SECOND ORDER DIFFERENTIAL OPERATORS WITH ALGEBRAIC SOLUTIONS

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We are surveying recent results that describe second order differential operators having only algebraic solutions in the sense of Galois theory. We call such operators algebraic. For hypergeometric operators, this problem was studied by Schwarz and Klein who also gave results that describe all second order linear differential operators with a full set of algebraic solutions. Starting from their work, we see algebraic operators as pull-backs of algebraic hypergeometric operators via Belyi functions. We are surveying some of the main results describing second order operators with a full set of algebraic solutions, especially those obtained by using the properties of the pull-back functions. Using the Grothendieck correspondence, these properties transfer to properties for their corresponding dessins d'enfants.

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1. INTRODUCTION

Consider an **ordinary linear differential operator** of **order** n:

$$L = \frac{\mathrm{d}^n}{\mathrm{d}\, z^n} + a_1(z)\frac{\mathrm{d}}{\mathrm{d}\, z} + \cdots + a_n(z), \tag{1}$$

with the coefficients $a_1, \ldots, a_n : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ being rational functions over $\mathbb{C}$. This operator corresponds to a homogeneous linear differential equation on $\mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$. If we change the variable $z \mapsto \frac{1}{z}$, we can see what happens around the point $z = \infty$ and we can look at this equation on the whole $\mathbb{P}^1_{\mathbb{C}}$. It behaves "regularly" around the points of $\mathbb{P}^1_{\mathbb{C}} \setminus S$, where S contains the finite set of the poles of the coefficients $a_1, \ldots, a_n$. It also includes ∞ if, after changing the variable as mentioned, 0 becomes a pole for at least one of the coefficient functions of the induced equation. More precisely, we have:

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Definition 1.1. Consider equation (2). A point $\alpha \in \mathbb{C}$ is called:

- **regular** if the limit $\lim_{z\to\alpha} a_i(z)$ exists and is finite for all the coefficients a_i, where $i \in \{1, 2, \ldots, n\}$ (in other words, all the coefficient functions a_i are holomorphic at α);
- **regular singular** if the limit $\lim_{z\to\alpha} (z-\alpha)^i a_i(z)$ exists and is finite for all the coefficients a_i, where $i \in \{1, 2, \ldots, n\}$ (the order of α as a possible pole of a_i is bounded by i).

The analogous conditions for the point $z = \infty$ are obtained by making the substitution $z = \frac{1}{x}$ and studying if $x = 0$ is a regular or a regular singular point of the new equation. A point in $\mathbb{P}^1(\mathbb{C})$ is called a **singular** point for (2) if it is not a regular point.

Let us notice that this is the approach that Lazarus Fuchs used for dealing with regular singular points. He proved that this condition is equivalent to the more classical one involving growth estimates for the solutions, emphasizing by this approach that the nature of this notion is algebraic. Moreover, if ζ is a uniformizing parameter at the point α, we denote $D = \zeta \frac{\mathrm{d}}{\mathrm{d}\,\zeta}$ and we rewrite

$$L = D^n + b_1(\zeta)D^{n-1} + \ldots + b_n(\zeta)$$

then $z = \alpha$ is a regular singular point if and only if all the functions $b_i(\zeta)$ are holomorphic at $\zeta = 0$.

Fuchs studied the equations that now are called Fuchsian equations:

Definition 1.2. An ordinary linear differential equation is called **Fuchsian** if any point in $\mathbb{P}^1(\mathbb{C})$ is a regular or a regular singular point for the equation.

For a detailed history of Fuchsian operators, see [18].

We are interested in the situation when the ordinary linear differential equation:

$$L(y(z)) = 0 \tag{2}$$

has a full set of solutions in an algebraic extension of $\mathbb{C}(x)$. In this case, we call the operator and the corresponding equation algebraic. Every algebraic operator is Fuchsian, hence, when looking for conditions for an operator to be algebraic, we will always suppose that any singularity of the operator is a regular singular point.

Questions related to the nature of the solutions have been studied since the 1850's. Riemann studied algebraic functions [38] and the hypergeometric operator [37] given by:

$$D = \frac{\mathrm{d}^2}{\mathrm{d}\, z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1}\right)\frac{\mathrm{d}}{\mathrm{d}\, z} + \frac{\alpha\beta}{z(z-1)}, \tag{3}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha + \beta + 1 = \gamma + \delta$.

Kummer listed substitutions for hypergeometric functions and compiled a list known as Kummer's 24 solutions to the hypergeometric equation [28]. Following the work of Weierstrass and Cauchy regarding the existence of solutions of differential equations, Bouquet and Briot studied singular points of differential equations [8] and [9].

From Cauchy's theory of differential equations, it follows that given α a regular point of an n-th order differential equation there exists a basis of solutions that locally look like:

$$f_i(z) = (z-\alpha)^i \cdot g_i, \quad g_i \in \mathbb{C}[[z-\alpha]], \quad g_i(\alpha) \neq 0$$

for all $i \in \{0, 1, \dots, n-1\}$.

In the attempt to recover similar results for regular singular points, the indicial equation was introduced. For further details, see [6], [36] and [43].

Definition 1.3. Let $\alpha \in \mathbb{P}^1(\mathbb{C})$ be a regular or a regular singular point of (2). Let

$$\alpha_i := \begin{cases} \lim\limits_{z\to\alpha} (z-\alpha)^i a_i(z), & \text{if } \alpha \in \mathbb{C} \\ \lim\limits_{z\to\infty} z^i a_i(z), & \text{if } \alpha = \infty, \end{cases} \quad \text{for all } i = \{1, 2, \dots, n\}.$$

The **indicial equation** is:

$$X(X-1)\dots(X-n+1) + \alpha_1 X(X-1)\dots(X-n+2) + \dots + \alpha_{n-1}X + \alpha_n = 0 \tag{4}$$

if $\alpha \in \mathbb{C}$;

$$X(X-1)\dots(X-n+1) - \alpha_1 X\dots(X-n+2) + \dots + (-1)^{n-1}\alpha_{n-1}X + (-1)^n\alpha_n = 0 \tag{5}$$

if $\alpha = \infty$.

