A STABILIZED FINITE ELEMENT METHOD FOR THE STOKES EIGENVALUE PROBLEM

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A stabilized finite element method is considered for the Stokes eigenvalue problem based on the lowest equal-order element pair, which does not need to choose stabilization parameter. Stabilization terms of the stabilized finite element method include not only the term related to the momentum equation but also the continuity equation. Furthermore, combined the approximation of compact operator with the analysis of Stokes source problem, error estimates of the present method are deduced. Finally, numerical tests are made to confirm effectively.

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1. INTRODUCTION

The Stokes eigenvalue problem is widely used in many application areas: structural mechanics, electromagnetic field and fluid mechanics. It is also one of the most important eigenvalue problems and plays an important role in analysis of the stability of nonlinear partial differential equations. In fact, the Stokes eigenvalue problem is known as one of the most significant problems in fluid mechanics. It is an important topic to solve effectively, which has attracted great attention in mathematical and physical fields. Recently, numerous works have been devoted to this problem [11, 12, 13, 14]. For example, Lovadina et al. [17] have presented an a posteriori error analysis for the finite element discretization of this problem by introducing and studying a suitable residual-based error indicator. Jia et al. [15] have applied some nonconforming finite element methods for the stream function-vorticity-pressure

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method of the Stokes eigenvalue problem. Meanwhile, some optimal error estimates have been shown under some conditions. Based on stabilized low-order mixed finite elements, Armentano and Moreno [1] have obtained a priori and a posteriori error estimates for Stokes eigenvalue problem. The applications of the multilevel correction method can be found in [16]. Huang [10] has presented a superconvergence result based on projection method for stabilized finite element approximation of the problem. Additionally, Türk et al. [19] have presented a stabilized finite element method for the problem at both the two-field (displacement-pressure) and three-field (stress-displacement-pressure) formulations.

Mixed finite element method is a natural choice to solve the Stokes eigenvalue problem because the term of velocity and pressure appeared in mixed form [5, 18]. However, it is unsuitable to choose some finite element spaces like equal-order finite element space pair ($P1-P1$ or $P2-P2$) which does not satisfy the inf-sup condition. Since this condition ensures stability and accuracy of underlying numerical schemes, naturally it is the key to fulfill the condition. In fact, many stabilized finite element methods have been developed and studied to overcome this difficulty. A common feature of these stabilization methods is that some mesh-dependent stabilization parameters are involved. The stabilization parameters play key roles in these methods, not only enhancing the numerical stability but also improving the accuracy in the finite element solutions. However, it will take time to choose suitable value of stabilization parameter in numerical tests. Thus, much attention has been devoted to the study of stabilized methods without stabilization parameter.

Inspired by the work in [9], our aim is to construct a stabilized finite element method based on the lowest equal-order finite element space pair for the Stokes eigenvalue problem, which does not need to choose stabilization parameter like [2, 3, 10, 19] and is formed by adding to the discrete counterpart of the weak formulation with the residuals of the partial differential equations. The proposed stabilized finite element method employs the C^0 piecewise linear elements for both the velocity field and pressure on the same mesh and uses the residuals of the momentum equation and the divergence-free equation to define the stabilization terms, which extends from the work on generalized Stokes problem [9]. We are interested in showing the advantages of the stabilized method for the generalized Stokes eigenvalue problem and demonstrating that the construction is very simple. In fact, by applying the stabilized method to the Stokes eigenvalue problem, some good results can be obtained, which will be found in numerical tests. In addition, the stabilization parameters are fixed and element-independent. However, the stabilized method for the eigenvalue problem leads to a non-symmetric right-hand side. So, combining with the

Babŭska-Osborn theory and using some ideas on non-symmetric eigenvalue problem from Section 9 of [4] technically, error estimates for eigenvalue and corresponding eigenfunction solving by the stabilized method will be presented by employing a theoretical analysis for its counterpart source problem. In the experiments, the numerical results computed by the proposed stabilized method are better than those by other stabilized methods, which indicates that the proposed stabilized method is suitable and effective for the Stokes eigenvalue problem. Besides, in this paper, we consider the case of single eigenvalues like many other authors.

The remainder of this paper is organized as follows. In the next section, we introduce the studied problem, notations and some well-known results used throughout the paper. The stabilized finite element scheme is given in Section 3. Then, we continue by error estimates in Section 4. Finally, numerical experiments are shown to confirm the theoretical results in Section 5.

2. PROBLEM STATEMENTS

Firstly, we introduce some notations. Let Ω be a bounded, convex and open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$. Usual notation will be applied to define the Hilbert spaces:

$$\mathbf{X} = H_0^1(\Omega)^2, \qquad W = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \,\mathrm{d}x = 0\}.$$

The space of square integrable functions in a domain Ω is denoted by $L^2(\Omega)$, which is equipped with the L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_{L^2}$ or $\|\cdot\|_0$. Standard definitions are used for the Sobolev spaces $W^{m,p}(\Omega)$, with the norm $\|\cdot\|_{m,p}$, $p > 0$. The space of function has distributional derivatives of order up to an integer $m \geq 0$ belonging to $L^2(\Omega)$ by $H^m(\Omega)$, then the space of function in $H^1(\Omega)$ vanishing on its boundary $\partial\Omega$ is denoted by $H_0^1(\Omega)$. The Hilbert space $\mathbf{X}$ is endowed with the scalar product $(\nabla\cdot,\nabla\cdot)$ and the norm $\|\nabla\cdot\|_0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$. Space consisting of vector-valued function is denoted in bold font.

