ESTIMATE FOR HIGHER MOMENTS OF CUSP FORM COEFFICIENTS OVER SUM OF TWO SQUARES

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Let f and g be two distinct primitive holomorphic cusp forms of even integral weights k_1 and k_2 for the full modular group $\Gamma = SL(2,\mathbb{Z})$, respectively. Denote by $\lambda_f(n)$ and $\lambda_g(n)$ the nth normalized Fourier coefficients of f and g, respectively. In this paper, we consider the summatory function

$$\sum_{n=a^2+b^2\leq x} \lambda_f(n)^i \lambda_g(n)^j,$$

for $x \geq 2$, where $a, b \in \mathbb{Z}$ and $i, j \geq 1$ are positive integers.

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1. INTRODUCTION

The Fourier coefficients of modular forms are interesting and important number-theoretic functions. Let H_k be the set of normalized primitive holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = SL(2,\mathbb{Z})$, which consists of the eigenfunctions for the all Hecke operators T_n. The Fourier coefficients of $f \in H_k$ at the cusp infinity admits the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \qquad \Im(z) > 0,$$

where $e(z) = e^{2\pi i z}$, and we normalize $\lambda_f(1) = 1$ and $\lambda_f(n) \in \mathbb{R}$ is the nth normalized Fourier coefficient (Hecke eigenvalue) of f. It is well-known that the Hecke eigenvalues $\lambda_f(n)$ satisfies the Hecke relation

$$\lambda_f(n)\lambda_f(m) = \sum_{\mathrm{d}|(m,n)} \lambda_f\left(\frac{mn}{\mathrm{d}^2}\right) \tag{1}$$

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for all integers $m, n \geq 1$. In 1974, Deligne [5] proved the celebrated Ramanujan-Petersson conjecture which asserts that

$$|\lambda_f(n)| \leq \mathrm{d}(n), \tag{2}$$

where $\mathrm{d}(n)$ denotes the classical divisor function.

Then the result (2) implies that for any prime number p, there exist two complex numbers $\alpha_f(p), \beta_f(p)$ such that

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \qquad |\alpha_f(p)| = |\beta_f(p)| = \alpha_f(p)\beta_f(p) = 1. \tag{3}$$

The distribution of the Fourier coefficients of cusp forms are interesting and has been investigated by lots of authors. In 1927, Hecke [9] proved that

$$S(x) = \sum_{n \leq x} \lambda_f(n) \ll_f x^{\frac{1}{2}}.$$

Subsequent improvements on $S(x)$ were made by a number of authors (see e.g., [5, 10, 27, 32]).

In the 1930's, Rankin [26] and Selberg [29] independently proved that

$$\sum_{n \leq x} \lambda_f^2(n) = c_f x + O_f(x^{\frac{3}{5}}),$$

where c_f is some suitable constant depending on f. Later, other authors considered the higher moments of the Fourier coefficients of cusp forms (see c.f., [18, 7, 22]). In particular, Lau and Lü [19] established a general formula for the summatory function

$$U_l(x) = \sum_{n \leq x} \lambda_f(n)^l, \quad l \geq 3$$

for the normalized Fourier coefficients of both holomorphic cusp forms and Maass cusp forms under suitable conditions.

Let $f \in H_{k_1}$ and $g \in H_{k_2}$ be Hecke eigenforms, and denote by $\lambda_f(n)$ and $\lambda_g(n)$ the nth normalized Fourier coefficients of f and g, respectively. Based on the work of Gelbart and Jacquet [8], we know that the automorphy of symmetric power lifting $\mathrm{sym}^j \pi_f$ attached to f is proved for $j = 2$, and similarly for g. Fomenko [7] considered the average behaviour of second moment of normalized Fourier coefficients involving two distinct cusp forms. More precisely, he proved that

$$\sum_{n \leq x} \lambda_f^2(n)\lambda_g(n) \ll_{f,g,\varepsilon} x^{\frac{5}{6}+\varepsilon}$$

and

$$\sum_{n \leq x} \lambda_f^2(n)\lambda_g^2(n) = c_{f,g} x + O_{f,g,\varepsilon}\left(x^{\frac{9}{10}+\varepsilon}\right) \tag{4}$$

for any $\varepsilon > 0$, here $c_{f,g} > 0$ is some positive constant depending on f, g. The result in (4) require the condition that $\mathrm{sym}^2\pi_f \not\cong \mathrm{sym}^2\pi_g$.