The solutions of this equation are called **local exponents**.

Fuchs proposed a theorem to replicate Cauchy's result [6]:

THEOREM 1.1 (Fuchs). *Let $\alpha \in \mathbb{P}^1(\mathbb{C})$ be a regular singularity of (2) and t be a local parameter at α. We distinguish two cases regarding the local exponents:*

1. *Assume that* ρ *is a local exponent at* α *such that none of the numbers* $\rho+1, \rho+2, \ldots$ *is a local exponent. In this case, there exists* $g(t) = \sum_{k\geq 0} g_k t^k$ *an analytic function around* $t = 0$*, with* $g_0 \neq 0$*, such that* $t^\rho \cdot g(t)$ *is a solution.*

2. *Let* $\rho_1, \ldots, \rho_n$ *be the set of local exponents ordered such that the local exponents that differ by an integer occur in a decreasing order. Then there exists a nilpotent matrix* $N = (N_{ij})_{1\leq i,j\leq n} \in nil(\mathcal{M}_{n\times n}(\mathbb{C}))$*,* $g_1, g_2, \ldots, g_n$ *analytic functions around* $t = 0$ *with* $g_k(0) \neq 0$ $(k \in \{1, \ldots, n\})$ *such that*
$$(t^{\rho_1} g_1, t^{\rho_2} g_2, \ldots, t^{\rho_n} g_n) t^N$$
is a basis of solutions. Moreover, if $N_{ij} \neq 0$ *it follows that* $i \neq j$ *and* $\rho_i - \rho_j \in \mathbb{N}$*.*

A Fuchsian differential operator of order n, having s singular points, is fully determined by the singular points, the ns exponents and a set of accessory parameters, whose cardinality a_L can be determined ([22], [15]):

$$a_L = \frac{(n-1)[n(s-2)-2]}{2}. \tag{6}$$

The monodromy group of a differential operator of order n is the image of the representation $\pi_1(\mathbb{P}^1 \setminus S) \to GL(n, \mathbb{C})$ given by the analytic continuation of n solutions in a basis along the closed paths that represent the elements of $\pi_1(\mathbb{P}^1 \setminus S)$. For a second order operator, the projective monodromy group is defined in a similar way, using the continuation of the ratio of solutions in a basis. These groups are well-defined up to conjugation.

If X is a higher genus Riemann surface and $U \subset X$ is an open subset, a differential equation on U is a pair (M, ∇), where M is a locally free coherent sheaf on U and ∇ is a connection
$$\nabla : M \longrightarrow M \otimes \Omega^1_{U/\mathbb{C}}.$$
The set S of singular points corresponds to $M \setminus U$ and the solutions of such an equation are the sections of the kernel of ∇. The regular singular points are defined for such a general equation in an analogous manner to the corresponding notion for an equation on $\mathbb{P}^1$. One can use growth estimates for the local holomorphic solutions, or Fuchs' algebraic approach, as explained by Deligne [13]: the point α is regular singular if there exists a locally free coherent algebraic sheaf $\overline{M}$ on X, extending M, on which the derivation $t\frac{d}{dt}$ acts stably through ∇ (here t is a uniformizing parameter at α) (see also [26]).

Before outlining the content of the paper, let us also mention the connection with Hilbert's 21st problem and with p-adic statements. Hilbert's 21st problem asks whether there always exists a Fuchsian linear differential equation with prescribed regular singular points and monodromy group. The reader can see Deligne's fundamental paper [13] and Katz's expository article [26]. One of the open questions raised by Katz concerns the algebraic description of the algebraic linear differential operators.

On the other hand, progress in studying algebraic differential operators has been stimulated by the famous "p-curvature conjecture" of Grothendieck. Suppose that, for the operator L on $\mathbb{P}^1_{\mathbb{C}}$, the coefficients belong to $K(x)$, where K is a number field. One can reduce these coefficients modulo almost all primes p of the ring of integers of K and ask about the properties of the newly obtained operators (eventually after completing the field of constants) and about the p-adic behaviour of the solutions. For example, if L is algebraic, then for almost all primes p the reduced operator has a full set of solutions or, equivalently, the invariant called "p-curvature" vanishes. Grothendieck's conjecture states that the converse is also true. Katz proved the conjecture for Picard-Fuchs operators in [25]. Other cases were proved by Farb and Kisin [16]. See also [24], [21] for properties of the p-curvature and relations with properties of the operator L, and [14], [15] for details on p-adic operators.

The next section of the paper concerns a brief survey of some general results describing second order differential operators with a full set of algebraic solutions. It includes some general properties of such operators, starting with the classical work of Klein and Schwarz, and the connection with the Belyi functions and the associated dessins d'enfants. The following part is devoted to more specific and explicit results for Lamé operators, obtained using these arithmetic and combinatorial tools. They include some finiteness properties, answers to questions raised by Baldassarri and by Dwork and some explicit results concerning these operators with four singular points. We emphasize the fact that, even if we consider the context of differential operators on the Riemann sphere, most of the notions and invariants involved (such as the monodromy group, local exponents etc.) can be easily defined in the framework of general Riemann surfaces, in sense explained hereinbefore. Many of the results presented hereafter are valid for linear differential operators defined on arbitrary Riemann surfaces. We will make this explicit for some finiteness results in the last section.

