\vec{a}),\beta(\vec{b})\big)$$

for each term $t(\vec{x},\vec{y})$ *and elements* $\vec{a} \in A$, $\vec{b} \in B$.

Proof. Elements of $\mathfrak{A} \vee \mathfrak{B}$ are of the form $t^{\mathfrak{A}\vee\mathfrak{B}}(\vec{a},\vec{b})$ for $\vec{a} \in A$ and $\vec{b} \in B$. As γ is a homomorphism that extends both α and β, we must have

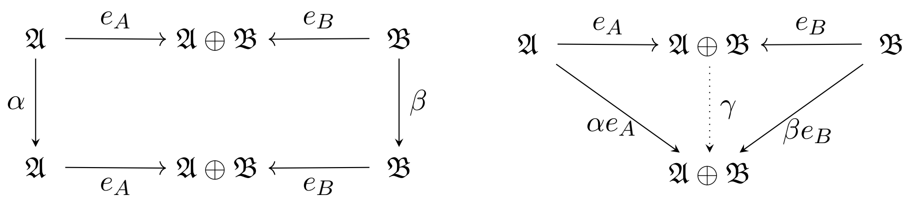
$$\gamma\big(t^{\mathfrak{A}\vee\mathfrak{B}}(\vec{a},\vec{b})\big) \;=\; t^{\mathfrak{A}\vee\mathfrak{B}}\big(\gamma(\vec{a}),\gamma(\vec{b})\big) \;=\; t^{\mathfrak{A}\vee\mathfrak{B}}\big(\alpha(\vec{a}),\beta(\vec{b})\big).$$

□

Let $\mathbf{K}$ be a class of similar algebras regarded as a category with homomorphisms as morphisms. Let $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathbf{K}$ and consider embeddings $e_i : \mathfrak{A}_i \to \mathfrak{C}$. Then $\mathfrak{C}$ is a coproduct of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in $\mathbf{K}$ iff $\mathfrak{C}$ has the following universal property with respect to $\mathbf{K}$: for any $\mathfrak{D} \in \mathbf{K}$ and homomorphisms $f_i : \mathfrak{A}_i \to \mathfrak{D}$ there is a homomorphism $g : \mathfrak{C} \to \mathfrak{D}$ such that $f_i = g \circ e_i$ $(i = 1, 2)$. The coproduct, if exists, is unique up to isomorphism. If $\mathbf{K}$ is clear from the context, we denote a coproduct of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by $\mathfrak{A}_1 \oplus \mathfrak{A}_2$. In what follows, we assume that $\mathfrak{A}$ and $\mathfrak{B}$ are (identified with) subalgebras of the coproduct $\mathfrak{A} \oplus \mathfrak{B}$.

Proposition 2.3. *Consider* $\mathfrak{A}$ *and* $\mathfrak{B}$ *as subalgebras of the coproduct* $\mathfrak{A} \oplus \mathfrak{B}$*. Then any pair of homomorphisms* $\alpha : \mathfrak{A} \to \mathfrak{A}$ *and* $\beta : \mathfrak{B} \to \mathfrak{B}$ *has a joint extension to a homomorphism* $\alpha \oplus \beta : \mathfrak{A} \oplus \mathfrak{B} \to \mathfrak{A} \oplus \mathfrak{B}$.

Proof. From the diagram below on the left-hand side, by composing arrows, one gets the diagram on the right-hand side which is a coproduct diagram. Therefore, a suitable γ with the dotted arrow exists and completes the proof.



□

Proposition 2.4. *Subalgebras* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of the coproduct* $\mathfrak{A} \oplus \mathfrak{B}$ *are subalgebra-independent provided* $\mathfrak{A} \vee \mathfrak{B} = \mathfrak{A} \oplus \mathfrak{B}$.