In 2013, Lü [20] established the following formula

$$\sum_{n\le x} \lambda_f^5(n)\lambda_g(n) \ll_{f,g,\varepsilon} x^{\frac{31}{32}+\varepsilon},$$

which can be regarded that the sequences $\{\lambda_f(n)^5\}$ and $\{\lambda_g(n)\}$ are asymptotically orthogonal as $x \to \infty$. Later, Lü and Sankaranarayanan [21] further proved that

$$\sum_{n\le x} \lambda_f^5(n)\lambda_g^2(n) \ll_{f,g,\varepsilon} x^{\frac{184}{187}+\varepsilon}, \quad \sum_{n\le x} \lambda_f(n)\lambda_g^6(n) \ll_{f,g,\varepsilon} x^{\frac{63}{64}+\varepsilon}. \tag{5}$$

They also established some other similar formulae analogue to (5).

On the other hand, Zhai [33] considered the asymptotic behaviour of the following sum

$$S_l(f;x) := \sum_{a^2+b^2\le x} \lambda_f(a^2+b^2)^l, \tag{6}$$

where $2 \le l \le 8$ and $a, b, l \in \mathbb{Z}$. In fact, he established the asymptotics for (6) by showing that

$$S_l(f;x) = xP_l(\log x) + O_{f,\varepsilon}(x^{\theta_l+\varepsilon}),$$

where the $P_2(t), P_4(t), P_6(t), P_8(t)$ are polynomials of degree $0, 1, 4, 13$, respectively. And $\deg P_j \equiv 0$ for $l = 3, 5, 7$. Here

$$\theta_2 = \frac{8}{11}, \qquad \theta_3 = \frac{17}{20}, \qquad \theta_4 = \frac{43}{46}, \qquad \theta_5 = \frac{83}{86},$$
$$\theta_6 = \frac{184}{187}, \qquad \theta_7 = \frac{355}{358}, \qquad \theta_8 = \frac{752}{755}.$$

Let $f \in H_{k_1}$ and $g \in H_{k_2}$ be two distinct Hecke eigenforms. Inspired by the above results, in this paper, we are interested in the asymptotic behaviour of the following sum

$$S_{i,j}(f,g;x) := \sum_{n=a^2+b^2\le x} \lambda_f(n)^i \lambda_g(n)^j$$

where $a, b \in \mathbb{Z}$ and $i, j \ge 1$ are positive integers. More precisely, we will prove the following theorems.

THEOREM 1.1. *Let $f \in H_{k_1}$ and $g \in H_{k_2}$ be two distinct Hecke eigenforms. Set $i_1, j_1 \ge 1$ be positive integers. Then for any $\varepsilon > 0$ we have*

$$S_{2i_1,2j_1}(f,g;x) = xP_{A_{i_1}A_{j_1}-1}(\log x) + O_{f,g,\varepsilon}\Big(x^{1-\frac{42}{2^{2i_1+2j_1+1}\cdot 21-8A_{i_1}A_{j_1}+29}+\varepsilon}\Big),$$

where $P_j(t)$ is a polynomial of t with degree j, and A_j is defined by

$$A_j = \frac{(2j)!}{j!(j+1)!}.$$

Theorem 1.2. *Let $f \in H_{k_1}$ and $g \in H_{k_2}$ be two distinct Hecke eigenforms. Let $i_2, j_2 \geq 1$ be positive integers with at least one of them odd. Then, for any $\varepsilon > 0$, we have*

$$S_{i_2,j_2}(f,g;x) \ll_{f,g,\varepsilon} x^{1-\frac{1}{2^{i_2+j_2}}+\varepsilon}.$$

2. AUXILIARY RESULTS

In this section, we review some relevant facts about the automorphic L-functions and collect some important lemmas which play an important role in the proof of the main results in this paper.