Next, we introduce the Stokes eigenvalue problem: Find $(\mathbf{u}, p; \lambda) \in \mathbf{X} \times W \times \mathbb{R}$ such that

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \lambda\mathbf{u} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\nu > 0$ is the viscosity, which is proportional to the inverse of Reynolds number, $\mathbf{u} \in \mathbf{X}$ represents the displacement or velocity filed, $p \in W$ is the pressure and $\lambda \in \mathbb{R}$ is the eigenvalue.

Next, we will show the weak formulation of the Stokes eigenvalue problem (1) as follows: Find $(\mathbf{u}, p; \lambda) \in \mathbf{X} \times W \times \mathbb{R}$ with $\|\mathbf{u}\|_0 = 1$ such that

$$
\left\{\begin{array}{ll} a(\mathbf{u}, \mathbf{v}) - d(\mathbf{v}, p) = \lambda r(\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{X}, \\ d(\mathbf{u}, q) = 0 & \forall q \in W, \end{array}\right. \tag{2}
$$

where two continuous bilinear forms $a(\cdot, \cdot)$ and $d(\cdot, \cdot)$ on $\mathbf{X} \times \mathbf{X}$ and $\mathbf{X} \times W$ are defined respectively, by

$$
\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &= \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) \quad && \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}, \\
d(\mathbf{v}, q) &= (q, \nabla \cdot \mathbf{v}) && \forall \mathbf{v} \in \mathbf{X},\ q \in W, \\
r(\mathbf{u}, \mathbf{v}) &= (\mathbf{u}, \mathbf{v}) && \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}.
\end{aligned}
$$

For simplicity, a generalized bilinear form $B((\cdot, \cdot); (\cdot, \cdot))$ on $(\mathbf{X} \times W) \times (\mathbf{X} \times W)$ is used which is defined by

$$
B((\mathbf{u}, p); (\mathbf{v}, q)) = a(\mathbf{u}, \mathbf{v}) - d(\mathbf{v}, p) - d(\mathbf{u}, q) \quad \forall (\mathbf{u}, p), (\mathbf{v}, q) \in \mathbf{X} \times W.
$$

In fact, the generalized bilinear form satisfies the continuity property and coercivity property [1]

$$
\begin{aligned}
|B((\mathbf{u}, p); (\mathbf{v}, q))| &\le C_1(\|\nabla \mathbf{u}\|_0 + \|p\|_0)(\|\nabla \mathbf{v}\|_0 + \|q\|_0), \\
\sup_{(\mathbf{v}, q) \in \mathbf{X} \times W} \frac{|B((\mathbf{u}, p); (\mathbf{v}, q))|}{\|\nabla \mathbf{v}\|_0 + \|q\|_0} &\ge C_2(\|\nabla \mathbf{u}\|_0 + \|p\|_0).
\end{aligned}
$$

Note that in this paper, we use C (with or without a subscript) to denote a generic positive constant, which is possibly different at different occurrences but always independent of mesh size.

With the above definition, the variational formulation (2) can be rewritten as: Find $(\mathbf{u}, p; \lambda) \in \mathbf{X} \times W \times \mathbb{R}$ with $\|\mathbf{u}\|_0 = 1$, such that

$$
B((\mathbf{u}, p); (\mathbf{v}, q)) = \lambda r(\mathbf{u}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{X} \times W. \tag{3}
$$

3. A STABILIZED FINITE ELEMENT METHOD

Let the finite element subspace pair $\mathbf{X}_h \times W_h \subset \mathbf{X} \times W$, which is associated with a regular triangulation K_h of Ω as follows:

$$
\begin{aligned}
\mathbf{X}_h &= \Big\{ \mathbf{u}_h = (u_{1h}, u_{2h}) \in C^0(\bar{\Omega})^2 \cap \mathbf{X} : u_{ih}|_K \in P_1(K), i = 1, 2, \ \ \forall K \in K_h \Big\}, \\
W_h &= \Big\{ q_h \in C^0(\bar{\Omega}) \cap W : q_h|_K \in P_1(K), \ \ \forall K \in K_h \Big\},
\end{aligned}
$$

where the mesh size parameter $h > 0$ is defined as $h = \max\{h_K : K \in K_h\}$ and $P_1(K)$ represents the set of all polynomials on K of degree no more than one. Then a Galerkin finite element approximation of the Stokes eigenvalue problem can be written as: Find $(\mathbf{u}_h, p_h; \lambda_h) \in \mathbf{X}_h \times W_h \times \mathbb{R}$ with $\|\mathbf{u}_h\|_0 = 1$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) - d(\mathbf{v}_h, p_h) = \lambda_h r(\mathbf{u}_h, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{X}_h, \\ d(\mathbf{u}_h, q_h) = 0 & \forall q_h \in W_h. \end{cases} \tag{4}$$