Proof. Immediate from Proposition 2.3. $\square$

It is clear that there is a canonical surjective homomorphism $q : \mathfrak{A} \oplus \mathfrak{B} \to \mathfrak{A} \vee \mathfrak{B}$. Take homomorphisms $\alpha : \mathfrak{A} \to \mathfrak{A}$ and $\beta : \mathfrak{B} \to \mathfrak{B}$ and consider the diagram below.

$$\begin{array}{ccc} \mathfrak{A} \oplus \mathfrak{B} & \xrightarrow{\ \alpha \oplus \beta\ } & \mathfrak{A} \oplus \mathfrak{B} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathfrak{A} \vee \mathfrak{B} & \underset{\gamma}{\dashrightarrow} & \mathfrak{A} \vee \mathfrak{B} \end{array}$$

Then the joint extension $\gamma : \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A} \vee \mathfrak{B}$ of α and β exists if and only if the mapping

$$\gamma(q(x)) = q((\alpha \oplus \beta)(x))$$

is well-defined, that is, $\alpha \oplus \beta$ is "compatible" with the kernel $\ker(q)$. We make use of this observation later on when we discuss the case of groups.

Let us see the examples without further ado.

2.1. **Sets**

Sets can be regarded as structures having the empty set as similarity type. If A and B are subsets of C, then the subset of C generated by A and B is simply their union $A \cup B$. It is straightforward to check that subset independence coincides with disjointness.

PROPOSITION 2.5. *For $A, B \subseteq C$, we have $A \mathbin{\smile\!\!\!\!\!\!\mid} B$ if and only if $A \cap B = \emptyset$.*

Proof. It is straightforward to check that A and B are independent if and only if they are disjoint as otherwise, one could take permutations of A and B that act differently on the intersection disallowing a joint extension of these permutations to $A \cup B$. $\square$

Let **Set** be the category of sets as objects and functions as morphisms. The coproduct $A \oplus B$ of two sets A and B exists and is equal (isomorphic) to the disjoint union of A and B. Hence, we get the following corollary.

COROLLARY 2.6. *For subsets $A, B \subseteq C$ we have $A \mathbin{\smile\!\!\!\!\!\!\mid} B$ if and only if $A \cup B \cong A \oplus B$.* $\square$

2.2. **Vector spaces**

Let $\mathbf{Vect}_{\mathbb{F}}$ be the class (category) of vector spaces over the field $\mathbb{F}$. Homomorphisms between vector spaces are precisely the linear mappings. Recall that two subspaces $\mathfrak{A}$ and $\mathfrak{B}$ of a vector space $\mathfrak{C}$ are linearly independent if and only if $A \cap B = \{0\}$.

We claim that subspace independence coincides with linear independence of subspaces.

PROPOSITION 2.7. *For subspaces $\mathfrak{A}, \mathfrak{B}$ of a vector space $\mathfrak{C}$, we have $\mathfrak{A} \downarrow \mathfrak{B}$ if and only if $A \cap B = \{0\}$.*

Proof. Take homomorphisms $\alpha : \mathfrak{A} \to \mathfrak{A}$ and $\beta : \mathfrak{B} \to \mathfrak{B}$. Then α and β act on the bases $\langle a_i : i \in I \rangle = \mathfrak{A}$ and $\langle b_j : j \in J \rangle = \mathfrak{B}$. Any function defined on bases can be extended to a linear mapping, therefore α and β have a common extension

$$\gamma : \langle a_i, b_j : i \in I, j \in J \rangle \to \langle a_i, b_j : i \in I, j \in J \rangle$$

if and only if they act on $A \cap B$ the same way. As α, β were arbitrary, the latter condition is equivalent to $A \cap B = \{0\}$. □

Coproduct in the category $\mathbf{Vect}_{\mathbb{F}}$ of vector spaces over the fixed field $\mathbb{F}$ coincides with the direct sum construction. Let us denote the direct sum (coproduct) of two subspaces $\mathfrak{A}$, $\mathfrak{B}$ by $\mathfrak{A} \oplus \mathfrak{B}$.