In a subsequent paper [35], we will provide a new illustration of this strategy, consisting in a detailed description of Heun operators with tetrahedral projective monodromy group. The same ideas can be similarly employed for studying Heun operators with any fixed, finite monodromy group.

2. SECOND ORDER DIFFERENTIAL OPERATORS WITH ALGEBRAIC SOLUTIONS

We will consider, from this point forward, only second order differential operators. Hence the monodromy group will be a subgroup of $GL(2,\mathbb{C})$ and the projective monodromy group a subgroup of $PGL(1,\mathbb{C})$. For each singular point, the indicial equation is quadratic and provides two local exponents ρ_1 and ρ_2. The assertions of Theorem 1.1 become:

- If the difference of the exponents $\rho_1 - \rho_2 \notin \mathbb{Z}$, then there is a basis of solutions of the form
$$\begin{aligned} g_1(t) &= t^{\rho_1}\cdot \sum_{i\geq 0} a_i t^i, \quad a_0 \neq 0 \\ g_2(t) &= t^{\rho_2}\cdot \sum_{i\geq 0} b_i t^i, \quad b_0 \neq 0. \end{aligned}$$

- If $\rho_1 - \rho_2 \in \mathbb{N}$, then there exists a nilpotent matrix
$$N = (N_{ij})_{1\leq i,j\leq 2} \in \mathrm{nil}(\mathcal{M}_{2\times 2}(\mathbb{C})),$$
and the functions g_1, g_2 analytic around $t = 0$ with $g_k(0) \neq 0$ ($k \in \{1,2\}$) such that
$$(t^{\rho_1} g_1, t^{\rho_2} g_2)t^N$$
is a basis of solutions.

As we have already mentioned, we are interested in operators having a full set of solutions included in an algebraic extension of $\mathbb{C}(z)$.

This property is closely related to the finiteness of the monodromy group [23]. More precisely, the following statements are equivalent (see, for example, [43]):

- The equation is algebraic.
- The monodromy group of the equation is finite.
- The projective monodromy group of the equation is finite and its Wronskian is algebraic.

It is easily seen that a full set of algebraic solutions can occur only if the local exponents at any point are rational (for example, one can argue as in [36], observing that the analytic continuation of a solution along the closed paths leads to an infinite number of determinations if the corresponding exponent is irrational, and hence contradicting the equivalence hereinbefore).

In what concerns the algebraicity of the Wronskian, more details can be found in [4]. In what follows, we will focus on the finiteness of the projective monodromy group. Properly speaking, this characterizes the algebraicity of the ratio of the solutions in a basis. It is then natural to define the following equivalence relation:

Definition 2.1 ([4]). Two differential operators L_1 and L_2 are said to be **projectively equivalent** if there exists θ a radical function (a product of powers of rational functions, rational exponents being allowed) such that $L_2 = \theta \circ L_1 \circ \theta^{-1}$.

PROPOSITION 2.1 ([4]). *Let L_1 and L_2 be* ***projectively equivalent*** *differential operators. The following are true:*

1. *If L_1 is Fuchsian, then L_2 is Fuchsian.*
2. *L_1 and L_2 have a common ratio of independent solutions at a point of the Riemann sphere.*
3. *L_1 and L_2 have isomorphic projective monodromy group.*
4. *Let ρ_1 and ρ_2 be the local exponents of L_1 in $\alpha \in \mathbb{P}^1(\mathbb{C})$. It follows that $\rho_2 - \rho_1$ is the difference of local exponents of both L_1 and L_2 at $\alpha \in \mathbb{P}^1(\mathbb{C})$.*

Then the second order linear differential operators can be reduced to their normalized form (see [17], [4], [10], [43]):

PROPOSITION 2.2. *Let $L = \frac{\mathrm{d}^2}{\mathrm{d}\,z^2} + a_1(z)\frac{\mathrm{d}}{\mathrm{d}\,z} + a_2(z)$ be a second order differential operator. Then it is projectively equivalent to one in normalized form:*

$$\frac{\mathrm{d}^2}{dz^2} + B(z), \quad \textit{where } B : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}).$$

An essential tool in our discussion is the pull-back of a differential operator via a rational function. In fact, one can find in the recent literature two types of pull-back relations:

Definition 2.2 ([43]). Let $L := \frac{\mathrm{d}^n}{\mathrm{d}\,z^n} + a_1(z)\frac{\mathrm{d}^{n-1}}{\mathrm{d}\,z^{n-1}} + \cdots + a_{n-1}(z)\frac{\mathrm{d}}{\mathrm{d}\,z} + a_n(z)$ be a Fuchsian operator with finite monodromy. If z is replaced by a non-constant rational function $f(x) \in \mathbb{C}(x)$, then L becomes:

$$L_f := \left(\frac{\mathrm{d}}{f'(x)\,\mathrm{d}\,x}\right)^n + \cdots + a_{n-1}(f)\frac{\mathrm{d}}{f'(x)\,\mathrm{d}\,x} + a_n(f). \tag{7}$$

L_f is an operator of order n (the derivation being with respect to x), called the **proper rational pull-back** of L by $z = f(x)$. If L' is a differential operator with respect to x that is projectively equivalent with L_f, L' is called a **rational pull-back** of L by $z = f(x)$.

