# ON THE NORMALITY AND ASSOCIATED PRIMES OF COVER IDEALS OF A CLASS OF IMPERFECT GRAPHS

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In this paper, we show that the cover ideals of Jahangir's graphs are normal, and also satisfy the (symbolic) (strong) persistence property. In addition, we determine when the unique homogeneous maximal ideal appears in the set of associated primes of powers of the cover ideals of Jahangir's graphs.

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## 1. INTRODUCTION AND PRELIMINARIES

The Jahangir's graph $\mathcal{J}_{n,m}$, for $m \geq 3$, is a graph on $nm+1$ vertices, i.e., a graph consisting of a cycle $C_{nm}$ with one additional vertex which is adjacent to $m$ vertices of $C_{nm}$ at distance $n$ to each other on $C_{nm}$. More explicitly, it consists of $m$ consecutive cycles of equal length $n+2$ such that all these cycles have one vertex in common and every pair of consecutive cycles has exactly one edge in common. Also, a vertex $v$ in the vertex set of $C_{nm}$ is called *radial* if $\{v, w\}$ is an edge of $\mathcal{J}_{n,m}$, where $w$ denotes the additional vertex of $\mathcal{J}_{n,m}$. Moreover, each edge $\{v, w\}$ is called a *spoke*. In fact, $\mathcal{J}_{n,m}$ has exactly $m$ spokes.

As an example, consider Figure 1. It presents the Jahangir's graphs $\mathcal{J}_{2,8}, \mathcal{J}_{5,4}$, and $\mathcal{J}_{3,5}$. The Jahangir's graph $\mathcal{J}_{2,8}$ appears on Jahangir's tomb in his mausoleum, which lays five kilometers north-west of Lahore, Pakistan across the River Ravi ([2]).

In general, Jahangir's graphs have been explored by many authors. In [9], algebraic and combinatorial properties and a computation of the number of the spanning trees are developed for a Jahangir's graph. Moreover, some algebraic and combinatorial characterizations of the spanning simplicial complex $\triangle_s(\mathcal{J}_{n,m})$ of the Jahangir's graph $\mathcal{J}_{n,m}$ are examined by Raza et al. [14]. In

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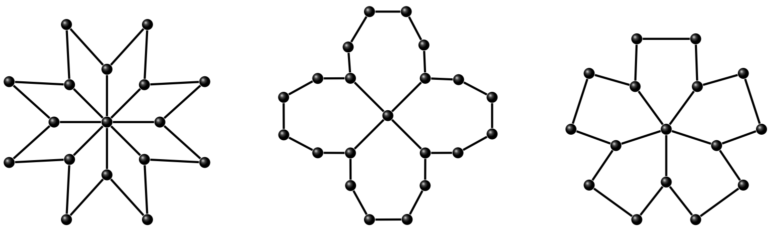

(a) $\mathcal{J}_{2,8}$ (b) $\mathcal{J}_{5,4}$ (c) $\mathcal{J}_{3,5}$

Figure 1 – Jahangir's graphs.

addition, it follows from [17, Theorem 7.7.14] that the edge ideals of Jahangir's graphs satisfy the strong persistence property. It should be noted that, based on [17, Corollary 10.5.9], the edge ideals of Jahangir's graphs are normal.

In this paper, we concentrate on the cover ideals of Jahangir's graphs. For this purpose, we need to recall some notions. Let $G$ be a finite simple graph with the vertex set $V(G)$ and the edge set $E(G)$. A subset $W \subseteq V(G)$ is called a vertex cover of $G$ if it intersects any edge of $G$. Furthermore, $W$ is called a minimal vertex cover of $G$ if it is a vertex cover and no proper subset of $W$ is a vertex cover of $G$. Let $W_1, \ldots, W_r$ be the minimal vertex covers of the graph $G$. Then, the cover ideal of $G$, denoted by $J(G)$, is defined as $J(G) = (X_{W_1}, \ldots, X_{W_r})$, where $X_{W_j} = \prod_{t \in W_j} x_t$ for each $j = 1, \ldots, r$. For more information, see [8, 17].