COROLLARY 2.8. *For subspaces $\mathfrak{A}, \mathfrak{B}$ of a vector space $\mathfrak{C}$, we have $\mathfrak{A} \downarrow \mathfrak{B}$ if and only if $\mathfrak{A} \vee \mathfrak{B} \cong \mathfrak{A} \oplus \mathfrak{B}$.* □

2.3. **Boolean algebras**

Let **Bool** be the category of Boolean algebras as objects with Boolean homomorphisms as morphisms. The Boolean algebra $\mathfrak{C}$ is the internal sum of the subalgebras $\mathfrak{A}, \mathfrak{B} \leq \mathfrak{C}$ just in case the union $A \cup B$ generates $\mathfrak{C}$ and whenever $a \in A$, $b \in B$ are non-zero elements, then $a \wedge b$ is non-zero (cf. Lemma 1 on p. 428 in [1]). This latter condition is called *Boole-independence*: two subalgebras $\mathfrak{A}, \mathfrak{B} \leq \mathfrak{C}$ are Boole-independent ($\mathfrak{A} \parallel \mathfrak{B}$ in symbols) if for all $a \in A$, $b \in B$, we have $a \cap b \neq 0$ provided $a \neq 0 \neq b$.

The internal sum construction coincides with the coproduct $\mathfrak{A} \oplus \mathfrak{B}$ in the category **Bool**. As before, $\mathfrak{A} \vee \mathfrak{B}$ is the subalgebra (of $\mathfrak{C}$) generated by $A \cup B$. Then, we have $\mathfrak{A} \vee \mathfrak{B} \cong \mathfrak{A} \oplus \mathfrak{B}$ precisely when $\mathfrak{A} \parallel \mathfrak{B}$.

We claim that Boolean subalgebra independence coincides with Boole-independence of subalgebras.

Proposition 2.9. *For Boolean subalgebras* $\mathfrak{A}, \mathfrak{B}$ *of a Boolean algebra* $\mathfrak{C}$ *we have*

$$\mathfrak{A} \perp \mathfrak{B} \iff \mathfrak{A} \parallel \mathfrak{B} \iff \mathfrak{A} \vee \mathfrak{B} \cong \mathfrak{A} \oplus \mathfrak{B}.$$

Proof. The second equivalence $\mathfrak{A} \parallel \mathfrak{B} \iff \mathfrak{A} \vee \mathfrak{B} = \mathfrak{A} \oplus \mathfrak{B}$ is clear. By Proposition 2.4 coproduct injections are always independent, therefore, we have

$$\mathfrak{A} \parallel \mathfrak{B} \Rightarrow \mathfrak{A} \perp \mathfrak{B}.$$

As for the converse implication assume $\mathfrak{A} \perp \mathfrak{B}$. By way of contradiction suppose there are non-zero elements $a \in A$, $b \in B$ so that $a \cap b = 0$. For an element x let x' stand for the Boolean negation (complement) of x. Take a homomorphism $\alpha : \mathfrak{A} \to \mathfrak{A}$ such that $\alpha(a) = 1 \in A$ and $\alpha(a') = 0 \in A$ (e.g. take an ultrafilter in $\mathfrak{A}$ that contains a, and send elements belonging to the ultrafilter to $1 \in A$). Take $\beta = \mathrm{id}_B$. These two homomorphisms cannot be jointly extended to a homomorphism $\gamma : \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A} \vee \mathfrak{B}$ because such a joint extension γ would satisfy $\gamma(a') = \alpha(a') = 0$ and $\gamma(b) = \beta(b) = b \neq 0$. As $b \subseteq a'$, it must follow that $\gamma(b) \subseteq \gamma(a') = 0$; contradiction. □

We remark that $\mathfrak{A} \parallel \mathfrak{B}$ implies $A \cap B = \{0, 1\}$ (for if $0 \neq a \neq 1$ was an element of $A \cap B$, then taking $a \in A$ and $a' \in B$ would witness non-Boole-independence). Thus, similarly to the previous cases, subalgebra-independence requires that the two subalgebras in question intersect in the minimal subalgebra.

Notice that Boolean independence coincides with logical independence if the Boolean algebras are viewed as the Lindenbaum–Tarski algebras of a classical propositional logic: $a \wedge b \neq 0$ entails that there is an interpretation on C that makes $a \wedge b$ hence, both a and b true; i.e., any two propositions that are not contradictions can be jointly true in some interpretation. Therefore, Boolean-subalgebra independence captures logical independence in the category **Bool**.