Proposition 2.3 ([43]). *Let L_f be a proper rational pull-back of L by f. If L is Fuchsian, then L_f is Fuchsian.*

Proposition 2.4 ([4]). *Let L' be a rational pull-back of the Fuchsian operator L by $z = f(x) \in \mathbb{C}(x)$. Let $\rho_1^L(\alpha)$ and $\rho_2^L(\alpha)$ be two local exponents of L in $\alpha = f(\tilde{\alpha}) \in \mathbb{P}^1(\mathbb{C})$. Let $e_{\tilde{\alpha}}$ be the ramification index of f in $\tilde{\alpha}$. If $\rho_1^{L'}(\tilde{\alpha})$ and $\rho_2^{L'}(\tilde{\alpha})$ are local exponents of L' in $\tilde{\alpha}$, then:*

$$\rho_1^{L'}(\tilde{\alpha}) - \rho_2^{L'}(\tilde{\alpha}) = e_{\tilde{\alpha}} \cdot \left(\rho_1^L(\alpha) - \rho_2^L(\alpha)\right). \tag{8}$$

We can define the global invariant

$$\Delta_L = \sum_{\alpha \in \mathbb{P}^1(\mathbb{C})} (|\rho_1(\alpha) - \rho_2(\alpha)| - 1), \tag{9}$$

where $\rho_1(\alpha), \rho_2(\alpha)$ are the local exponents of L in α. Then, we have:

Proposition 2.5 ([4]). *Consider L, L' two differential operators such that L' is the pull-back of L by the function f. Then*

$$\Delta_{L'} + 2 = \deg f \cdot (\Delta_L + 2). \tag{10}$$

One can extend the definition of the pull-back for general operators defined over arbitrary (open subsets of) Riemann surfaces (as described in the previous section), and Proposition 2.4 holds. If L and L' are differential operators on the Riemann surfaces X and X' respectively and $f : X' \to X$ is a morphism such that L' is the pull-back of L through f, then the formula (10) becomes ([4])

$$\Delta_{L'} + 2(1 - g') = \deg f \cdot [\Delta_L + 2(1 - g)] \tag{11}$$

where g and g' are the genus of X and X', respectively.

Formulas (10) and (11) easily follow from Proposition 2.4 and the Hurwitz genus formula.

Let us return to the algebraicity problem. The first case to look at is that of operators with three singular points (we can suppose that they are $0, 1$ and ∞) that is, the minimal number leading to a non-trivial situation. These are the hypergeometric operators/equations given by (3). Their normal form is

$$H_{\lambda,\mu,\nu} = \frac{\mathrm{d}^2}{\mathrm{d}\, z^2} + \frac{1-\lambda^2}{4z^2} + \frac{1-\mu^2}{4(z-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{4z(z-1)} \tag{12}$$

where $\lambda, \mu, \nu \in \mathbb{C}$, while the local exponents at the singular points are:

$$\left(\begin{array}{ccc|c} 0 & 1 & \infty & z \\ \hline 0 & 0 & 0 & \\ \lambda & \mu & \nu & \end{array}\right) \tag{13}$$

Such a table, including, on the first row, the singular points, and the local exponents on the corresponding columns is called the *Riemann scheme* of the operator. It can be observed that λ, μ, ν are also the differences of the local exponents. It is worth mentioning that the hypergeometric operators are completely determined by their associated Riemann scheme (13), and any differential operator with three singular points is projectively equivalent to a hypergeometric one (see also formula (6)). This case was fully solved in the second half of the 19th century, when Schwarz determined completely the equations that have only algebraic solutions by studying the algebraicity of the ratio of two independent solutions [39]. F. C. Klein introduced in [27] equivalences between operators and reduced the original list of Schwarz to what is called the "basic Schwarz list":

(14)

(λ, μ, ν)	Projective monodromy group
$(1/n, 1, 1/n)$	$C_N,\ N \in \mathbb{N}^*$
$(1/2, 1/n, 1/2)$	$D_N,\ N \in \mathbb{N}^*$
$(1/2, 1/3, 1/3)$	A_4
$(1/2, 1/3, 1/4)$	S_4
$(1/2, 1/3, 1/5)$	A_5

.

Moreover, these operators proved to be essential in studying all second order equations with a full set of algebraic solutions. Klein proved the following theorem and provided an efficient strategy for studying the algebraicity of a second order differential operator ([27], [4]):

THEOREM 2.6 (Klein). *Let L be a second order differential operator in normal form on $\mathbb{P}^1(\mathbb{C})$ with finite projective monodromy group G. Then there exists a unique hypergeometric operator H belonging to* (14)*, having G as projective monodromy group, such that L is a rational pull-back of H via a rational function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Moreover, the function f is also unique, modulo Möbius transformations leaving the operator H invariant and permuting its singular points.*

The analogous result for general second order operators on Riemann surfaces is proved in [1].

We can say more if the operator has no apparent singularity (that is, at each singular point the exponent difference is a non-integer rational number):

PROPOSITION 2.7 ([31]). *Let L be a second order differential operator with finite projective monodromy and without apparent singularities, and let S be its set of singular points. Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map that realizes L as a rational pull-back of a hypergeometric operator H from* (14)*. The following hold true:*

- *f is only ramified above* 0, 1 *and* ∞.

- $f(S) \subset \{0, 1, \infty\}$.

- *Let* $\alpha \in \{0, 1, \infty\}$ *with* $\rho_1^H(\alpha), \rho_2^H(\alpha)$ *local exponents of* H *at* α, *and* $\tilde{\alpha} \notin S$ *with* $f(\tilde{\alpha}) = \alpha$. *Then the ramification index of* f *at* $\tilde{\alpha}$ *is*

$$e_{\overline{\alpha}} = \frac{1}{|\rho_1^H(\alpha) - \rho_2^H(\alpha)|}.$$

Proof. The statements follow easily from Proposition 2.4. Let $\alpha, \tilde{\alpha} \in \mathbb{P}^1(\mathbb{C})$ such that $f(\tilde{\alpha}) = \alpha$. Let $\rho_1^H(\alpha), \rho_2^H(\alpha))$, $\rho_1^L(\tilde{\alpha}), \rho_2^L(\tilde{\alpha})$ be local exponents of H and L at α and $\tilde{\alpha}$, respectively. Let $e_{\tilde{\alpha}}$ be the ramification index of f in $\tilde{\alpha}$.