Since the pair of the finite element space $\mathbf{X}_h \times W_h$ does not satisfy the discrete inf-sup condition [8], the counterpart discrete scheme (4) of the weak formulation (2) is generally unstable. Hence, we add stabilization terms to tackle it as usual, but note that this stabilized method is fairly different from the stabilized method in [10]. Next, we construct the stabilized finite element method for the Stokes eigenvalue problem as follows: Find $(\mathbf{u}_h, p_h; \lambda_h) \in \mathbf{X}_h \times W_h \times \mathbb{R}$ with $\|\mathbf{u}_h\|_0 = 1$ such that

$$B_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) = \lambda_h L_h(\mathbf{v}_h, q_h) \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h \times W_h, \tag{5}$$

where the generalized bilinear form $B_h((\cdot,\cdot);(\cdot,\cdot))$ and the linear form $L_h(\cdot,\cdot)$ are, respectively, defined as follows:

$$\begin{aligned} B_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) = {} & B((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) - \sum_{K \in K_h} (\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h)_{0,K} \\ & + \sum_{K \in K_h} \frac{h^2}{12\nu} (-\nu \Delta \mathbf{u}_h + \nabla p_h, -\nu \Delta \mathbf{v}_h + \nabla q_h)_{0,K}, \end{aligned} \tag{6}$$

$$L_h(\mathbf{v}_h, q_h) = (\mathbf{u}_h, \mathbf{v}_h) - \sum_{K \in K_h} \frac{h^2}{12\nu} (\mathbf{u}_h, -\nu \Delta \mathbf{v}_h + \nabla q_h)_{0,K}. \tag{7}$$

Note that the stabilization terms include the term related to not only the momentum equation but also the continuity equation, actually, which are formed by adding to the discrete counterpart of the weak formulation (4) with the residuals of the partial differential equations. Obviously, our stabilization method retains the property of symmetry of the associated bilinear form B_h but right-hand side L_h is not so. In addition, the choice of the stabilization parameter $\frac{h^2}{12\nu}$ in the proposed stabilized finite element method (5) is inspired by the literature [8, 9]. Besides, the term $\Delta \mathbf{v}_h|_K = 0$ for all $K \in K_h$ and $\mathbf{v}_h \in \mathbf{X}_h$ in (6) and (7). However, for the completeness of construction, we still retain the terms here.

All the terms involving second order derivatives are excluded from the following analysis.

4. ERROR ESTIMATES

As mentioned earlier, our aim is to show error estimates of the stabilized finite element scheme (5) for the Stokes eigenvalue problem (1) based on the finite element analysis of the corresponding source problem. To achieve this, firstly, we will present the source problem in this section for the Stokes eigenvalue problem (1) and its convergence result.

The source problem can be written as: Given $\mathbf{f} \in L^2(\Omega)^2$, find $(\mathbf{u}, p) \in \mathbf{X} \times W$ such that

$$B((\mathbf{u}, p); (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{X} \times W, \tag{8}$$

and the corresponding stabilized finite element formulation can be written as: Find $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times W_h$ such that for all $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times W_h$

$$B_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) = \widetilde{r}(\mathbf{f}, \mathbf{v}_h), \tag{9}$$

where $B_h((\cdot,\cdot);(\cdot,\cdot))$ is defined in (6) and

$$\widetilde{r}(\mathbf{f}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) - \sum_{K \in K_h} \frac{h^2}{12\nu} (\mathbf{f}, \nabla q_h)_{0,K}.$$

Define the induced energy norm $\|\cdot\|_h$ on $\mathbf{X}_h \times W_h$ by

$$\|(\mathbf{v}_h, q_h)\|_h^2 = \nu \|\nabla \mathbf{v}_h\|_0^2 + \frac{h^2}{12\nu} \|\nabla q_h\|_0^2 + \|\nabla \cdot \mathbf{v}_h\|_0^2. \tag{10}$$

With the defined energy norm, we give the continuity and weak coercivity property of the general bilinear form $B_h((\cdot,\cdot);(\cdot,\cdot))$ for the finite element pair $\mathbf{X}_h \times W_h$.

THEOREM 4.1. *The bilinear form $B_h((\cdot,\cdot);(\cdot,\cdot))$ is continuous and weakly coercive on $(\mathbf{X}_h \times W_h) \times (\mathbf{X}_h \times W_h)$, that is*

$$|B_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h))| \le C_3 \|(\mathbf{u}_h, p_h)\|_h \|(\mathbf{v}_h, q_h)\|_h, \tag{11}$$

$$\sup_{(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times W_h} \frac{|B_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h))|}{\|(\mathbf{v}_h, q_h)\|_h} \ge \|(\mathbf{u}_h, p_h)\|_h. \tag{12}$$

Proof. By the continuous property of the bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$, the continuous property of $B_h((\cdot,\cdot);(\cdot,\cdot))$ is easily obtained. Then, we mainly focus on the weakly coercive property. Given $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times W_h$, from (6), we get

$$B_h((\mathbf{u}_h, p_h); (\mathbf{u}_h, -p_h)) = \nu \|\nabla \mathbf{u}_h\|_0^2 + \frac{h^2}{12\nu} \|\nabla p_h\|_0^2 + \|\nabla \cdot \mathbf{u}_h\|_0^2 = \|(\mathbf{u}_h, p_h)\|_h^2,$$

which, combining with the fact that $\|(\mathbf{u}_h, -p_h)\|_h = \|(\mathbf{u}_h, p_h)\|_h$, implies that (12) is valid. $\square$

Next, we take account of the following auxiliary boundary value problem:

$$\begin{cases} -\nu\Delta\mathbf{w} = \mathbf{g} & \text{in } \Omega, \\ \mathbf{w} = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

where $\mathbf{g} \in L^2(\Omega)^2$ is a given source function. The corresponding stabilization finite element method on the space $\mathbf{X}_h$ is defined as follows: Find $\mathbf{w}_h \in \mathbf{X}_h$ such that for all $\mathbf{v}_h \in \mathbf{X}_h$

$$B_{aux}(\mathbf{w}_h, \mathbf{v}_h) = L_{aux}(\mathbf{v}_h), \tag{14}$$

where the bilinear form $B_{aux}(\cdot,\cdot)$ and the linear form $L_{aux}(\cdot)$ are, respectively, given by

$$B_{aux}(\mathbf{w}_h, \mathbf{v}_h) = \nu(\nabla\mathbf{w}_h, \nabla\mathbf{v}_h), \tag{15}$$

$$L_{aux}(\mathbf{v}_h) = (\mathbf{g}, \mathbf{v}_h). \tag{16}$$

Moreover, we show the error estimates of the stabilization finite element solution $\mathbf{w}_h$ of (14).