Let $f \in H_{k_1}$ and $g \in H_{k_2}$ be two distinct Hecke eigenforms. We can define the jth symmetric power L-function attached to f by

$$L(\mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j} \left(1 - \frac{\alpha_f(p)^{j-m}\beta_f(p)^m}{p^s}\right)^{-1} \tag{7}$$

for $\Re(s) > 1$. We can rewrite it as a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) &= \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \ldots + \frac{\lambda_{\mathrm{sym}^{jk} f}(p)}{p^{ks}} + \ldots\right) \\ &:= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}, \quad \Re(s) > 1. \end{aligned} \tag{8}$$

It is well-known that $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. In particular, $L(\mathrm{sym}^0 f, s)$ and $L(\mathrm{sym}^1 f, s)$ correspond to the Riemann zeta-function $\zeta(s)$ and the Hecke L-function $L(f,s)$. And from (3), (7), (8) and the theory of Hecke operators, we have

$$\lambda_f(p^j) = \sum_{m=0}^{j} \alpha_f(p)^{j-2m} = \lambda_{\mathrm{sym}^j f}(p), \ j \geq 1. \tag{9}$$

The Rankin-Selberg L-function attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined by

$$\begin{aligned} &L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) \\ &:= \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} \left(1 - \alpha_f(p)^{i-2m}\alpha_g(p)^{j-2m'}p^{-s}\right)^{-1}, \quad \Re(s) > 1. \end{aligned} \tag{10}$$

Similarly, for $\Re(s) > 1$, we have

$$\begin{aligned} & L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) \\ (11) \qquad = & \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)}{n^s} = \prod_p \left(1 + \sum_{k\geq 1} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p^k)}{p^{ks}}\right), \end{aligned}$$

where $\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)$ is real and multiplicative. Similarly, by (9)-(11), for $i, j \geq 1$, we have

$$\begin{aligned} \lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p) &= \sum_{m=0}^{i} \sum_{m'=0}^{j} \alpha_f(p)^{i-2m} \alpha_g(p)^{j-2m'}(p) \\ (12) \qquad &= \lambda_{\mathrm{sym}^i f}(p) \lambda_{\mathrm{sym}^j g}(p). \end{aligned}$$

Let π_f be a automorphic cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$. It is well-known that an automorphic cuspidal representation π of $GL_2(\mathbb{A}_{\mathbb{Q}})$ is associated to a primitive form f, hence an automorphic function $L(\pi_f, s)$ which coincides with $L(f, s)$. Denote by $\mathrm{sym}^j \pi_f$ the jth symmetric power lift of π_f. For $2 \leq j \leq 8$, the automorphy of $\mathrm{sym}^j \pi_f$ were proved by a series of important works of Gelbart and Jacquet [8], Kim and Shahidi [14, 15, 16], Dieulefait [6], and Clozel and Thorne [2, 3, 4]. Very recently, Newton and Thorne [23, 24] showed that there exists a cuspidal automorphy representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ whose L-function equals $L(\mathrm{sym}^j f, s)$ for all $j \geq 1$. Hence for $j \geq 1$, the L-function $L(\mathrm{sym}^j f, s)$ can be extended to the whole complex plane as an entire function and satisfies a functional equation of degree $j + 1$. Furthermore, based on the works of Jacquet-Shalika [12, 13], Shahidi [30, 31], Rudnick-Sarnak [28], Lau-Wu [17], the L-function $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ can be analytically continued to the whole complex plane as an entire function and satisfies a certain functional equation of degree $(i + 1)(j + 1)$.