Suppose that $\alpha \notin \{0, 1, \infty\}$ and α is a critical point of f. Then α is a regular point for H, hence $\mid \rho_1^H(\alpha) - \rho_2^H(\alpha)) \mid = 1$. As α is a critical point, there exists $\tilde{\alpha} \in \mathbb{P}^1(\mathbb{C})$ such that and the ramification index of f at $\tilde{\alpha}$ is $e_{\tilde{\alpha}} > 1$. It follows from (8) that the exponent difference of L at $\tilde{\alpha}$ is

$$\mid \rho_1^L(\tilde{\alpha}) - \rho_2^L(\tilde{\alpha}) \mid = e_{\tilde{\alpha}} > 1.$$

As $e_{\tilde{\alpha}} \in \mathbb{N}$, this contradicts the assumption that L doesn't have apparent singularities.

Hence any $\alpha \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ is not critical for f. It is also a regular point for H, so relation (8) implies that, for every $\tilde{\alpha}$ such that $f(\tilde{\alpha}) = \alpha$,

$$\mid \rho_1^L(\tilde{\alpha}) - \rho_2^L(\tilde{\alpha}) \mid = 1.$$

So $\tilde{\alpha} \notin S$, in other words $f(S) \subset \{0, 1, \infty\}$.

Finally, the last assertion of the statement follows also from (8), as the hypothesis $\tilde{\alpha} \notin S$ implies that the difference of the local exponents of L at $\tilde{\alpha}$ is 1. □

Hence, the basic Schwarz list gives a complete picture for second order differential operators with three singular points. Klein's work also opened the way for studying more general second order operators with algebraic solutions, as he proved that they are pull-backs of hypergeometric operators from this list. In 1979-1980, F. Baldassari and B. Dwork started studying the problem for second order operators with four singular points by studying Lamé operators. Based on some work of F. Brioschi, they gave necessary conditions for a Lamé operator to be algebraic [4], [1], [2].

3. BELYI FUNCTIONS, DESSINS D'ENFANTS AND GENERAL CONSEQUENCES

In [30] and [31] these results have been recovered in a new way, using Grothendieck's theory of Belyi functions and dessins d'enfants. This approach is possible due to Theorem 2.6 and Proposition 2.7, which provide the main technical tool for our approach.

Definition 3.1 ([20], [40]). Given X a compact Riemann surface, a **Belyi function** is a morphism of Riemann surfaces $f : X \to \mathbb{P}^1(\mathbb{C})$ with at most three critical values. The pair (X, f) is called a **Belyi pair**.

Up to Möbius transformations, we can assume that the critical points are 0, 1, and ∞. Moreover, a Belyi function f is called *clean* if every point in the fibre $f^{-1}(1)$ is ramified with the ramification index 2. Even if this may seem a significant restriction, there is a standard procedure for associating a clean function to an arbitrary Belyi function f: consider $f' = 4f(1-f)$.

A celebrated theorem of Belyi [5] states that there exists a function $f : X \to \mathbb{P}^1(\mathbb{C})$ with at most three critical values if and only if the Riemann surface X can be defined over a number field. Grothendieck [20] emphasized the arithmetic importance of this result and associated to f a particular type of graph embedded in the topological surface X. By Belyi's theorem, these graphs, which he called *dessins d'enfants*, have a profound arithmetic nature, beside the topological and the combinatorial ones. We mention that sometimes the term multi-graph is used when describing the dessins d'enfants, as they may have loops (edges for which the two end vertices coincide), or multiple edges (that is, edges that have the same end vertices).

In order to clarify the aforementioned correspondence, we recall some useful concepts. For further details on Belyi functions, dessins d'enfants and their topological and arithmetical properties and applications, see for example [40],[29], [19].

Definition 3.2. A **map** is a pair $(X, \mathcal{D})$ where $\mathcal{D}$ is a graph embedded into a surface X such that

- the vertices are distinct points of the surface;
- the edges are curves on the surface that intersect only at the vertices;
- the set $X \setminus \mathcal{D}$ is a disjoint union of connected components, called **faces**, each homeomorphic to an open disk.

The map $(X, \mathcal{D})$ is called a **dessin d'enfants** (or **hypermap**) if $\mathcal{D}$ is a bi-coloured graph. We shall consider the two colours to be white and black.

A dessin d'enfant that has all its black vertices of order 2 is called a **clean dessin** ([40],[32]). In this case, we can simplify the drawing, by only representing the white vertices (but keeping in mind that on each edge there is an old "black vertex"). It should be mentioned that in this case the graph can also have loops. The original graph can be recovered by adding a black vertex somewhere on each edge. An *abstract dessin* is an isomorphism class of dessins, where two dessins are called isomorphic if there exists a homeomorphism between the underlying topological surfaces that induces homeomorphisms between the topological spaces obtained as the union of the edges, on one side, and of the vertices, on the other.

Now, we can state the correspondence between Belyi pairs and dessins d'enfants ([20]):

Grothendieck correspondence. *There is a bijective correspondence between the set of clean abstract dessins d'enfants and the isomorphism classes of clean Belyi pairs.*

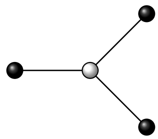
In short, if one starts with a clean Belyi couple (X, f), then $f^{-1}([0,1])$ is a graph embedded on X, the vertices being the elements of the set $f^{-1}(\{0,1\})$ (whence the two colours). The other way around, if one has a clean dessin embedded in a surface X, it is easy to construct (topologically) a map $f : X \to \mathbb{P}^1(\mathbb{C})$ (seen as the topological sphere) with three branching points. Then the Riemann Existence Theorem guarantees the existence of a Riemann surface structure on X such that f becomes a rational function with three critical values. Moreover, Belyi's Theorem says that this Riemann surface can be defined over a number field.