LEMMA 4.2. *Let* $\mathbf{w} \in \mathbf{X} \cap H^2(\Omega)^2$ *be the solution of the auxiliary boundary value problem* (13). *Then the stabilized finite element solution* $\mathbf{w}_h$ *of* (14) *has the following error estimate*

$$h\|\nabla(\mathbf{w} - \mathbf{w}_h)\|_0 + \|\mathbf{w} - \mathbf{w}_h\|_0 \le Ch^2\|\mathbf{w}\|_2. \tag{17}$$

Proof. Firstly, we introduce an interpolation operator I_h, which is $\mathbf{X}_h$ of C^0 piecewise linear finite element over the triangulation K_h. The standard interpolation theory [6, 7] ensures that if $\mathbf{w} \in \mathbf{X} \cap H^2(\Omega)^2$, then there exists an interpolation $I_h\mathbf{w} \in \mathbf{X}_h$ such that

$$\|\mathbf{w} - I_h\mathbf{w}\|_{0,K} + h_K\|\mathbf{w} - I_h\mathbf{w}\|_{1,K} + h_K^2\|\mathbf{w} - I_h\mathbf{w}\|_{2,K} \le Ch_K^2\|\mathbf{w}\|_{2,K}. \tag{18}$$

Secondly, define $\boldsymbol{\eta} = \mathbf{w} - I_h\mathbf{w}$ and $\mathbf{e}_h = I_h\mathbf{w} - \mathbf{w}_h$, then $\mathbf{e} = \mathbf{w} - \mathbf{w}_h = \boldsymbol{\eta} + \mathbf{e}_h$. From (14) and Young inequality, we have

$$\begin{aligned} \nu\|\nabla\mathbf{e}_h\|_0^2 &= B_{aux}(\mathbf{e}_h, \mathbf{e}_h) = B_{aux}(\mathbf{e}_h - \mathbf{e}, \mathbf{e}_h) = -B_{aux}(\boldsymbol{\eta}, \mathbf{e}_h) \\ &= -\nu(\nabla\boldsymbol{\eta}, \nabla\mathbf{e}_h) \le \nu\|\nabla\boldsymbol{\eta}\|_0\|\nabla\mathbf{e}_h\|_0 \\ &\le \frac{\nu}{2}\|\nabla\mathbf{e}_h\|_0^2 + \frac{\nu}{2}\|\nabla\boldsymbol{\eta}\|_0^2. \end{aligned}$$

The above second equation is true due to the fact that

$$B_{aux}(\mathbf{w} - \mathbf{w}_h, \mathbf{v}_h) = 0.$$

Note that the last term of previous inequality is vanished. Therefore, according to (18), the following result is obtained

$$\|\nabla(\mathbf{w} - \mathbf{w}_h)\|_0^2 \le C_4h^2\|\mathbf{w}\|_2^2.$$

Next, we will prove the L^2-norm error for $\mathbf{w}_h$ by the duality technique. Consider the following equation

$$\begin{cases} -\nu\Delta\tilde{\mathbf{w}} = \mathbf{w} - \mathbf{w}_h & \text{in } \Omega, \\ \tilde{\mathbf{w}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Choosing $\mathbf{w} - \mathbf{w}_h$ to test previous equation and setting $\tilde{\mathbf{w}}_h \in \mathbf{X}_h$ be the corresponding numerical solution, we get

$$\begin{aligned} \|\mathbf{w} - \mathbf{w}_h\|_0^2 &= B_{aux}(\tilde{\mathbf{w}}, \mathbf{w} - \mathbf{w}_h) = B_{aux}(\tilde{\mathbf{w}} - \tilde{\mathbf{w}}_h, \mathbf{w} - \mathbf{w}_h) \\ &\le Ch^2\|\tilde{\mathbf{w}}\|_2\|\mathbf{w}\|_2 \\ &\le Ch^2\|\mathbf{w} - \mathbf{w}_h\|_0\|\mathbf{w}\|_2. \end{aligned}$$

The proof is finished. $\square$

Further, let $\mathbf{u} \in \mathbf{X} \cap H^2(\Omega)^2$ be the exact solution of problem (8). Then we define a source function $\mathbf{g}$ by

$$\mathbf{g} := -\nu\Delta\mathbf{u}. \tag{19}$$

In fact, $\mathbf{u}$ solves the auxiliary boundary value problem (13) with this given source function. The following lemma can be obtained as a consequence of Lemma 4.2.