We state some basic definitions and analytic properties of general L-functions. Let $L(\phi, s)$ be a Dirichlet series (associated with the object ϕ) that admits an Euler product of degree $m \geq 1$, namely

$$L(\phi, s) = \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s} = \prod_{p<\infty} \prod_{j=1}^{m} \left(1 - \frac{\alpha_\phi(p, j)}{p^s}\right)^{-1},$$

where $\alpha_\phi(p, j), j = 1, 2, \cdots, m$ are the local parameters of $L(\phi, s)$ at a finite prime p. Suppose that this series and its Euler product are absolutely convergent for $\Re(s) > 1$. We denote the gamma factor by

$$L_\infty(\phi, s) = \prod_{j=1}^{m} \pi^{-\frac{s+\mu_\phi(j)}{2}} \Gamma\left(\frac{s + \mu_\phi(j)}{2}\right)$$

with local parameters $\mu_\phi(j), j = 1, 2, \cdots, m$ of $L(\phi, s)$ at ∞. The complete L-function $\Lambda(\phi, s)$ is defined by

$$\Lambda(\phi, s) = q(\phi)^{\frac{s}{2}} L_\infty(\phi, s) L(\phi, s),$$

where $q(\phi)$ is the conductor of $L(\phi, s)$. We assume that $\Lambda(\phi, s)$ admits an analytic continuation to the the whole complex plane $\mathbb{C}$ and is holomorphic everywhere except for possible poles of finite order at $s = 0, 1$. Furthermore, it satisfies a functional equation of the type

$$\Lambda(\phi, s) = \epsilon_\phi \Lambda(\tilde{\phi}, 1 - s)$$

where ϵ_ϕ is the root number with $|\epsilon_\phi| = 1$ and $\tilde{\phi}$ is dual of ϕ such that $\lambda_{\tilde{\phi}}(n) = \overline{\lambda_\phi(n)}, L_\infty(\tilde{\phi}, s) = L_\infty(\phi, s)$ and $q(\tilde{\phi}) = q(\phi)$. We call $\phi \in S_e^{\#}$ if it satisfies the above conditions. We say the L-function $L(\phi, s)$ satisfies the Ramanujan conjecture if $\lambda_\phi(n) \ll n^\varepsilon$ for any ε.

Here, we state a very general theorem due to Lau and Lü [19].

LEMMA 2.1 ([19]). *Let $L(f, s)$ is a product of two L-functions $L_1, L_2 \in S_e^{\#}$ with both $\deg L_i \geq 2, i = 1, 2$ and $L(f, s)$ satisfies the Ramanujan conjecture. Then for any $\varepsilon > 0$, we have*

$$\sum_{n \leq x} \lambda_f(n) = M(x) + O\big(x^{1 - \frac{2}{m} + \varepsilon}\big),$$

where $M(x) = \mathrm{Res}_{s=1}\{L(f, s) x^s / s\}$ and $m = \deg L$.

We define

$$r_2(n) = \#\{(n_1, n_2) \in \mathbb{Z}^2 \mid n = n_1^2 + n_2^2\}.$$

It is well-known that $r_2(n)$ is a multiplicative function and satisfies the relation

$$r_2(n) = 4 \sum_{\mathrm{d} \mid n} \chi_4(\mathrm{d}),$$

where χ_4 is the non-trivial Dirichlet character modulo 4. We denote $r(n) := \sum_{\mathrm{d}|n} \chi_4(\mathrm{d})$ and $\chi := \chi_4$. In particular, one has

$$r(p) = \sum_{\mathrm{d} \mid p} \chi(\mathrm{d}) = 1 + \chi(p).$$

It is not difficult to find that

$$\begin{aligned} S_{i,j}(f, g; x) &= \sum_{n \leq x} \lambda_f(n)^i \lambda_g(n)^j \sum_{n = a^2 + b^2} 1 \\ &= 4 \sum_{n \leq x} \lambda_f(n)^i \lambda_g(n)^j r(n), \end{aligned}$$

where $a, b \in \mathbb{Z}$ are integers.

Define

$$L_{i,j}(f,g;s) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)^i \lambda_g(n)^j r(n)}{n^s}$$

for $\Re(s) > 1$, where $i, j \geq 1$ are positive integers.