This correspondence is the framework in which a dictionary can be established, between the ramification data of the Belyi function f, on one side, and the combinatorial data of the associated dessin, on the other.

Dessin d'enfant	**Belyi function with critical values** $\{0, 1, \infty\}$
White vertices	$f^{-1}(\{0\})$
Black vertices	$f^{-1}(\{1\})$
Faces of the graph	$f^{-1}(\{\infty\})$
Degree of white vertex v_w	Branching order of the point $P_{v_w} \in f^{-1}(\{0\})$
Degree of black vertex v_b	Branching order of the point $P_{v_b} \in f^{-1}(\{1\})$
Order of the face φ	Branching order of the point $P_\varphi \in f^{-1}(\{\infty\})$
Number of edges	Degree of f
Edges of the graph	Sheets of $f^{-1}([0,1])$

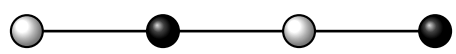
In order to offer a better picture of this correspondence, we will provide some examples.

Example 1. A common example is the star graph that corresponds to the power function. Below, we give the graph with three edges that corresponds to the function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}), f(z) = z^3$.



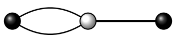
The white vertex corresponds to $f^{-1}(0) = \{0^3\}$[1] and the three black ones represent the complex roots of unity that are counterimages of 1, $f^{-1}(1) = \{1, \varepsilon, \overline{\varepsilon}\}$. The graph has only one face, corresponding to $f^{-1}(\infty) = \{\infty^3\}$.

Example 2. Other standard examples are Chebyshev polynomials of the first kind corresponding to path graphs. We give a graph with three edges that corresponds to $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}), f(z) = z(4z-3)^3$. (We applied a Möbius transformation in order to keep the critical points in the set $\{0, 1, \infty\}$).



The white vertices correspond to $f^{-1}(0) = \{0, \frac{3}{4}^2\}$ the black ones to $f^{-1}(1) = \{1, \frac{1}{4}^2\}$, while the only face corresponds to $f^{-1}(\infty) = \{\infty^3\}$.

Example 3. Another graph with three edges is the following:



corresponding to the function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, $f(z) = \frac{z^3}{3z-2}$. This time, the graph has two faces, corresponding to the points in the ramified fiber $f^{-1}(\infty)$: one of order 1, corresponding to $\frac{2}{3}$ and another of order 2, corresponding to ∞^2. The only white vertex represents 0^3 while the two black ones represent 1^2 and -2.

On the other side, the equalities (8), (10), (11) relate the ramification data of a Belyi function f to the invariants of a second order algebraic operator L without apparent singularities, which is realized as a pull-back of a hypergeometric operator in the Schwarz list through f; this always happens, following Klein's Theorem 2.6 and Proposition 2.7. This is the main point of our strategy.

We have the following finiteness result:

[1]The exponent denotes the ramification index, if greater than 1.

Theorem 3.1 ([32]). *Let m be a fixed positive real number. The set of Belyi pairs (X, f) (up to automorphisms of X and of $\mathbb{P}^1$) of degree at most m is finite.*

In particular, the set of Belyi functions $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that the critical values are $0, 1$ and ∞ and $f(\{0, 1, \infty\}) \subset \{0, 1, \infty\}$, of bounded degree, is finite.

If X and X' are two Riemann surfaces endowed with second order linear differential operators L and L' respectively, we say that the couples (X, L) and (X', L') are isomorphic if there exists an isomorphism of Riemann surfaces $f : X \to X'$ such that L is a rational pull-back of L' by f.

We obtain immediately:

Proposition 3.2. *If M is a fixed constant and G is one of the finite groups appearing in the Schwarz list, there are finitely many isomorphism classes of couples (X, L) where L has the projective monodromy group G, no apparent singularity and $\Delta_L \leq M$.*

Proof. Let H be the hypergeometric operator having the projective monodromy G. The differences of the local exponents at the three singular points are given by the Riemann scheme (13), hence the invariant $\Delta(H)$ is fixed.

If (X, L) is a couple as in the statement of the proposition then, according to Klein's Theorem (Theorem 2.6), there exists a morphism $f : X \to \mathbb{P}^1$ such that L is a rational pull-back by f. Moreover, f is a Belyi function, as L has no apparent singularity. Then, as $\Delta_L \leq M$, the relation (11) implies that the degree of f is bounded. Theorem 3.1 implies the result. $\square$

We obtain:

Theorem 3.3 ([31]). *The set of isomorphism classes of couples (X, L), where L has finite monodromy and no apparent singularity, is countable.*

4. LAMÉ OPERATORS WITH FINITE MONODROMY

The case of second order operators (on $\mathbb{P}^1$) with three singular points being completely and explicitly solved, the next situation to look at is that of second order operators with four singular points. A fundamental difficulty that arises is the following. For operators with three singular points we can suppose, modulo an isomorphism of $\mathbb{P}^1$, that the singular points are $0, 1$ and ∞. Moreover, the local exponent differences at these points completely determine the operator, modulo the projective equivalence defined hereinbefore. This is no longer the case for operators with at least four singular points. The location of the singular points and local exponent differences do not determine the differential operator any more. It depends, moreover, on some accessory parameters, namely one accessory parameter for operators with four singular points, according to (6).

In a series of papers ([30], [31], [33], [34],[12], [44]), the techniques described in the previous section were used for studying the Lamé operators:

$$L_n = \frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + \frac{1}{2}\sum_{i=1}^{3} \frac{1}{x-e_i}\frac{\mathrm{d}}{\mathrm{d}\,x} - \frac{n(n+1)x+B}{4(x-e_1)(x-e_2)(x-e_3)}. \tag{15}$$

We survey in what follows the main results.