The first main aim of this paper is to study the normality of the cover ideals of Jahangir's graphs. To achieve this, we should review the required notions. Let $R$ be a ring and $I$ an ideal in $R$. An element $f \in R$ is *integral* over $I$, if there exists an equation

$$f^k + c_1 f^{k-1} + \cdots + c_{k-1} f + c_k = 0 \quad \text{with} \quad c_i \in I^i.$$

The set of elements $\overline{I}$ in $R$ which are integral over $I$ is the *integral closure* of $I$. The ideal $I$ is *integrally closed*, if $I = \overline{I}$, and $I$ is *normal* if all powers of $I$ are integrally closed. When $I$ is a monomial ideal in a polynomial ring $R$, then $\overline{I}$ is the monomial ideal generated by all monomials $u \in R$ for which there exists an integer $k$ such that $u^k \in I^k$, see [8, Theorem 1.4.2]. It has been shown in [16] that the edge ideals of bipartite graphs are normal and that the cover ideals of perfect graphs are normal [17, Corollary 14.6.25]. In this direction, we show that the cover ideals of Jahangir's graphs are normal, see Theorem 2.3.

The second main goal of this paper is to study the associated primes of the powers of cover ideals of Jahangir's graphs. We here give some background on this. Suppose that $R$ is a commutative Noetherian ring and $I$ an ideal of $R$.

A prime ideal $\mathfrak{p} \subset R$ is an *associated prime* of $I$ if there exists an element $v$ in $R$ such that $\mathfrak{p} = (I :_R v)$, where $(I :_R v) = \{r \in R \ : \ rv \in I\}$. The set of associated primes of $I$, denoted by $\mathrm{Ass}_R(R/I)$, is the set of all prime ideals associated to $I$. Brodmann [4] showed that the sequence $\{\mathrm{Ass}_R(R/I^k)\}_{k \geq 1}$ of associated prime ideals is stationary for large $k$, that is, there exists a positive integer $k_0$ such that $\mathrm{Ass}_R(R/I^k) = \mathrm{Ass}_R(R/I^{k_0})$ for all integers $k \geq k_0$. The minimum such $k_0$ is called the *index of stability* of $I$ and $\mathrm{Ass}_R(R/I^{k_0})$ is called the *stable set* of associated prime ideals of $I$, which is denoted by $\mathrm{Ass}^{\infty}(I)$. Many questions can be asked in the context of Brodmann's theorem. An ideal $I$ of $R$ satisfies the *persistence property* if $\mathrm{Ass}_R(R/I^k) \subseteq \mathrm{Ass}_R(R/I^{k+1})$ for all positive integers $k$. In addition, an ideal $I$ of $R$ has the *strong persistence property* if $(I^{k+1} :_R I) = I^k$ for all $k \geq 1$. It is well-known that the strong persistence property implies the persistence property, see Proposition 2.4. Suppose now that $I$ is a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$ and $x_1, \ldots, x_n$ are indeterminates. In general, finding classes of monomial ideals which have the (strong) persistence property is tricky and complicated. Nonetheless, according to [17, Example 7.7.18], there exist square-free monomial ideals which do not satisfy the persistence property. It is also known by [17, Theorem 7.7.14] that all edge ideals of finite simple graphs have the strong persistence property; even so, this result is true for every finite graph with loops [15]. Furthermore, based on [6], the cover ideals of perfect graphs satisfy the persistence property. Note that a graph is perfect if and only if it contains no odd cycle of length at least five, or its complement, as an induced subgraph [3, Theorem 14.18]. However, little is known about the cover ideals of imperfect graphs. More recently, it has been established in [11] that the cover ideals of some imperfect graphs have the strong persistence property. Thus, another motivation of studying Jahangir's graph $\mathcal{J}_{n,m}$ is that, under certain $m, n$, it is imperfect, and so we can identify another class of imperfect graphs which satisfy the (strong) persistence property.