Now, we establish some lemmas concerning the decomposition of the L-function $L_{i,j}(f,g;s)$.

Lemma 2.2. *Let $f \in H_{k_1}$ and $g \in H_{k_2}$ be distinct Hecke eigenforms. Let $i_1, j_1 \geq 1$ be positive integers. Then, we have*

$$L_{2i_1,2j_1}(f,g;s) = \zeta(s)^{A_{i_1}A_{j_1}} H_{i_1,j_1}(f,g;s,\chi) U_{i_1,j_1}(f,g;s),$$

where the constant A_j is given by

$$A_j = \frac{(2j)!}{j!(j+1)!}, \tag{13}$$

here $H_{i_1,j_1}(f,g;s)$ is an L-function which can be represented as the product of some automorphic L-functions $L(sym^{l_1} f, s), L(sym^{r_1} g, s)$ and $L(sym^{l_2} f \times sym^{r_2} g, s)$ with $l_1, r_1, l_2, r_2 \geq 1$ and its twisted L-functions, and the function $U_{i_1,j_1}(f,g;s)$ for which the associated Dirichlet series converges uniformly and absolutely in the half-plane $\Re(s) \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

Proof. In view of (1) and the multiplicative property of $r(n)$, we know that $\lambda_f(n)^i \lambda_g(n)^j r(n)$ is multiplicative and satisfies the trivial bound $O(n^\varepsilon)$ for any $\varepsilon > 0$. Hence, we can write $L_{2i_1,2j_1}(f,g;s)$ as

$$L_{2i_1,2j_1}(f,g;s) = \prod_p \left(1 + \sum_{k\geq 1} \frac{\lambda_f(p^k)^{2i_1} \lambda_g(p^k)^{2j_1} r(p^k)}{p^{ks}}\right) \tag{14}$$

for $\Re(s) > 1$. In the half-plane $\Re(s) > \frac{1}{2}$, the corresponding coefficients of p^{-s} determine the analytic properties of $L_{2i_1,2j_1}(f,g;s)$.

By the result of Lau and Lü [19, Lemma 7.1], then we know that

$$\lambda_f(p)^{2j} = A_j + \sum_{1\leq r\leq j-1} C_j(r)\lambda_{\mathrm{sym}^{2r} f}(p) + \lambda_{\mathrm{sym}^{2j} f}(p),$$

where A_j is defined as in (13), and $C_j(r)$ are some suitable constants. By relation (12), we can rewrite (14) in the following

$$\begin{aligned} &L_{2i_1,2j_1}(f,g;s) \\ = \ &\zeta(s)^{A_{i_1}A_{j_1}} \prod_{l,r} L(\mathrm{sym}^l f, s)^{d_1} L(\mathrm{sym}^r g, s)^{e_1} L(\mathrm{sym}^l f \times \mathrm{sym}^r g, s)^{f_1} \end{aligned}$$

$$\times L(\mathrm{sym}^l f \times \chi, s)^{d_1} L(\mathrm{sym}^r g \times \chi, s)^{e_1} L(\mathrm{sym}^l f \times \mathrm{sym}^r g \times \chi, s)^{f_1}$$
$$\times U_{i_1,j_1}(f, g; s),$$

where $1 \leq l \leq 2i_1, l \leq r \leq 2j_1$ are some suitable constants, and d_1, e_1, f_1 are some constants which need not be specified in this occurrence, the function $U_{i_1,j_1}(f, g; s)$ is some Dirichlet series for which converges uniformly and absolutely for $\Re(s) > \frac{1}{2}$. This completes the proof of Lemma 2.2. □