Such an operator has four singular points (e_1, e_2, e_3 and ∞) and local exponents as follows:

$$\left(\begin{array}{cccc|c} e_1 & e_2 & e_3 & \infty & x \\ \hline 0 & 0 & 0 & -\frac{n}{2} & \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{n+1}{2} & \end{array}\right)$$

The accessory parameter is denoted by B. Modulo a homography, we can suppose that $e_1 = 0$ and $e_2 = 1$. In this case, we will denote λ the fourth singular point, $\lambda \in \mathbb{C} \setminus \{0, 1\}$, and the formula (15) becomes:

$$L_n = \frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-\lambda}\right)\frac{\mathrm{d}}{\mathrm{d}\,x} - \frac{n(n+1)x+B}{4x(x-1)(x-\lambda)}. \tag{16}$$

We can see a Lamé operator as being defined on a suitable elliptic curve [3]. Let E_λ be the elliptic curve described by the equation:

$$y^2 = 4x(x-1)(x-\lambda).$$

Then the pull-back of L_n by the projection

$$\pi : E_\lambda \to \mathbb{P}^1, \quad \pi(x, y) = x$$

is a second order operator

$$\mathcal{L}_n = D^2 - [n(n+1)x + B] \tag{17}$$

having one singular point, 0_E, with the local exponents $(-n, n+1)$. Here, $D = y\frac{\mathrm{d}}{\mathrm{d}\,x}$.

Let us remark first that $L_n = L_{-n-1}$, so we can suppose $n > -\frac{1}{2}$. Then, if $n \in \mathbb{Z} + \frac{1}{2}$, the difference of the local exponents at ∞ is an integer, so either L_n has an apparent singularity, or a logarithmic solution (which is not algebraic). On the other hand, Brioschi proved that if $n \in \mathbb{Z} + \frac{1}{2}$ then L_n is algebraic if and only if its projective monodromy group is the dihedral group of order four D_2 (the Klein "Vierergruppe"). See [36], [2], [15].

The following theorem fully classifies the Lamé operators with finite monodromy and no apparent singularity:

THEOREM 4.1 ([2], [31]). *Suppose that* $n \notin \mathbb{N} + \frac{1}{2}$.

1. *There is no Lamé operator with cyclic projective monodromy group.*
2. *There is no Lamé operator with tetrahedral projective monodromy group.*
3. *If the projective monodromy group of* L_n *is octahedral, then*
$$n \in \frac{1}{2}\left(\mathbb{Z} + \frac{1}{2}\right) \cup \frac{1}{3}\left(\mathbb{Z} + \frac{1}{2}\right).$$
4. *If the projective monodromy group of* L_n *is icosahedral, then*
$$n \in \frac{1}{3}\left(\mathbb{Z} + \frac{1}{2}\right) \cup \frac{1}{5}\left(\mathbb{Z} + \frac{1}{2}\right).$$
5. *If the projective monodromy group of* L_n *is dihedral, then* $n \in \mathbb{Z}$. *If the projective monodromy group of* L_n *is finite and* $n \in \mathbb{Z}$, *then this group is dihedral of order at least* 6.

Proof. We give here just the main idea, following [31]. As L_n has no apparent singularity, it is the pull-back, by a Belyi function f, of a hypergeometric operator H in the basic Schwarz list, having the same projective monodromy. Moreover, as this projective monodromy group is not D_2, this pull-back is unique (this remark is exploited in the explicit results that will be mentioned hereafter). Hence, using the Grothendieck correspondence, to such an operator one can associate a unique abstract dessin d'enfants on the Riemann sphere. The singular points and the differences of the local exponents of both the Lamé operator and the corresponding hypergeometric operator give information, by relation (8), about the ramification data of the pull-back function f. This corresponds, further on, to the combinatorial data of the dessin. By analysing these data in this combinatorial context, we obtain restrictions on the existence of the dessin, hence of the function f, and we arrive to the conclusion.

We illustrate this strategy by giving some more details for the case 2 in the statement of the theorem. If the projective monodromy group is the tetrahedral group $\mathcal{A}_4$, then the Riemann scheme of the hypergeometric operator H is

$$\left(\begin{array}{ccc|c} 0 & 1 & \infty & x \\ \hline 0 & 0 & 0 & \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \end{array}\right). \tag{18}$$

Then, by (8), the ramification data of the Belyi function f realizing the pull-back is summarized in the following table:

	0	1	λ	∞	a_i
0	1	1	1	$0, 2n+1$	$0, 2$
1	0	0	0	$0, 3n+\frac{3}{2}$	$0, 3$
∞	0	0	0	$0, 3n+\frac{3}{2}$	$0, 3$

This table should be read as follows. On the first column, there are the critical values of f (0, 1 and ∞ - the singular points of H); on the first line, there are all the points in the ramified fibers: 0, 1, λ, ∞ (the singular points of L) and some other points a_i (that are regular points for L). The rest of the table contains the possible ramification indices, as they follow from (8). For example, the relation (8) forces $f(0) = f(1) = f(\lambda) = 0$ with ramification index 1 (unramified points). The image of ∞ can be either 0 (with the ramification index $2n+1$), or 1 (with the ramification index $3n+\frac{3}{2}$), or ∞ (again, with the ramification index $3n+\frac{3}{2}$). For the other points a_i, as they are regular points of L, their ramification index must "kill" the difference of the exponents of the image, so it must be 2 if $f(a_i) = 0$ and it must be 3 if $f(a_i) \in \{0, \infty\}$.