Finally, we say that $I$ has the *symbolic strong persistence property* if $(I^{(k+1)} : I^{(1)}) = I^{(k)}$ for all $k$, where $I^{(k)}$ denotes the $k$-th symbolic power of $I$, cf. [12, 10]. As the second main result of this paper, we conclude that the cover ideals of Jahangir's graphs satisfy the (symbolic) (strong) persistence property, see Corollary 2.5. In the third main result of this paper, we determine when the unique homogeneous maximal ideal appears in the set of associated primes of powers of the cover ideals of Jahangir's graphs (Theorem 2.9).

Throughout this paper, we denote the unique minimal set of monomial generators of a monomial ideal $I$ by $\mathcal{G}(I)$. Also, $R = K[x_1, \ldots, x_n]$ is a polynomial ring over a field $K$ and $x_1, \ldots, x_n$ are indeterminates. A simple graph $G$ means that $G$ has no loop and no multiple edge. All graphs in this paper are undirected. Moreover, if $G$ is a finite simple graph, then $J(G)$ stands for the cover ideal of $G$.

## 2. MAIN RESULT

We begin this section with the following auxiliary results, which we require to establish Theorem 2.3.

PROPOSITION 2.1. *The cover ideals of cycle graphs are normal.*

*Proof.* Let $C_n$ denote a cycle graph of order $n$. If $n$ is even, then $C_n$ is bipartite. It follows from [7, Corollary 2.6] that $J(C_n)$ is normally torsion-free. Now, [8, Theorem 1.4.6] yields that $J(C_n)$ is normal. If $n$ is odd, then $C_n$ is an odd cycle graph, and the claim can be deduced immediately from [1, Theorem 1.10]. $\square$

THEOREM 2.2. *Let $I$ and $H$ be two normal square-free monomial ideals in a polynomial ring $R = K[x_1, \ldots, x_n]$ such that $I + H$ is normal. Let $x_\ell \in \{x_1, \ldots, x_n\}$ be a variable with $\gcd(v, x_\ell) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$. Let $\mathcal{G}(I) = \{u_1, \ldots, u_s\}$ and $\mathcal{G}(H) = \{h_1, \ldots, h_r\}$ such that $\gcd(u_1, \ldots, u_s) = 1$ and $\gcd(h_1, \ldots, h_r) = 1$. Then $L := I + x_\ell H$ is normal.*

*Proof.* Let $\mathcal{G}(I) = \{u_1, \ldots, u_s\}$ and $\mathcal{G}(H) = \{h_1, \ldots, h_r\}$. As $\gcd(v, x_\ell) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$, without loss of generality, we may assume that $x_\ell = x_1 \in K[x_1]$ and

$$\mathcal{G}(I) \cup \mathcal{G}(H) = \{u_1, \ldots, u_s, h_1, \ldots, h_r\} \subseteq K[x_2, \ldots, x_n].$$

Our aim is to show that $\overline{L^t} = L^t$ for all integers $t \geq 1$. To do this, it is sufficient to prove that $\overline{L^t} \subseteq L^t$. Let $\alpha$ be a monomial in $\overline{L^t}$ and write $\alpha = x_1^b \delta$ with $x_1 \nmid \delta$ and $\delta \in R$. It follows from [8, Theorem 1.4.2] that $\alpha^k \in L^{tk}$ for some integer $k \geq 1$. Thus, one can write

$$\alpha^k = x_1^{bk} \delta^k = \prod_{i=1}^{s} u_i^{p_i} x_1^{q+\varepsilon} \prod_{j=1}^{r} h_j^{q_j} \beta, \tag{1}$$

with $\sum_{i=1}^{s} p_i = p$, $\sum_{j=1}^{r} q_j = q$, $p + q = tk$, $\varepsilon \geq 0$, and $\beta$ is some monomial in $R$ such that $x_1 \nmid \beta$. Since $x_1 \nmid \beta$, $x_1 \nmid \delta$, and $\gcd(v, x_1) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$, one can derive that $bk = q + \varepsilon$. Accordingly, by virtue of (1), we get

$$\delta^k = \prod_{i=1}^{s} u_i^{p_i} \prod_{j=1}^{r} h_j^{q_j} \beta \in (I + H)^{tk}.$$

