LIMITING DIRECTIONS FOR ENTIRE SOLUTIONS OF A CERTAIN CLASS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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Let us consider f as being an entire solution of the differential-difference equation $G(z, f) + h(z)f^m(z) = 0, (m \in \mathbb{N})$, where $h(z)$ is a transcendental entire function and $G(z, f)$ is a differential-difference polynomial in f with entire coefficients. By considering the order and deficiency of $h(z)$ and such coefficients, we mainly study the radial distribution of f, and establish a lower bound of measure for the set of common limiting directions of the Julia sets of derivatives and primitives of its shifts.

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1. INTRODUCTION AND MAIN RESULTS

A function $f(z)$ is called meromorphic if it is nonconstant and analytic in the complex plane except at possible isolated poles. A family $\mathcal{F}$ of meromorphic functions on G is said to be normal, in the sense of Montel, if every sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of G to a function f which is meromorphic or identically ∞, the convergence being with respect to the spherical metric $d\sigma = 2|dw|/(1 + |w|^2)$. The family $\mathcal{F}$ is said to be normal at a point $z_0 \in G$, if there exists a neighborhood of z_0 in which $\mathcal{F}$ is normal. Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a transcendental meromorphic function in the complex plane. For $n \in \mathbb{N}$, we define the n-th iterate of f as follows:

$$\widetilde{f}^0(z) = z, \cdots, \widetilde{f}^n(z) = f \circ \widetilde{f}^{n-1}(z).$$

Let $\mathcal{F}(f)$ be the set of all points z such that the family $\{\widetilde{f}^n(z)\}_{n=1}^{\infty}$ is normal at z. We also say $\mathcal{F}(f)$ is the Fatou set of $f(z)$, and its complement

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$\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$ is called the Julia set of $f(z)$. It is well-known that $\mathcal{F}(f)$ is open and completely invariant under f, and $\mathcal{J}(f)$ is closed and non-empty. Some basic knowledge of complex dynamics of meromorphic functions and related results can be found in [2]–[4],[14]–[16],[21, 22].

Let α, β be two numbers such that $0 \leq \alpha < \beta < 2\pi$, and we set

$$\Omega(\alpha,\beta) = \{z \in \mathbb{C} | \alpha < \arg z < \beta\}, \quad \Omega(r;\alpha,\beta) = \Omega(\alpha,\beta) \cap \{z \in \mathbb{C} : |z| > r\}.$$

Assume that $f(z)$ is a transcendental meromorphic function in $\mathbb{C}$. The ray $\arg z = \theta$ from the origin is said to be a limiting direction of the Julia set of $f(z)$ if $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap \mathcal{J}(f)$ is unbounded for any $\varepsilon > 0$. Then, the set of all limiting directions of the Julia set of $f(z)$ is denoted by

$$\Theta(f) = \{\theta \in [0, 2\pi) | \text{the ray } \arg z = \theta \text{ is a limiting direction of } \mathcal{J}(f)\},$$

and set

$$E(f) = \bigcap_{n \in \mathbb{Z}} \Theta(f^{(n)}),$$

where $f^{(n)}(z)$ denotes the n-th derivative of $f(z)$ for $n \geq 0$ or the n-th integral primitive of $f(z)$ for $n < 0$. Clearly, $\Theta(f)$ and $E(f)$ are closed and measurable. For brevity, we call a limiting direction of the Julia set of f a limiting direction of f in this paper, and denote by mes$\Theta(f)$ and mes$E(f)$ for their linear measure, respectively. The following example is intended to help readers understand the definition of $\Theta(f)$ intuitively.

Example 1.1 ([5]). Consider the map $f_\lambda(z) = \lambda e^z, \lambda > 0$. If $\lambda > \frac{1}{e}$, then $\mathcal{J}(f_\lambda)$ is the whole complex plane, hence $\Theta(f_\lambda) = [0, 2\pi)$. If $0 < \lambda \leq \frac{1}{e}$, then f_λ has a repelling fixed point $x > 1$, $\mathcal{J}(f_\lambda)$ is a Cantor set of curves in $\{z : Re z > x\}$, and $\Theta(f_\lambda) = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi)$.