2.4. Abelian groups

The category **AbGrp** contains commutative groups as objects and group homomorphisms as arrows. The commutative group $\mathfrak{G}$ is the internal direct sum of its two subgroups $\mathfrak{H}$ and $\mathfrak{F}$ if and only if $\mathfrak{G}$ is generated by $H \cup F$ and $H \cap F = \{e\}$ (here and later on, e is the unit element of the group). (Internal) direct sums are precisely the coproducts, denoted by $\mathfrak{A} \oplus \mathfrak{B}$, in the category **AbGrp**.

We claim that abelian-subgroup independence coincides with having the trivial group as the intersection.

Proposition 2.10. *For subgroups $\mathfrak{A}, \mathfrak{B}$ of the commutative group $\mathfrak{C}$, we have*

$$\mathfrak{A} \downarrow \mathfrak{B} \iff A \cap B = \{e\} \iff \mathfrak{A} \vee \mathfrak{B} \cong \mathfrak{A} \oplus \mathfrak{B}.$$

Proof. As $\mathfrak{A} \vee \mathfrak{B}$ is the subgroup of $\mathfrak{C}$ generated by $A \cup B$, the equivalence

$$A \cap B = \{e\} \iff \mathfrak{A} \vee \mathfrak{B} \cong \mathfrak{A} \oplus \mathfrak{B}$$

is clear. Since summands of a coproduct are always independent (Proposition 2.4), we also have

$$A \cap B = \{e\} \implies \mathfrak{A} \downarrow \mathfrak{B}.$$

As for the other direction suppose, by way of contradiction, that there is $e \neq g \in A \cap B$. Take $\alpha : \mathfrak{A} \to \mathfrak{A}$, $\alpha(x) = e$ and $\beta = \mathrm{id}_B$. These two homomorphisms cannot have a joint extension to $\mathfrak{A} \vee \mathfrak{B}$ as $\alpha(g) \neq \beta(g)$; contradicting $\mathfrak{A} \downarrow \mathfrak{B}$. □

Independence of subgroups $\mathfrak{A}, \mathfrak{B}$ of $\mathfrak{C}$ was defined in [7] by the condition $A \cap B = \{e\}$. In the case of Abelian groups, subgroup independence gives back this exact notion, however, the case of general groups is much more complicated.

2.5. **Groups**

Consider the category **Grp** of groups with homomorphisms. Coproducts in this category exist and are isomorphic to free products. Recall that the free product of two groups is infinite and non-commutative even if both groups are finite or commutative (unless one of them is trivial as in this case the free product is isomorphic to one of the two groups). Suppose $\mathfrak{A}, \mathfrak{B} \leq \mathfrak{C}$. The proof of Proposition 2.10 shows that $\mathfrak{A} \downarrow \mathfrak{B}$ implies $A \cap B = \{e\}$.

Proposition 2.11. *If $\mathfrak{A} \downarrow \mathfrak{B}$, then $A \cap B = \{e\}$.*

On the other hand, consider the subgroups $\mathbb{Z}_2, \mathbb{Z}_3$ of $\mathbb{Z}_6$ (here, $\mathbb{Z}_n$ is the modulo n group with addition). These subgroups are independent as Abelian groups, and since any homomorphic image of a commutative group is commutative, they are independent as groups, too. But the free product (coproduct) $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is infinite, thus it is not isomorphic to $\mathbb{Z}_2 \vee \mathbb{Z}_3 = \mathbb{Z}_6$. This is an example for an algebraic category where subalgebra independence and being an internal coproduct are not equivalent.

Using the next proposition, we can draw some useful sufficient conditions for subgroup independence.

Proposition 2.12. *$\mathfrak{A} \perp \mathfrak{B}$ if and only if for all homomorphisms $\alpha : \mathfrak{A} \to \mathfrak{A}$ and $\beta : \mathfrak{B} \to \mathfrak{B}$ and elements $a_i \in A$, $b_i \in B$, we have*

$$\prod a_i b_i = e \quad \textit{implies} \quad \prod \alpha(a_i)\beta(b_i) = e.$$

Proof. Consider the diagram below and let $\mathfrak{N}$ be the normal subgroup of $\mathfrak{A} \oplus \mathfrak{B}$ corresponding to the kernel $\ker(q)$.