This leads to $\delta \in \overline{(I+H)^t}$. Because $I+H$ is normal, we obtain that $\overline{(I+H)^t} = (I+H)^t$, and so $\delta \in (I+H)^t$. Consequently, one can write

$$\delta = \prod_{i=1}^{s} u_i^{l_i} \prod_{j=1}^{r} h_j^{z_j} \gamma, \tag{2}$$

with $\sum_{i=1}^{s} l_i = l$, $\sum_{j=1}^{r} z_j = z$, $l+z=t$, and $\gamma$ is some monomial in $R$. Note that $x_1 \nmid \gamma$ since $x_1 \nmid \delta$. Thanks to $x_1^{bk}\delta^k \in L^{tk}$, one can deduce immediately from (2) that

$$\prod_{i=1}^{s} u_i^{l_i k} x_1^{bk} \prod_{j=1}^{r} h_j^{z_j k} \gamma^k \in L^{tk} = (I + x_1 H)^{tk}.$$

Therefore, we conclude that $bk \geq zk$, that is, $b \geq z$. This leads us to

$$x_1^b \delta = \prod_{i=1}^{s} u_i^{l_i} x_1^b \prod_{j=1}^{r} h_j^{z_j} \gamma \in (I + x_1 H)^t,$$

and the proof is complete. □

We are now in a position to state the first main result of this paper in the following theorem.

THEOREM 2.3. *The cover ideals of the Jahangir's graphs are normal.*

*Proof.* Assume that $\mathcal{J}_{n,m}$ is the Jahangir's graph with the vertex set $V(\mathcal{J}_{n,m}) = \{0, 1, 2, \ldots, nm\}$, 0 is the additional vertex, and the following edge set

$$\begin{aligned} E(\mathcal{J}_{n,m}) = &\{\{i, i+1\} : i = 1, \ldots, nm-1\} \cup \{\{nm, 1\}\} \\ &\cup \{\{0, 1+rn\} \ : \ r = 0, 1, \ldots, m-1\}. \end{aligned}$$

Let $R = K[x_1, \ldots, x_{nm}]$, $u := \prod_{r=0}^{m-1} x_{1+rn}$, and $S = R[x_0]$. Let $J(\mathcal{J}_{n,m})$ (respectively, $J(C_{nm})$) denote the cover ideal of $\mathcal{J}_{n,m}$ (respectively, $C_{nm}$). To simplify notation, set $L := J(\mathcal{J}_{n,m})$, $I := J(C_{nm})$, and $F := (I :_S u)$. Clearly, $L = I \cap (u, x_0) \subset S$. Because $I \cap (u) = u\,(I :_S u)$ and $I \cap (x_0) = x_0 I$, we derive that $L = u\,(I :_S u) + x_0 I$; thus, $L = uF + x_0 I$. Moreover, based on Proposition 2.1, the ideal $I$ is normal. On account of $I$ is square-free, then the ideal $F$ is obtained from $I$ by putting $x_{1+rn} = 1$ for all $r = 0, 1, \ldots, m-1$. Hence, $F$ is normal by [17, Proposition 12.2.3], and so $uF$ is normal according to [1, Remark 1.2]. Due to $F = (I :_S u)$, one gains $uF + I = I$; therefore, $uF + I$ is normal. Since $\gcd(v, x_0) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(uF)$, it follows from Theorem 2.2 that $L$ is normal. This finishes the proof. □

The next proposition is a well-known result, but we prove it in a different way.