Value distribution theory plays an important role in studying transcendental meromorphic functions. Some standard notations and basic results can be found in [8, 9, 12, 13, 20, 23]. Given a meromorphic function $f(z)$ in $\mathbb{C}$, we define the following three functions: the *proximity function*

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \mathrm{d}\theta,$$

where $\log^+ x := \max\{0, \log x\}$; the *integrated counting function*

$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t} \mathrm{d}t + n(0, f) \log r,$$

where $n(t, f)$ denotes the number of poles of $f(z)$ in $\{|z| \leq t\}$ counting multiplicities; the *characteristic function* $T(r, f) := m(r, f) + N(r, f)$.

We further denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside a possibly exceptional set of finite linear measure. In addition, the meromorphic function $\alpha(z)$ is said to be a small function of $f(z)$ if $T(r,\alpha)=S(r,f)$. The order $\rho(f)$ and the lower order $\mu(f)$ of $f(z)$ are, respectively, defined by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+T(r,f)}{\log r},\quad \mu(f)=\liminf_{r\to\infty}\frac{\log^+T(r,f)}{\log r}.$$

For entire functions, $T(r,f)$ in the above two definitions can be replaced by $\log^+M(r,f)$, which is based on a relationship between the maximum modulus $M(r,f)=\max_{|z|=r}|f(z)|$ and $T(r,f)$. We denote by $\lambda(f)$ and $\delta(a,f)$, respectively, the convergence exponent of zero sequence and the deficiency of $a(\in\mathbb{C})$ for f. They are defined as follows:

$$\lambda(f)=\limsup_{r\to\infty}\frac{\log^+N(r,1/f)}{\log r},\quad \delta(a,f)=1-\limsup_{r\to\infty}\frac{N(r,1/(f-a))}{T(r,f)}.$$

We say $a\in\mathbb{C}$ is a Borel exceptional value of $f(z)$ if $\lambda(f-a)<\rho(f)$, and a is a Nevanlinna exceptional value or deficient value of $f(z)$ if $\delta(a,f)>0$.

As a very active subject, the research on limiting directions of Julia sets of entire solutions for complex differential or difference equations has generated a lot of interest, and many interesting results have been established (see, e.g., [5, 10, 11, 17, 18]). In 2012, Huang and Wang [10] investigated the limiting directions of the product of linearly independent solutions of differential equation

$$f^{(n)}+A(z)f=0, n\in\mathbb{N}, n\geq 2, \tag{1}$$

where $A(z)$ is a transcendental entire function with finite order, and obtained the following result.

Theorem A ([10]). *Suppose that* $\{f_1,f_2,\cdots,f_n\}$ *is a solution base of* (1), *then* $\operatorname{mes}\Theta(f_1f_2\cdots f_n)\geq\min\{2\pi,\pi/\rho(A)\}$.

After that, many authors made a further study for the solutions of higher order linear differential equation

$$f^{(n)}+\sum_{i=0}^{n-1}A_i(z)f^{(i)}=0, \tag{2}$$

where $A_0(z)$ is a transcendental entire function and $A_i(z)$ are entire functions satisfying $T(r,A_i)=o(T(r,A_0))(i=1,2,\cdots,n-1)$ as $r\to\infty$. For instance, Huang and Wang [11] prove that every non-trivial solution f of (2) satisfies $\operatorname{mes}\Theta(f)\geq\min\{2\pi,\pi/\mu(A_0)\}$. Wang and Chen [17] confirmed that

$$\operatorname{mes}E(f)\geq\min\{2\pi,\pi/\mu(A_0)\},$$

and there exists a closed interval $I \subseteq E(f)$ such that mes$I \geq \min\{2\pi, \pi/\rho(A_0)\}$ provided that $\rho(A_i) < \rho(A_0)(i = 1, 2, \cdots, n-1)$.