LEMMA 4.3. *Let $\mathbf{u} \in \mathbf{X} \cap H^2(\Omega)^2$ be the solution of problem* (8) *and $\mathbf{w}_h \in \mathbf{X}_h$ be the solution of R* (14) *associated with the source function $\mathbf{g}$ given in* (19). *Then there exists a constant C_5 such that*

$$\|\mathbf{u} - \mathbf{w}_h\|_0 + h\|\nabla(\mathbf{u} - \mathbf{w}_h)\|_0 \le C_5 h^2\|\mathbf{u}\|_2. \tag{20}$$

Since $\mathbf{w}_h$ is the stabilized finite element solution of (14) with the source function (19), we have for all $\mathbf{v}_h \in \mathbf{X}_h$ that

$$a(\mathbf{w}_h, \mathbf{v}_h) = (\mathbf{g}, \mathbf{v}_h) = (-\nu\Delta\mathbf{u}, \mathbf{v}_h). \tag{21}$$

In addition, we also need the following result about the pressure p for error estimate of the stabilized scheme (9).

LEMMA 4.4. *([9]) Assume that $p \in H^1(\Omega) \cap W$. Let $\widetilde{p}_h \in W_h$ be the H^1-seminorm projection of p on the finite element space W_h, that is*

$$(\nabla\widetilde{p}_h, \nabla q_h) = (\nabla p, \nabla q_h) \quad \forall q_h \in W_h. \tag{22}$$

Then there exists a constant C_6 such that

$$\|p - \widetilde{p}_h\|_0 \le C_6 h\|p\|_1. \tag{23}$$

Based on the above numerical analysis results, we now derive the error estimate of the considered source problem (9) using the stabilized finite element method.

THEOREM 4.5. *Let* $(\mathbf{u}, p) \in \mathbf{X} \cap H^2(\Omega)^2 \times W \cap H^1(\Omega)$ *be the solution of problem* (8) *and* $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times W_h$ *be the corresponding stabilized finite element solution given by* (9). *Then there exists a constant* C_7 *such that*

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_0 \leq C_7 h(\|p\|_1 + \|\mathbf{u}\|_2), \tag{24}$$

where $C_7 = \max\{C_5, \frac{C_5}{\sqrt{\nu}}, \sqrt{12}C_5 + C + \frac{C_5}{\sqrt{\nu}}\}$.

Proof. Let $\mathbf{w}_h \in \mathbf{X}_h$ and $\widetilde{p}_h \in W_h$ be the functions stated in Lemma 4.3 and 4.4, respectively. Utilizing (12), we have

$$\begin{aligned}
\|(\mathbf{u}_h - \mathbf{w}_h, p_h - \widetilde{p}_h)\|_h &\leq \sup_{(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times W_h} \frac{B_h((\mathbf{u}_h - \mathbf{w}_h, p_h - \widetilde{p}_h), (\mathbf{v}_h, q_h))}{\|(\mathbf{v}_h, q_h)\|_h} \\
&= \sup_{(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times W_h} \frac{B_h((\mathbf{u} - \mathbf{w}_h, p - \widetilde{p}_h), (\mathbf{v}_h, q_h))}{\|(\mathbf{v}_h, q_h)\|_h},
\end{aligned} \tag{25}$$

due to the fact $B_h((\mathbf{u} - \mathbf{u}_h, p - p_h); (\mathbf{v}_h, q_h)) = 0$.

Following from (22) and (21), we have

$$\begin{aligned}
&B_h((\mathbf{u} - \mathbf{w}_h, p - \widetilde{p}_h), (\mathbf{v}_h, q_h)) \\
&= a(\mathbf{u} - \mathbf{w}_h, \mathbf{v}_h) - d(p - \widetilde{p}_h, \mathbf{v}_h) - \sum_{K \in K_h} \frac{h^2}{12\nu}(-\nu\Delta\mathbf{u} + \nabla(p - \widetilde{p}_h), \nabla q_h)_{0,K} \\
&\quad - \sum_{K \in K_h} (\nabla \cdot (\mathbf{u} - \mathbf{w}_h), \nabla \cdot \mathbf{v}_h)_{0,K} - d(\mathbf{u} - \mathbf{w}_h, q_h) \\
&= a(\mathbf{u} - \mathbf{w}_h, \nabla\mathbf{v}_h) + (\nabla(p - \widetilde{p}_h), \mathbf{v}_h) - \sum_{K \in K_h} \frac{h^2}{12\nu}(-\nu\Delta\mathbf{u} + \nabla(p - \widetilde{p}_h), \nabla q_h)_{0,K} \\
&\quad - \sum_{K \in K_h} (\nabla \cdot (\mathbf{u} - \mathbf{w}_h), \nabla \cdot \mathbf{v}_h)_{0,K} + (\mathbf{u} - \mathbf{w}_h, \nabla q_h) \\
&= (\widetilde{p}_h - p, \nabla \cdot \mathbf{v}_h) + (\mathbf{u} - \mathbf{w}_h, \nabla q_h) - \frac{h^2}{12\nu}(-\nu\Delta\mathbf{u}, \nabla q_h) - (\nabla \cdot (\mathbf{u} - \mathbf{w}_h), \nabla \cdot \mathbf{v}_h).
\end{aligned}$$

Applying Hölder inequality on $B_h((\mathbf{u} - \mathbf{w}_h, p - \widetilde{p}_h), (\mathbf{v}_h, q_h))$ with (20) and (23), we get

$$\begin{aligned}
&B_h((\mathbf{u} - \mathbf{w}_h, p - \widetilde{p}_h), (\mathbf{v}_h, q_h)) \\
&\leq \Big(\|p - \widetilde{p}_h\|_0 + \frac{\sqrt{12\nu}}{h}\|\mathbf{u} - \mathbf{w}_h\|_0 + h\sqrt{\nu}\|\Delta\mathbf{u}\|_0 \\
&\quad + \|\nabla \cdot (\mathbf{u} - \mathbf{w}_h)\|_0\Big)\|(\mathbf{v}_h, q_h)\|_h \\
&\leq \Big(C_5\|p\|_1 + (\sqrt{12\nu}C_5 + C\sqrt{\nu} + C_5)\|\mathbf{u}\|_2\Big)h\|(\mathbf{v}_h, q_h)\|_h.
\end{aligned}$$