Proposition 2.4. *Let $I$ be an ideal in a commutative Noetherian ring $S$ such that it satisfies the strong persistence property. Then $I$ has the persistence property.*

*Proof.* Fix $k \geq 1$, and select an arbitrary element $\mathfrak{p} \in \mathrm{Ass}_S(S/I^k)$. This yields that $\mathfrak{p} = (I^k :_S h)$ for some $h \in S$. Because $I$ satisfies the strong persistence property, we obtain $(I^{k+1} :_S I) = I^k$, and so $\mathfrak{p} = ((I^{k+1} :_S I) :_S h)$. Let $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$. We therefore have $\mathfrak{p} = (I^{k+1} :_S h\sum_{i=1}^{m} u_iS) = \bigcap_{i=1}^{m}(I^{k+1} :_S hu_i)$. Hence, one gains $\mathfrak{p} = (I^{k+1} :_S hu_i)$ for some $1 \leq i \leq m$. Accordingly, $\mathfrak{p} \in \mathrm{Ass}_S(S/I^{k+1})$. This implies that $I$ has the persistence property, as claimed. $\square$

As an application of Theorem 2.3, we can present the second main result of this paper in the next corollary.

Corollary 2.5. *Let $\mathcal{J}_{n,m}$ be the Jahangir's graph, and $J(\mathcal{J}_{n,m})$ its cover ideal. Then, the following statements hold:*

(i) *$J(\mathcal{J}_{n,m})$ has the strong persistence property.*

(ii) *$J(\mathcal{J}_{n,m})$ has the persistence property.*

(iii) *$J(\mathcal{J}_{n,m})$ has the symbolic strong persistence property.*

*Proof.* (i) Thanks to [13, Theorem 6.2], every normal monomial ideal has the strong persistence property, and so the claim follows readily from Theorem 2.3.

(ii) We can combine together Proposition 2.4 and (i) to obtain the assertion.

(iii) By [15, Theorem 11], the strong persistence property implies the symbolic strong persistence property, and thus the claim is an immediate consequence of (i). $\square$

Here, we give the following proposition, which helps us to demonstrate Lemma 2.7.

Proposition 2.6. *Let $\mathcal{J}_{n,m}$ be the Jahangir's graph, $n$ odd, and $W$ a vertex cover set of $\mathcal{J}_{n,m}$. If $W$ does not contain the additional vertex, then $|W| \geq m(n+1)/2$. If $W$ contains the additional vertex, then $|W| \geq (nm+3)/2$ if $m$ is odd, or $|W| \geq (nm+2)/2$ if $m$ is even.*

*Proof.* Let $w$ denote the additional vertex of $\mathcal{J}_{n,m}$. First, assume that $w \notin W$. Let $v_1, \ldots, v_m$ be the radial vertices of $\mathcal{J}_{n,m}$. It follows from $w \notin W$ that $v_i \in W$ for $i = 1, \ldots, m$. Since $W \cap V(C_{nm})$ is a vertex cover of $C_{nm}$, and $v_i, v_{i+1} \in W \cap V(C_{nm})$, we deduce that in the path from $v_i$ to $v_{i+1}$, one needs at least $(n-1)/2$ vertices for $i = 1, \ldots, m$, where $v_{m+1}$ represents $v_1$. We therefore gain $|W| \geq m + m(n-1)/2$, and so $|W| \geq m(n+1)/2$. Now, let $w \in W$. Then $W \cap V(C_{nm})$ must be a vertex cover of $C_{nm}$. If $m$ is odd, then $C_{nm}$ is an odd cycle, and so $W \cap V(C_{nm})$ has at least $(nm+1)/2$ elements; thus, we get $|W| \geq 1 + (nm+1)/2$, and hence, $|W| \geq (nm+3)/2$. If $m$ is even, then $C_{nm}$ is an even cycle. Accordingly, $W \cap V(C_{nm})$ has at least $nm/2$ elements, and so $|W| \geq (nm+2)/2$. This completes the proof. □

To prove Theorem 2.9, we need the subsequent lemma.