LEMMA 2.3. *Let $f \in H_{k_1}$ and $g \in H_{k_2}$ be distinct Hecke eigenforms. Let $i_2, j_2 \geq 1$ be positive integers with at least one of them odd. Then, we have*

$$L_{i_2,j_2}(f, g; s) = H_{i_2,j_2}(f, g; s, \chi) U_{i_2,j_2}(f, g; s),$$

where $H_{i_2,j_2}(f, g; s)$ is an L-function which can be represented as the product of some automorphic L-functions $L(sym^{l'_1} f, s), L(sym^{r'_1} g, s)$ and $L(sym^{l'_2} f \times sym^{r'_2} g, s)$ with $l'_1, r'_1, l'_2, r'_2 \geq 1$ and its twisted L-functions, and the function $U_{i_2,j_2}(f, g; s)$ for which the associated Dirichlet series converges uniformly and absolutely in the half-plane $\Re(s) \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

Proof. The proof follows essentially the same argument as Lemma 2.2. □

LEMMA 2.4. *We have*

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \mathrm{d}t \ll T^{1+\varepsilon} \tag{15}$$

uniformly for $T \geq 2$. Furthermore,

$$\zeta(\sigma + it) \ll \left(1 + |t|\right)^{\max\{\frac{13}{42}(1-\sigma), 0\}+\varepsilon} \tag{16}$$

uniformly for $\frac{1}{2} + \epsilon \leq 2$ and $|t| \geq 1$.

Proof. The first result (15) is a classical result, and second one (16) is the new breakthrough of Bourgain [1]. □

From the above, we observe that $L(\mathrm{sym}^j f, s), L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ and its twisted L-functions for all $i, j \geq 1$ are general L-functions in the sense of Perelli [25]. For the general functions, we have the following averaged or individual convexity bounds.

Lemma 2.5. *Suppose that $\mathfrak{L}(s)$ is a general function of degree m. Then for any $\varepsilon > 0$, we have*

$$\int_1^T \left|\mathfrak{L}(\sigma+it)\right|^2 \mathrm{d}t \ll T^{m(1-\sigma)+\varepsilon}, \tag{17}$$

uniformly for $\frac{1}{2} \le \sigma \le 1$ and $T \ge 1$, and

$$\mathfrak{L}(\sigma+it) \ll \big(1+|t|\big)^{\max\{\frac{m}{2}(1-\sigma),0\}+\varepsilon} \tag{18}$$

uniformly for $\frac{1}{2} \le \sigma \le 1+\varepsilon$ and $|t| \ge 1$.

Proof. This follows the results of Perelli's mean value theorem and convexity bound for general L-function in [25]. $\square$

3. PROOF OF THEOREM 1.1

Recalling Lemma 2.2, and then applying Perron's formula (see [11, Proposition 5.54]) to the generating function $L_{2i_1,2j_1}(f,g;s)$, then we can obtain

$$\sum_{n\le x} \lambda_f(n)^{2i_1}\lambda_g(n)^{2j_1} r(n) = \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT} L_{2i_1,2j_1}(f,g;s)\frac{x^s}{s}\mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big),$$

where $\eta = 1+\varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later.

By shifting the line of integration to the parallel segment with $\Re(s) = \frac{1}{2}+\varepsilon$ and invoking Cauchy's residue theorem, then we have

$$\begin{aligned} &\sum_{n\le x} \lambda_f(n)^{2i_1}\lambda_g(n)^{2j_1} r(n) \\ =\ & \mathrm{Res}_{s=1}\Big\{L_{2i_1,2j_1}(f,g;s)\frac{x^s}{s}\Big\} \\ &+\frac{1}{2\pi i}\Big\{\int_{\kappa-iT}^{\kappa+iT} + \int_{\kappa+iT}^{\eta+iT} + \int_{\eta-iT}^{\kappa-iT}\Big\} L_{2i_1,2j_1}(f,g;s)\frac{x^s}{s}\mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) \\ (19){:=}\ & xP_{A_{i_1}A_{j_1}-1}(\log x) + J_1 + J_2 + J_3 + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \end{aligned}$$

where $\kappa := \frac{1}{2}+\varepsilon$ and $P_j(t)$ is a polynomial of t with degree j.