Using the above estimate and (25), we get

$$\|(\mathbf{u}_h - \mathbf{w}_h, p_h - \widetilde{p}_h)\|_h \le h\Big(C_5\|p\|_1 + (\sqrt{12\nu}C_5 + C\sqrt{\nu} + C_5)\|\mathbf{u}\|_2\Big). \tag{26}$$

According to the definition of energy norm (10), it implies

$$\|\nabla(\mathbf{u}_h - \mathbf{w}_h)\|_0 \le h\Big(\frac{C_5}{\sqrt{\nu}}\|p\|_1 + (\sqrt{12}C_5 + C + \frac{C_5}{\sqrt{\nu}})\|\mathbf{u}\|_2\Big). \tag{27}$$

Finally, combining the triangle inequality with (20) yields the conclusion. □

THEOREM 4.6. *Assume that $\mathbf{u}_h \in \mathbf{X}_h$ is the stabilized finite element solution given by* (9)*. If the continuous problem* (8) *satisfies the regularity condition*

$$\|\mathbf{u}\|_2 + \|p\|_1 \le C\|\boldsymbol{f}\|, \tag{28}$$

then

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^2(\|\mathbf{u}\|_2 + \|p\|_1). \tag{29}$$

Proof. To do this, firstly, let $(\mathbf{Q}, \Pi) \in \mathbf{X} \times W$ and consider the following problem:

$$\begin{cases} -\nu\Delta\mathbf{Q} + \nabla\Pi = \mathbf{u} - \mathbf{u}_h & \text{in } \Omega, \\ \nabla\cdot\mathbf{Q} = 0 & \text{in } \Omega, \\ \mathbf{Q} = 0 & \text{on } \partial\Omega. \end{cases} \tag{30}$$

Secondly, test (30) with $\mathbf{u} - \mathbf{u}_h$ and $p - p_h$. Then, we have

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_0^2 &= \nu(\nabla\mathbf{Q}, \nabla(\mathbf{u} - \mathbf{u}_h)) - (\Pi, \nabla(\mathbf{u} - \mathbf{u}_h)) - (p - p_h, \nabla\cdot\mathbf{Q}) \\ &= B([\mathbf{u} - \mathbf{u}_h, p - p_h], [\mathbf{Q}, \Pi]) \\ &= B_h([\mathbf{u} - \mathbf{u}_h, p - p_h], [\mathbf{Q}, \Pi]) + \sum_{K\in K_h}(\nabla\cdot(\mathbf{u} - \mathbf{u}_h), \nabla\cdot\mathbf{Q})_{0,K} \\ &\quad + \sum_{K\in K_h}\frac{h^2}{12\nu}(-\Delta(\mathbf{u} - \mathbf{u}_h) + \nabla(p - p_h), -\nu\Delta\mathbf{Q} + \nabla\Pi)_{0,K}. \end{aligned}$$

The last term in the above equation vanishes because $\nabla\cdot\mathbf{Q} = 0$.

Now, if we let $(\mathbf{Q}_h, \Pi_h)$ be the best approximation to $(\mathbf{Q}, \Pi)$, then the first term can be bounded. In fact, we have

$$\begin{aligned} B_h((\mathbf{u}, p); (\mathbf{Q}_h, \Pi_h)) &= (\mathbf{f}, \mathbf{Q}_h) - \sum_{K\in K_h}\frac{h^2}{12\nu}(\mathbf{f}, \nabla\Pi_h)_{0,K} + (\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{Q}_h) \\ &\quad + \sum_{K\in K_h}\frac{h^2}{12\nu}(-\Delta\mathbf{u} + \nabla p, \nabla\Pi_h)_{0,K}. \end{aligned} \tag{31}$$

Using (9) in (31) and noting that $\nabla \cdot \mathbf{u} = 0$, it yields

$$
\begin{aligned}
B_h((\mathbf{u}-\mathbf{u}_h, p-p_h); (\mathbf{Q}_h, \Pi_h)) &= \sum_{K \in K_h} \frac{h^2}{12\nu}(-\Delta \mathbf{u} + \nabla p, \nabla \Pi_h)_{0,K} \\
&\leq Ch^2(\|\mathbf{u}\|_2 + \|p\|_1)\|\Pi\|_1.
\end{aligned} \tag{32}
$$

Finally, from the elliptic regularity assumption, we have

$$
\|\mathbf{u}-\mathbf{u}_h\|_0^2 \leq Ch^2(\|\mathbf{u}\|_2 + \|p\|_1)\|\mathbf{u}-\mathbf{u}_h\|_0,
$$

which finishes the proof. □

In the following part, we deduce an optimal error estimate for the Stokes eigenvalue problem by the stabilized finite element scheme (5). An argument is similar to that in Section 9 of [4] about non-symmetric eigenvalue problem and yields the following theorem. According to the existence and uniqueness of the solutions (3) and (5), we define operators $\psi, \psi_h : \mathbf{X}_\psi \to \mathbf{X}_\psi$ such that for any $\mathbf{f} \in \mathbf{X}_\psi, \psi\mathbf{f} = \mathbf{u}$ and $\psi_h\mathbf{f} = \mathbf{u}_h$ are the displacement components of the solutions to (8) and (9), respectively, where $\mathbf{X}_\psi$ can be either $(H_0^1(\Omega))^2$ or $(L^2(\Omega))^2$. It is obvious that Theorem 4.5 implies the following convergence in norm