LEMMA 2.7. *Let $\mathcal{J}_{n,m}$ be the Jahangir's graph with the vertex set $V(\mathcal{J}_{n,m}) = \{0, 1, 2, \ldots, nm\}$, 0 be the additional vertex, and the following edge set*

$$E(\mathcal{J}_{n,m}) = \{\{i, i+1\} : i = 1, \ldots, nm-1\} \cup \{\{nm, 1\}\} \cup \{\{0, 1+kn\} : k = 0, 1, \ldots, m-1\}.$$

*Let $C_{2e_1+1}, \ldots, C_{2e_z+1}$ be all induced odd cycle subgraphs of $\mathcal{J}_{n,m}$. Then, for each positive integer $s$, if $x_0^{\ell_0} x_1^{\ell_1} \cdots x_{nm}^{\ell_{nm}} \in J(\mathcal{J}_{n,m})^s$, then the following conditions hold:*

(i) $\ell_\alpha + \ell_\beta \geq s$ *for each* $\{\alpha, \beta\} \in E(\mathcal{J}_{n,m})$;

(ii) $\displaystyle\sum_{\{\alpha,\beta\} \in E(C_{2e_j+1})} ((\ell_\alpha + \ell_\beta) - s) \geq s$ *for each* $j = 1, \ldots, z$;

(iii) *If $n$ and $m$ are odd, then* $((m-1)/2)\ell_0 + \displaystyle\sum_{i=1}^{nm} \ell_i \geq sm(n+1)/2$.

*Proof.* Fix $s \geq 1$. To verify the claim, consider $x_0^{\ell_0} x_1^{\ell_1} \cdots x_{nm}^{\ell_{nm}} \in J(\mathcal{J}_{n,m})^s$. Take an arbitrary edge $\{\alpha, \beta\}$ in $E(\mathcal{J}_{n,m})$. Since $J(\mathcal{J}_{n,m}) \subseteq (x_\alpha, x_\beta)$, this implies that $x_0^{\ell_0} x_1^{\ell_1} \cdots x_{nm}^{\ell_{nm}} \in (x_\alpha, x_\beta)^s$, and so $\ell_\alpha + \ell_\beta \geq s$. This proves condition (i). By virtue of $J(\mathcal{J}_{n,m})^s \subseteq J(C_{2e_j+1})^s$ for all $j$, we have $\prod_{i \in V(C_{2e_j+1})} x_i^{\ell_i} \in J(C_{2e_j+1})^s$ for all $j$, where $J(C_{2e_j+1})$ is the cover ideal of $C_{2e_j+1}$. Therefore, condition (ii) follows directly from [11, Lemma 3.2]. It follows from $x_0^{\ell_0} x_1^{\ell_1} \cdots x_{nm}^{\ell_{nm}} \in J(\mathcal{J}_{n,m})^s$ that there exist monomials $g_1, \ldots, g_s \in \mathcal{G}(J(\mathcal{J}_{n,m}))$ and some monomial $h$ in $K[x_0, x_1, \ldots, x_{nm}]$ such that

$$x_0^{\ell_0} x_1^{\ell_1} \cdots x_{nm}^{\ell_{nm}} = g_1 \cdots g_s h.$$

Note that each $g_i$ corresponds to a minimal vertex cover set of $\mathcal{J}_{n,m}$. After reordering, there exists some integer $\lambda \leq \ell_0$ such that $x_0 \mid g_i$ for $i = 1, \ldots, \lambda$, and $x_0 \nmid g_i$ for $i = \lambda + 1, \ldots, s$. If $n$ and $m$ are odd, then one can derive from Proposition 2.6 that $\deg(g_i) \geq (nm + 3)/2$ for $i = 1, \ldots, \lambda$, and also $\deg(g_i) \geq m(n+1)/2$ for $i = \lambda+1, \ldots, s$. Note that $m \geq 3$ and $\lambda \leq \ell_0$. Hence, we get the following

$$\begin{aligned} 2\sum_{i=0}^{nm} \ell_i &\geq \lambda(nm + 3) + (s - \lambda)m(n + 1) \\ &= smn + sm - \lambda(m - 3) \\ &\geq smn + sm - \ell_0(m - 3). \end{aligned}$$

Consequently, we deduce that $((m-1)/2)\ell_0 + \sum_{i=1}^{nm} \ell_i \geq sm(n+1)/2$. Therefore, one concludes that condition (iii) holds. This finishes the proof. □