$$\begin{array}{ccc} \mathfrak{A} \oplus \mathfrak{B} & \xrightarrow{\alpha \oplus \beta} & \mathfrak{A} \oplus \mathfrak{B} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathfrak{A} \vee \mathfrak{B} & \xrightarrow[\gamma]{\cdots\cdots} & \mathfrak{A} \vee \mathfrak{B} \end{array}$$

The joint extension $\gamma : \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A} \vee \mathfrak{B}$ of α and β exists if and only if $(\alpha \oplus \beta)(\mathfrak{N}) \subseteq \mathfrak{N}$ as this is equivalent to that the mapping

$$\gamma(q(x)) = q((\alpha \oplus \beta)(x))$$

is well-defined. $\square$

Observe that Proposition 2.2 implies that whenever α and β has a joint extension γ, then γ is given by the equation

$$\gamma\Big(\prod a_i b_i\Big) \;=\; \prod \alpha(a_i)\beta(b_i)$$

for every element $\prod a_i b_i$ of $\mathfrak{A} \vee \mathfrak{B}$ (where $a_i \in A$, $b_i \in B$).

Proposition 2.13. *If $\mathfrak{A}$ and $\mathfrak{B}$ are normal subgroups, such that $A \cap B = \{e\}$, then $\mathfrak{A} \perp \mathfrak{B}$.*

Proof. If $\mathfrak{A}$ and $\mathfrak{B}$ are normal subgroups with $A \cap B = \{e\}$, then $ab = ba$ holds for all $a \in A$ and $b \in B$, for $a(ba^{-1}b^{-1}) \in A$ and $(aba^{-1})b^{-1} \in B$, and thus, $aba^{-1}b^{-1} \in A \cap B = \{e\}$. Let us apply Proposition 2.12. Take homomorphisms α and β and elements $a_i \in A$ and $b_i \in B$. Write $a = \prod a_i$ and $b = \prod b_i$. By the first observation $\prod a_i b_i = ab$ follows. Thus, if $\prod a_i b_i = e$, then $ab = e$. As $a \in A$, $b \in B$ and $A \cap B = \{e\}$, we have $a = b = e$. Therefore, $\alpha(a)\beta(b) = e$. Using the homomorphism property and reordering the product, we get $\prod \alpha(a_i)\beta(b_i) = e$ as desired. $\square$

However, if one of the subgroups is normal but the other is not, then they cannot be subgroup independent.

Proposition 2.14. *If $\mathfrak{A}$ and $\mathfrak{B}$ are subgroups such that $\mathfrak{A}$ is normal but $\mathfrak{B}$ is not normal in their join, then $\mathfrak{A} \not\perp \mathfrak{B}$.*

Proof. We can assume $A \cap B = \{e\}$ as this condition is necessary for subgroup independence.

Note first that given the assumptions there must exist $a \in A$ and $b \in B$ such that $ab \neq ba$. Otherwise, we would have $aBa^{-1} = B$ for all $a \in A$, and thus

$$gBg^{-1} = a_1 b_1 ... a_n b_n B b_n^{-1} a_n^{-1} ... b_1^{-1} a_1^{-1}$$

would yield B, contradicting B being not normal.

Pick $a \in A$ and $b \in B$ with $ab \neq ba$. Then $bab^{-1} \neq a$, but $bab^{-1} \in A$ since $\mathfrak{A}$ is a normal subgroup. Therefore, $bab^{-1} = a' \neq a$ and $a' \in A$. Let $\alpha : \mathfrak{A} \to \mathfrak{A}$ be the identity function and $\beta : \mathfrak{B} \to \mathfrak{B}$ be such that $\beta(x) = e$. If σ was a joint extension of α and β, then we would get

$$\sigma(bab^{-1}) = \sigma(b)\sigma(a)\sigma(b^{-1}) = eae = a, \tag{1}$$

$$\sigma(a') = a'. \tag{2}$$

Hence, $\sigma(bab^{-1}) \neq \sigma(a')$ which contradicts $bab^{-1} = a'$. □

One might be tempted to think that because normal subgroups are independent, and if exactly one of the subgroups is normal, then they are not independent, it could also be the case that two non-normal subgroups cannot be independent. Unfortunately, this is not so, as indicated by the example below.