$$
\|\psi - \psi_h\|_{\mathcal{L}(\mathbf{X}_\psi)} \to 0 \quad \text{as} \quad h \to 0,
$$

when $\mathbf{X}_\psi = (H_0^1(\Omega))^2$, where $\mathcal{L}(\mathbf{X}_\psi)$ is the space of all bounded operators on $\mathbf{X}_\psi$. This implies that the eigensolutions of the discrete problem (5) converge to those of (2) with no spurious solutions. Similarly, the results about $(L^2(\Omega))^2$ can be obtained based on Theorem 4.6. Closely combining with theorems about non-symmetric eigenvalue problem in Section 9 of [4] and the finite element space choosing, we show the error estimates in the following theorem for the eigenvalues and eigenfunction by omitting the proof.

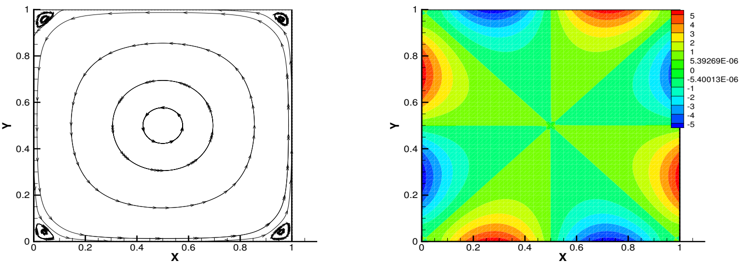
THEOREM 4.7. *Given an eigenpair* $(\mathbf{u}, p; \lambda) \in \mathbf{X} \cap H^2(\Omega)^2 \times W \cap H^1(\Omega) \times \mathbb{R}$*, solution of* (2)*, there exists a discrete eigenpair* $(\mathbf{u}_h; \lambda_h) \in \mathbf{X}_h \times \mathbb{R}$*, solution of* (5)*, such that*

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_0 + h\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0 &\leq Ch^2(\|p\|_1 + \|\mathbf{u}\|_2), \\
|\lambda - \lambda_h| &\leq Ch^2(\|p\|_1 + \|\mathbf{u}\|_2).
\end{aligned}
$$

Proof. The proof follows from Corollary 9.8 of [4]. □

5. NUMERICAL RESULTS

In this section, we provide some numerical tests to illustrate the efficiency of the stabilized finite element method for the Stokes eigenvalue problem. Two different domains are chosen in the following tests. Besides, the case $\nu = 1$ is chosen for all computations. In order to show the effectiveness of the proposed method, we also compare the numerical eigenvalue results by another five stabilized finite element methods based on the $P1 - P1$ element, which include the local Gauss integration stabilized method λ_G in [10], the regular stabilized method λ_R with stabilization parameter $\alpha = 8$ in [10], the Barrenechea–Valentin stabilized method λ_{BV} with stabilization parameter $\tau_\kappa = \frac{h^2}{12\nu}$ in [2], the Bochev–Gunzburger–Lehoucq stabilized method λ_{BGL} with stabilization parameter $\delta = 0.05$ in [3] and the Türk–Boffi–Codina stabilized method λ_{TBC} with stabilization parameter $\alpha_1 = \frac{1}{4},\ \alpha_2 = \frac{1}{10}$ in [19].



(a) (b)

Figure 1 – Velocity streamlines (a) and pressure contours (b) on a square domain.

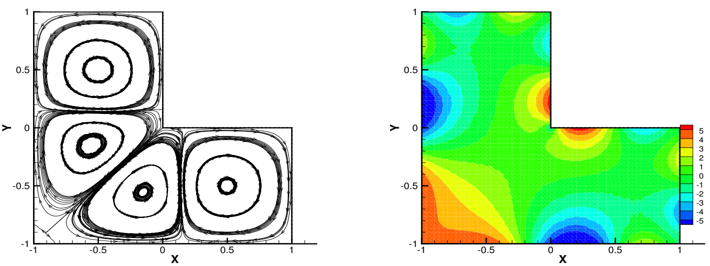


Figure 2 – Velocity streamlines (a) and pressure contours (b) on an L-shaped domain.

Test 1. Let the computation be carried out in the unit square domain $[0,1] \times [0,1] \in \mathbb{R}^2$, which is uniformly divided by the triangulation of mesh h. As it was mentioned, the exact solution is unknown. Generally, we take $\lambda = 52.3447$ as a reference of the minimum eigenvalue for the Stokes eigenvalue problem on this domain. Here, we pick six values of h, i.e., $1/h = 10, 20, 30, 40, 50, 60$.