*Remark* 2.8. Assume that $\mathcal{J}_{n,m}$ is the Jahangir's graph with the vertex set $V(\mathcal{J}_{n,m}) = \{0, 1, 2, \ldots, nm\}$ with labeling counterclockwise, 0 is the additional vertex, $\mathfrak{m} = (x_0, x_1, \ldots, x_{nm})$ the unique homogeneous maximal ideal, and $J(\mathcal{J}_{n,m})$ its cover ideal. If $n$ is an even number, then $\mathcal{J}_{n,m}$ has no any odd cycle, and so $\mathcal{J}_{n,m}$ is a bipartite graph. This implies that $J(\mathcal{J}_{n,m})$ is normally torsion-free, and so the unique homogeneous maximal ideal does not appear in the set of associated primes of powers of the cover ideal of the Jahangir's graph, that is, $\mathfrak{m} \notin \mathrm{Ass}(J(\mathcal{J}_{n,m})^s)$ for all $s \geq 1$.

Now, assume that $n$ is an odd number and $m$ is an even number. Our computations by Macaulay2 software show that the unique homogeneous maximal ideal $\mathfrak{m}$ does not appear in the set of associated primes of powers of the cover ideal of the Jahangir's graph, that is, $\mathfrak{m} \notin \mathrm{Ass}(J(\mathcal{J}_{n,m})^s)$ for all $s \geq 1$. In particular, it should be noted that in this case, $J(\mathcal{J}_{n,m})/\{x_0\}$ is exactly the cover ideal of the rim, and since $m$ is even, this yields that the rim is an even cycle graph, and hence $J(\mathcal{J}_{n,m})/\{x_0\}$ is normally torsion-free. Nevertheless, we need to discuss the case in which $n$ is an odd number and $m$ is an even number separately. In what follows, we explore the case in which $m, n$ are odd numbers.

We are ready to give the third main result of this paper in the next theorem. Indeed, we determine when the unique homogeneous maximal ideal appears in the set of associated primes of powers of the cover ideals of Jahangir's graphs.

THEOREM 2.9. *Let $\mathcal{J}_{n,m}$ be the Jahangir's graph, $J(\mathcal{J}_{n,m})$ its cover ideal, and $m, n$ odd numbers. Then, for all $s \geq 3$,*

$$(x_i \ : \ i \in V(\mathcal{J}_{n,m})) \in \mathrm{Ass}(J(\mathcal{J}_{n,m})^s) \setminus (\mathrm{Ass}(J(\mathcal{J}_{n,m})) \cup \mathrm{Ass}(J(\mathcal{J}_{n,m})^2)).$$

*Proof.* Without loss of generality, assume that $\mathcal{J}_{n,m}$ is the Jahangir's graph with the vertex set $V(\mathcal{J}_{n,m}) = \{0, 1, 2, \ldots, nm\}$ with labeling counterclockwise, 0 is the additional vertex, $\mathfrak{m} = (x_0, x_1, \ldots, x_{nm})$, and the following edge set

$$\begin{aligned} E(\mathcal{J}_{n,m}) =& \{\{i, i+1\} : i = 1, \ldots, nm-1\} \cup \{\{nm, 1\}\} \\ & \cup \{\{0, 1+kn\} : k = 0, 1, \ldots, m-1\}. \end{aligned}$$

Set $L := J(\mathcal{J}_{n,m})$. Since $L$ is a square-free monomial ideal, we have $\mathrm{Ass}(L) = \mathrm{Min}(L)$, and so $\mathfrak{m} \notin \mathrm{Ass}(L)$. In addition, it follows from [5, Corollary 3.4] that $\mathfrak{m} \notin \mathrm{Ass}(L^2)$. According to Corollary 2.5, the ideal $L$ satisfies the persistence property. Hence, it is sufficient for us to show that $\mathfrak{m} \in \mathrm{Ass}(J(\mathcal{J}_{n,m})^3)$. To accomplish this, set

$$h := x_0^2 \prod_{k=0}^{m-1} \prod_{j=1}^{n-1} x_{1+kn}^2 x_{1+kn+j}^{\lambda}, \text{ where } \lambda = \begin{cases} 1 & \text{if } j \text{ is odd,} \\ 2 & \text{if } j \text{ is even.} \end{cases}$$