We can immediately see that $f(\infty) \neq 0$, otherwise the ramification index would be $2n+1 \in \mathbb{Z}$, and this contradicts the hypothesis $n \notin \mathbb{N} + \frac{1}{2}$. Then we can suppose $f(\infty) = 1$ (the other case, $f(\infty) = \infty$, reduces to this one modulo a homography). Then, using the dictionary in the previous section, we can see the table hereinbefore as containing the combinatorial data of the associated dessin d'enfants and one can prove that such a graph cannot exist (it would be a bipartite regular graph with a cell whose valency is too high by respect to the total number of vertices). So, we obtain that there is no Lamé operator with tetrahedral projective monodromy. □

As a consequence, we obtain the following theorem, providing an answer to a question raised by Dwork:

Theorem 4.2 ([33]). *Let $n \notin \mathbb{Z} + \frac{1}{2}$ and G be a finite group. The set of isomorphism classes of elliptic curves on which there exists a Lamé operator $\mathcal{L}_n$ with projective monodromy G is finite, and on each such curve there exist finitely many such operators.*

Corollary 4.3 ([33]). *If $n \notin \frac{1}{2}\mathbb{Z}$, there are finitely many Lamé operators L_n with a full set of algebraic solutions.*

Using the combinatorics of dessins d'enfants, some explicit computations for various families of Lamé operators can be found in [30], [31], [34], [12]. In [30], the number of Lamé operators L_1 with given finite dihedral monodromy is explicitly computed using this method:

Theorem 4.4. *(i) The number* $\mathcal{C}(1, N)$ *of non-homographic covers* $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ *which transform by pull-back a hypergeometric operator* H *with dihedral projective monodromy group of order* $2N$ *into a Lamé operator* L_1 *is*

$$\mathcal{C}(1, N) = \frac{(N-1)(N-2)}{6} + \frac{2\varepsilon}{3}$$

where $\varepsilon = 1$ *if* $3 \mid N$ *and* $\varepsilon = 0$ *if not.*

(ii) If $\mathcal{L}(1, N)$ *is the number of non-homographic Lamé operators* L_1 *with dihedral projective monodromy group of order* $2N$*, then*

$$\mathcal{C}(1, N) = \sum_{N' \mid N, N' \neq 2} \mathcal{L}(1, N').$$

Explicit computations for the dihedral case with $n = 1$ were also performed in [10], with other techniques. The similar computation for $n = 2$ and dihedral projective monodromy can be found in [31] (and also in [11], with the same techniques as in [10]). In [34] the octahedral and icosahedral projective monodromies are considered. For dihedral monodromy, a general result is given in [12], generalizing Theorem 4.4 :

Theorem 4.5. *Let* $\mathcal{L}(n, N)$ *is the number of non-homographic Lamé operators* L_n *with dihedral projective monodromy group of order* $2N$*. Then*

$$\sum_{N' \mid N, N' \neq 2} \mathcal{L}(n, N') = \frac{n(n+1)}{12}(N-1)(N-2) + \frac{2}{3}\varepsilon(n, N)$$

where

$$\varepsilon(n, N) = \begin{cases} 1 & \text{if} \quad 3 \mid N\ ,\ n \equiv 1 \pmod 3 \\ 0 & \text{otherwise} \end{cases}.$$

One can use Euler's totient function and its 2-dimensional analogue for obtaining an explicit formula for $\mathcal{L}(n, N)$.

In [7], another study of the Lamé differential operators with algebraic solutions is realized, based mainly on the group theoretic properties of the corresponding finite monodromy group. All the possible cases for the monodromy group are listed.

Baldassarri, in [3], relates algebraic Lamé equations L_1 to torsion points on the associated elliptic curve and recovers several results (for example, that L_0 is never algebraic, the Klein group never occurs as the projective monodromy of L_1 etc.). More precisely, let us consider the operator L_n given by (16) and the elliptic curve E_λ given by the equation:

$$y^2 = 4x(x-1)(x-\lambda).$$

Denote by P one of the two points of E_λ such that $\pi(P) = B$, where $\pi : E_\lambda \to \mathbb{P}^1$ is the canonical degree two cover. If $n = 1$, one can see ([44]) that this provides a bijection between the equivalence classes of Lamé operators L_1 and equivalence classes of pointed elliptic curves (E, P).

Suppose now that L_1 is algebraic. Then, we have already seen that there is a (unique) Belyi function $f : \mathbb{P}^1 \to \mathbb{P}^1$ that realizes L_1 as a rational pull-back of a hypergeometric operator with the same projective monodromy. The point $P \in E_\lambda$ that we have described is the unique pole of the rational function $g = f \circ \pi$ ($-P$ being its only zero) and the differential form $\mathrm{d}\, g$ has a double zero at 0_E. Baldassarri proved that the reverse is also true, that is, if there exists a rational function g on the elliptic curve E with the properties hereinbefore, then L_1 is algebraic. Moreover, in this case, P is a torsion point of E_λ.

This approach is also taken by Zapponi in [44], where the field of moduli of Lamé operators L_1 with dihedral monodromy is studied and the relation between the algebraicity of the solutions and zeroes of certain modular forms is emphasized.

The methods described in this paper, combining data coming from the ramification properties of the pull-back function and from the combinatorics of the associated dessin d'enfants, can be used for general second order operators with four singular points - the Heun operators. In [41], Filipuk and Viduñas found 61 Belyi functions of degree less than 12, that realize pull-backs from hypergeometric to Heun operators, and their associated dessins d'enfants. Viduñas studies in [42] Heun equations that are pull-backs of hypergeometric equations with cyclic or dihedral monodromy. Another illustration of the method presented in our article can be found in [35], where a complete study of second order operators with four singular points, that are pull-backs of the hypergeometric operators with tetrahedral monodromy, is realized.

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