Example 2.15. Consider the group D_∞ given by the presentation $D_\infty = \langle x, y \mid x^2 = y^2 = e \rangle$. Let A and B be its subgroups generated respectively by x and y. Clearly, $A \cong B \cong \mathbb{Z}_2$. None of A and B are normal subgroups of D_∞, yet $A \perp_{D_\infty} B$ since the only homomorphisms $A \to A$ and $B \to B$ are either the identical or the trivial mappings, each can be extended to a joint homomorphism $D_\infty \to D_\infty$.

In the previous example, D_∞ is the free product of its subgroups A and B. The next example shows that two non-normal subgroups can be subgroup independent in finite groups too.

Example 2.16. Let $A = \{e, (12)\}$ and $B = \{e, (13)(24)\}$ be subgroups of the symmetric group on four elements. The subgroup generated by $A \cup B$ is isomorphic to the dihedral group D_4. None of A or B are normal subgroups, still $A \perp B$ for the same reason as in the previous example.

We do not yet have any nice group theoretical characterization of subgroup independence, and we leave it as an open problem.

2.6. **Graphs**

Let us see a non-algebraic example. A graph is a structure of the form $\mathfrak{G} = (V, E)$, where V is a set and E is a binary relation $E \subseteq V \times V$. There are at least two different types of homomorphisms between graphs: weak and strong homomorphisms. Let us recall the definitions.

Definition 2.17. Given two graphs (V, E) and (W, F) the mapping $f : V \to W$ is a (weak) homomorphism if

$$(u, v) \in E \implies (f(u), f(v)) \in F, \tag{3}$$

and a strong homomorphism, if

$$(u, v) \in E \iff (f(u), f(v)) \in F. \tag{4}$$

Subgraphs can be understood in the graph theoretic way (that is, embeddings are weak homomorphisms) or as substructures (i.e., we take inclusions as strong embeddings; this corresponds to spanned subgraphs in the graph theoretic terminology).

Let $\mathbf{Gra}_w$ and $\mathbf{Gra}_s$ respectively, be the category of graphs with weak or strong homomorphisms as arrows. In both cases the coproduct of two graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ exists and is (isomorphic to) their disjoint union, denoted by $\mathfrak{G}_1 \oplus \mathfrak{G}_2$. By Proposition 2.4, it is clear that $\mathfrak{G}_1 \downarrow_{\mathfrak{G}_1 \oplus \mathfrak{G}_2} \mathfrak{G}_2$. But not the other way around:

Example 2.18. Call a graph $\mathfrak{G}$ rigid if the identity is its only (weak) homomorphism. There are arbitrarily large rigid graphs [4, 3]. Take two rigid graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ such that their underlying sets are not disjoint. Then $\mathfrak{G}_1 \downarrow_{\mathfrak{G}_1 \cup \mathfrak{G}_2} \mathfrak{G}_2$ are independent, nevertheless, $\mathfrak{G}_1 \cup \mathfrak{G}_2$ is not the coproduct of $\mathfrak{G}_1$ and $\mathfrak{G}_2$.

3. **JOINT EXTENSION OF CONGRUENCES**

A property that is strongly related to subalgebra independence is the joint extension property of congruences. Suppose $\alpha : \mathfrak{A} \to \mathfrak{A}$ and $\beta : \mathfrak{B} \to \mathfrak{B}$ are homomorphisms and there is a joint extension $\gamma : \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A} \vee \mathfrak{B}$ such that the diagram in Definition 2.1 commutes. This implies a relation between the kernels of the homomorphisms:

$$\ker(\gamma) \cap (A \times A) = \ker(\alpha), \qquad \text{and} \qquad \ker(\gamma) \cap (B \times B) = \ker(\beta) \tag{5}$$

If $\mathfrak{A} \downarrow \mathfrak{B}$, then (5) is the case for all congruences that are kernels of the appropriate endomorphisms. This motivates the following definition.