Table 1 – Results obtained from the proposed method and the local Gauss integration method on a square domain

$1/h$	λ_h	$\frac{\lvert\lambda_h-\lambda\rvert}{\lambda}$	rate	λ_G [10]	$\frac{\lvert\lambda_G-\lambda\rvert}{\lambda}$	rate
10	54.1508	3.450e-02	—	55.5958	6.211e-02	—
20	52.8057	8.806e-03	1.9700	53.1614	1.560e-02	1.9933
30	52.5504	3.930e-03	1.9898	52.7077	6.935e-03	1.9994
40	52.4606	2.213e-03	1.9963	52.5489	3.901e-03	1.9999
50	52.4189	1.418e-03	1.9947	52.4754	2.497e-03	1.9994
60	52.3962	9.847e-04	2.0001	52.4354	1.733e-03	2.0033

Table 2 – Results obtained from the regular method and the Barrenechea–Valentin method on a square domain

$1/h$	λ_R [10]	$\frac{\lvert\lambda_R-\lambda\rvert}{\lambda}$	rate	λ_{BV} [2]	$\frac{\lvert\lambda_{BV}-\lambda\rvert}{\lambda}$	rate
10	55.1964	5.448e-02	—	55.4066	5.849e-02	—
20	53.0749	1.395e-02	1.9655	53.1237	1.488e-02	1.9748
30	52.6707	6.229e-03	1.9885	52.6921	6.636e-03	1.9916
40	52.5284	3.509e-03	1.9949	52.5403	3.737e-03	1.9960
50	52.4623	2.247e-03	1.9975	52.4700	2.393e-03	1.9975
60	52.4264	1.561e-03	1.9979	52.4317	1.663e-03	1.9961

Table 3 – Results obtained from the Bochev–Gunzburger–Lehoucq method and the Türk–Boffi–Codina method on a square domain

$1/h$	λ_{BGL} [3]	$\frac{\lvert\lambda_{BGL}-\lambda\rvert}{\lambda}$	rate	λ_{TBC} [19]	$\frac{\lvert\lambda_{TBC}-\lambda\rvert}{\lambda}$	rate
10	55.6149	6.247e-02	—	55.8688	6.732e-02	—
20	53.1671	1.571e-02	1.9915	53.2514	1.732e-02	1.9598
30	52.7104	6.987e-03	1.9983	52.7498	7.739e-03	1.9964
40	52.5504	3.930e-03	2.0002	52.5729	4.360e-03	1.9453
50	52.4763	2.515e-03	2.0004	52.4908	2.791e-03	2.0255
60	52.4361	1.747e-03	1.9985	52.4462	1.939e-03	2.1268

From Tables 1-3, we can see that the above stabilized methods all work well and keep the convergence rates just like the theoretical analysis. As expected, we have an interesting observation that the proposed method has the best relative error. Meanwhile, it does not need to choose stabilization parameter like some methods. Besides, we present the velocity streamlines and the pressure contours in Figure 1. From this figure, we can observe that the velocity streamlines and pressure contours are in good agreement with the previously published results [1, 19].

Test 2. In the previous test, we have considered a convex domain and showed the best performance among all the stabilization methods. Next, we want to examine a test case with an L-shaped domain $[-1,1] \times [-1,1]/[0,1] \times [0,1]$, which has a re-entrant corner. We consider $\lambda = 48.9844$ as the reference value for the fourth eigenvalue. Similarly, we give the numerical eigenvalues, relative errors, as well as convergence rates, in Tables 4-6 with mesh sizes $1/h = 10, 15, 20, 25, 30$. From these tables, we can obtain some desired results as in Test 1. Further, Figure 2 shows numerical velocity streamlines and pressure contours on this domain. From this figure, we can see that there is no obviously spurious oscillation. Hence, our method can capture this classical non-convex domain model well.

Table 4 – Results obtained from the proposed method and the local Gauss integration method on an L-shaped domain

$1/h$	λ_h	$\frac{\lvert\lambda_h-\lambda\rvert}{\lambda}$	rate	λ_G [10]	$\frac{\lvert\lambda_G-\lambda\rvert}{\lambda}$	rate
10	50.5491	3.194e-02	—	52.0851	6.329e-02	—
15	49.6986	1.458e-02	1.9343	50.3669	2.822e-02	1.9921
20	49.3883	8.246e-03	1.9811	49.7626	1.589e-02	1.9976
25	49.2433	5.286e-03	1.9930	49.4825	1.017e-02	1.9998
30	49.1642	3.670e-03	2.0005	49.3302	7.059e-03	2.0014

Table 5 – Results obtained from the regular method and the Barrenechea–Valentin method on an L-shaped domain

$1/h$	λ_R [10]	$\frac{\lvert\lambda_R-\lambda\rvert}{\lambda}$	rate	λ_{BV} [2]	$\frac{\lvert\lambda_{BV}-\lambda\rvert}{\lambda}$	rate
10	51.6975	5.539e-02	—	51.9101	5.972e-02	—
15	50.2098	2.502e-02	1.9603	50.2985	2.683e-02	1.9739
20	49.6771	1.414e-02	1.9826	49.7262	1.514e-02	1.9879
25	49.4285	9.067e-02	1.9920	49.4598	9.705e-03	1.9938
30	49.2931	6.299e-03	1.9975	49.3147	6.742e-03	1.9975

Table 6 – Results obtained from the Bochev–Gunzburger–Lehoucq method and the Türk–Boffi–Codina method on an L-shaped domain

$1/h$	λ_{BGL} [3]	$\frac{\lvert\lambda_{BGL}-\lambda\rvert}{\lambda}$	rate	λ_{TBC} [19]	$\frac{\lvert\lambda_{TBC}-\lambda\rvert}{\lambda}$	rate
10	52.1077	6.376e-02	—	51.8885	5.931e-02	—
15	50.3775	2.844e-02	1.9912	50.3119	2.712e-02	1.9313
20	49.7689	1.601e-02	1.9964	49.7384	1.543e-02	1.9646
25	49.4866	1.025e-02	1.9986	49.4692	9.931e-03	1.9800
30	49.3331	7.119e-03	2.0003	49.3218	6.912e-03	1.9801

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