In what follows, our aim is to show that $h \notin L^3$ and $x_i h \in L^3$ for each $i = 0, 1, \ldots, nm$. Suppose, on the contrary, that $h \in L^3$. One can promptly deduce from Lemma 2.7(iii) that

$$((m-1)/2)(2) + \sum_{i=1}^{nm} \deg_{x_i} h \geq 3m(n+1)/2. \tag{3}$$

Note that there exist exactly $n-1$ numbers between any two consecutive radial vertices, and by virtue of $n$ is odd, this implies that there exist $(n-1)/2$ odd numbers and also $(n-1)/2$ even numbers between any two consecutive radial vertices. We thus get

$$\begin{aligned} ((m-1)/2)(2) + \sum_{i=1}^{nm} \deg_{x_i} h =& m - 1 + 2m + 2m(n-1)/2 + m(n-1)/2 \\ =& (3m(n+1) - 2)/2, \end{aligned}$$

which contradicts (3). Accordingly, we obtain $h \notin L^3$. To complete the proof, one has to verify that $x_i h \in L^3$ for each $i = 0, 1, \ldots, nm$. To achieve this, fix $1 \leq t \leq nm$. We relabel the vertices of the odd cycle $C_{nm}$ starting from $t = y_1$ counterclockwise $y_2, y_3, \ldots, y_{nm}$. Now, set

$$u_1 := x_0 \prod_{i \text{ is odd}} y_i, \; u_2 := x_0 y_1 \prod_{i \text{ is even}} y_i, \; u_3 := \prod_{k=0}^{m-1} \prod_{\substack{j=1,\ldots,n-1, \\ \text{where } j \text{ is even}}} x_{1+kn} x_{1+kn+j}.$$

It is not hard to check that $x_t h = u_1 u_2 u_3$, and also $u_1$, $u_2$, and $u_3$ correspond to vertex cover sets of $\mathcal{J}_{n,m}$. This yields that $u_1, u_2, u_3 \in L$, and so $x_t h \in L^3$.

It remains to show that $x_0h \in L^3$. To do this, put

$$v_1 := x_0 \prod_{i \text{ is odd}} x_i,\ v_2 := x_0 \prod_{\substack{i=n+1,\ldots,nm,\\ \text{where } i \text{ is even}}} x_i \prod_{\substack{i=1,\ldots,n+1,\\ \text{where } i \text{ is odd}}} x_i,$$

$$v_3 := x_0 \prod_{\substack{i=1,\ldots,n,\\ \text{where } i \text{ is even}}} x_i \prod_{k=1}^{m-1} \prod_{\substack{j=1,\ldots,n-1,\\ \text{where } j \text{ is even}}} x_{1+kn}x_{1+kn+j}.$$

It is routine to see that $x_0h = v_1v_2v_3$, and also $v_1$, $v_2$, and $v_3$ correspond to vertex cover sets of $\mathcal{J}_{n,m}$. We thus gain $v_1, v_2, v_3 \in L$, and hence $x_0h \in L^3$, as required. □

As an immediate consequence of Theorem 2.9, we give the following result:

COROLLARY 2.10. *Let $G = (V(G), E(G))$ be a finite simple graph with the vertex set $V(G) = \{x_1, \ldots, x_n\}$, and let $G$ have a subgraph $\mathcal{J}_{n,m}$ as the Jahangir's graph, where $m, n$ be odd numbers. Let $R = K[x_1, \ldots, x_n]$ and $J$ denote the cover ideal of $G$. Then, for all $s \geq 3$,*

$$(x_i\ :\ i \in V(\mathcal{J}_{n,m})) \in \mathrm{Ass}(R/J^s) \setminus (\mathrm{Ass}(R/J) \cup \mathrm{Ass}(R/J^2)).$$

*Proof.* This claim can be deduced from [6, Lemma 2.11] and Theorem 2.9. □

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