Now, we begin to handle the three terms J_1, J_2 and J_3. For J_1, using the Cauchy-Schwarz inequality, and Lemma 2.4 for $\zeta(s)$ along with (17), we have

$$\begin{aligned} J_1 \ll\ & x^{\kappa} \max_{1\le T_1\le T} T_1^{-1} \max_{T_1\le t\le 2T_1} \left|\zeta(\kappa+it)\right|^{A_{i_1}A_{j_1}-1} \\ & \times\Big(\int_{T_1}^{2T_1} \left|\zeta(\kappa+it)\right|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \Big(\int_{T_1}^{2T_1} \left|H_{i_1,j_1}(f,g;\kappa+it)\right|^2 \mathrm{d}t\Big)^{\frac{1}{2}} \end{aligned}$$

$$\ll \quad x^{\kappa} T^{\frac{13}{84}(A_{i_1}A_{j_i}-1)-1} T^{\frac{1}{2}} T^{\frac{1}{2}\times\frac{1}{2}\times(2^{2i_1+2j_1+1}-A_{i_1}A_{j_1})+\varepsilon}$$

$$\ll \quad x^{\kappa} T^{2^{2i_1+2j_1-1}-\frac{2}{21}A_{i_1}A_{j_1}-\frac{55}{84}+\varepsilon}. \tag{20}$$

The estimates for the integrals over the horizontal segments are similar. From (16) and (18), we have

$$\begin{aligned} J_2+J_3 &\ll \int_{\kappa}^{\eta} x^{\sigma}\left|\zeta(\sigma+it)^{A_{i_1}A_{j_1}} H_{i_i,j_1}(f,g;\sigma+it)\right| T^{-1}\mathrm{d}\sigma \\ &\ll \max_{\kappa\le\sigma\le\eta} x^{\sigma} T^{\{\frac{13}{42}A_{i_1}A_{j_1}+\frac{2^{2i_1+2j_1+1}-A_{1_1}A_{j_1}}{2}\}(1-\sigma)+\varepsilon} T^{-1} \\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{\kappa} T^{2^{2i_1+2j_1-1}-\frac{2}{21}A_{i_1}A_{j_1}-1+\varepsilon}. \end{aligned} \tag{21}$$

Therefore, from (19), (20) and (21), we have

$$\begin{aligned} \sum_{n\le x} \lambda_f(n)^{2i_1}\lambda_g(n)^{2j_1} r(n) &= xP_{A_{i_1}A_{j_1}-1}(\log x) \\ &+O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(x^{\frac{1}{2}+\varepsilon} T^{2^{2i_1+2j_1-1}-\frac{2}{21}A_{i_1}A_{j_1}-\frac{55}{84}+\varepsilon}\right). \end{aligned} \tag{22}$$

On taking $T = x^{\frac{42}{2^{2i_1+2j_1+1}\cdot 21-8A_{i_1}A_{j_1}+29}}$ in (22), we can get

$$\begin{aligned} &\sum_{n\le x} \lambda_f(n)^{2i_1}\lambda_g(n)^{2j_1} r(n) \\ &= xP_{A_{i_1}A_{j_1}-1}(\log x) + O\left(x^{1-\frac{42}{2^{2i_1+2j_1+1}\cdot 21-8A_{i_1}A_{j_1}+29}+\varepsilon}\right), \end{aligned}$$

which completes the proof of Theorem 1.1.

4. **PROOF OF THEOREM** 1.2

As we know, $L_{i_2,j_2}(f,g;s)$ is a general L-function of degree $2^{i_2+j_2+1}$ in the sense of Lemma 2.1 by noting Lemma 2.3. From Lemma 2.3 and the assumption on i_2, j_2, we can derive that $L_{i_2,j_2}(f,g;s)$ can be analytically continued to the half-plane $\Re(s) > \frac{1}{2}$ without any poles, thus, by Lemma 2.1, we can obtain

$$\sum_{n\le x} \lambda_f(n)^{i_2}\lambda_g(n)^{j_2} r(n) \ll_{f,g,\varepsilon} x^{1-\frac{1}{2^{i_2+j_2}}+\varepsilon}.$$

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