Definition 3.1. Let $\mathfrak{X}$ be an algebra and $\mathfrak{A}, \mathfrak{B}$ be subalgebras of $\mathfrak{X}$. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *congruence-independent in* $\mathfrak{X}$ if for any congruences $\theta_A \in \mathrm{Con}(\mathfrak{A})$ and $\theta_B \in \mathrm{Con}(\mathfrak{B})$ there is a congruence $\theta \in \mathrm{Con}(\mathfrak{A} \vee \mathfrak{B})$ such that

$$\theta \cap (A \times A) = \theta_A, \qquad \text{and} \qquad \theta \cap (B \times B) = \theta_B$$

. We write $\mathfrak{A} \mathop{\smile\!\!\!\!\!\mid}^{c}_{\mathfrak{X}} \mathfrak{B}$ when $\mathfrak{A}$ and $\mathfrak{B}$ are congruence-independent in $\mathfrak{X}$, and we might omit the subscript $\mathfrak{X}$ when it is clear from the context.

Notice that $\mathfrak{A} \mathop{\smile\!\!\!\!\!\mid}^{c} \mathfrak{B}$ implies $|A \cap B| \leq 1$. For if $|A \cap B| \geq 2$, take the two congruences $\theta_A = \mathrm{id}_A$ and $\theta_B = B \times B$ (or $\theta_A = A \times A$ and $\theta_B = \mathrm{id}_B$). Then no θ can have the property

$$\theta \cap (A \times A) = \theta_A, \qquad \text{and} \qquad \theta \cap (B \times B) = \theta_B$$

as in that case we would have

$$\theta \cap (A \cap B)^2 = \theta_A \cap (A \cap B)^2 = \mathrm{id}_{A\cap B} \neq (A \cap B)^2 = \theta_B \cap (A \cap B)^2 = \theta \cap (A \cap B)^2.$$

The connection between subalgebra independence and congruence independence is subtle, and already the sets show that none implies the other. Take, for example, $A = \{a\}$ and $B = \{a, b\}$ as subsets of a set. Then $A \mathop{\smile\!\!\!\!\!\mid}^{c} B$ but $A \not\mathop{\smile\!\!\!\!\!\mid} B$ witnessed by $\alpha = \mathrm{id}_A$ and $\beta : B \to B$, $\beta(x) = b$. However, a proposition similar to Proposition 2.3 can be formulated.

Proposition 3.2. *Consider* $\mathfrak{A}$ *and* $\mathfrak{B}$ *as subalgebras of the coproduct* $\mathfrak{A} \oplus \mathfrak{B}$. *Then for any congruences* $\theta_A \in \mathrm{Con}(\mathfrak{A})$ *and* $\theta_B \in \mathrm{Con}(\mathfrak{B})$ *there is a congruence* $\theta \in \mathrm{Con}(\mathfrak{A} \oplus \mathfrak{B})$ *such that*

$$\theta \cap (A \times A) = \theta_A, \qquad \textit{and} \qquad \theta \cap (B \times B) = \theta_B$$

.

Proof. Let $\alpha : \mathfrak{A} \to \mathfrak{A}/\theta_A$ and $\beta : \mathfrak{B} \to \mathfrak{B}/\theta_B$ be the quotient mappings. Using the universal property of the coproduct, there is a homomorphism γ making the diagram below commute.

$$\begin{array}{ccccc}
\mathfrak{A} & \xrightarrow{e_A} & \mathfrak{A} \oplus \mathfrak{B} & \xleftarrow{e_B} & \mathfrak{B} \\
\downarrow \alpha & & \vdots\,\gamma & & \downarrow \beta \\
\mathfrak{A}/\theta_A & \xrightarrow[e_A]{} & \mathfrak{A}/\theta_A \oplus \mathfrak{B}/\theta_B & \xleftarrow[e_B]{} & \mathfrak{B}/\theta_B
\end{array}$$

Then $\theta = \ker(\gamma)$ is suitable. $\square$

Proposition 3.3. *Subalgebras* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of the coproduct* $\mathfrak{A} \oplus \mathfrak{B}$ *are congruence-independent provided* $\mathfrak{A} \vee \mathfrak{B} = \mathfrak{A} \oplus \mathfrak{B}$.

Proof. Immediate from Proposition 3.2. □

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