Recently, Wang et al. [18] considered the following differential equation

$$P(z, f) + h(z) f^m(z) = 0, \tag{3}$$

where $P(z, f) = \sum_{j=1}^{s} a_j f^{n_{0j}} (f')^{n_{1j}} \cdots (f^{(k)})^{n_{kj}}$, $a_j(z)$ are some meromorphic functions and s, m, n_{ij} are non-negative integers satisfying $\gamma = \min\limits_{1 \leq j \leq s} \sum_{i=0}^{k} n_{ij} \geq m$ for $0 \leq i \leq k, 1 \leq j \leq s$. We restate it as follows:

Theorem B ([18]). *Suppose that m, n are integers, $h(z)$ is a transcendental entire function of finite lower order, and that $P(z, f)$ is a differential polynomial in f with $\gamma \geq m$, where all coefficients $a_j(z)(j = 1, \cdots, s)$ are polynomials if $\mu(h) = 0$, or all $a_j(z)(j = 1, \cdots, s)$ are entire functions satisfying $\rho(a_j) < \mu(h)$. Then, for every nonzero transcendental entire solution f of* (3), *one has* mes$\Theta(f^{(n)}) \geq \min\{2\pi, \pi/\mu(h)\}$.

Similar to the differential equation, the case of the difference equation is also deeply studied (see, e.g., [5]). For $j \in \{1, 2, \cdots, n\}$, we set $P_j(z, f) = \sum_{\lambda = (k_1, \cdots, k_m) \in \Lambda_j} a_\lambda \prod_{i=1}^{m} f^{k_i}(z + c_i)$, where Λ_j denotes a finite subset of $\mathbb{N}^m$ and $c_i (1 \leq i \leq m)$ are distinct complex numbers. Chen et al. [5] considered the following equation

$$\sum_{j=1}^{n} A_j(z) P_j(z, f) = A_0(z), \tag{4}$$

where $A_j(z)(j = 0, 1, \cdots, n)$ are $n+1$ entire functions and A_0 is transcendental with finite lower order so that $T(r, A_j) = o(T(r, A_0))(j = 1, \cdots, n)$ as $r \to \infty$. Set $\Theta_1(f) = \bigcap\limits_{i \in \mathbb{M}} \Theta(f(z + \eta_i))$, where $\mathbb{M}$ denotes a countable subset of $\mathbb{N}^+$ and $\eta_i (i \in \mathbb{M})$ are some distinct complex numbers. Indeed, Chen et al. [5] obtained the following results.

Theorem C ([5]). *For any non-trivial entire solution f of* (4), *we obtain*

$$\mathrm{mes}\Theta_1(f) \geq \min\{2\pi, \pi/\mu(A_0)\}.$$

Moreover, if $\rho(A_0) > \max\{\rho(A_1), \cdots, \rho(A_n)\}$, then there exists a closed interval $[\theta_1, \theta_2]$ of $\Theta_1(f)$ such that $\theta_2 - \theta_1 \geq \min\{2\pi, \pi/\rho(A_0)\}$.

Noting that, in Theorem C, $A_j(z)(0 \leq j \leq n)$ are assumed $n + 1$ entire functions and $A_0(z)$ is a dominant term. For the special case of $n = 2$ in (4), they also proved that for every non-trivial entire solution f of (4), if A_0 is transcendental satisfying $T(r, A_0) \sim \kappa \log M(r, A_0)(0 < \kappa \leq 1)$ as $r \to \infty$

outside a set of logarithmic density zero, A_1 has a finite deficient value a and $\mu(A_1) < \infty$, then

$$\mathrm{mes}\Theta_1(f) \geq \max\{0, \sigma\},$$

where $\sigma = \min\left\{2\pi\kappa, \frac{4}{\mu(A_1)} \arcsin\sqrt{\frac{\delta(a,A_1)}{2}} - 2\pi(1-\kappa)\right\}$.

Motivated by these works, we consider entire solutions f of a class of more general differential-difference equation

$$G(z,f) + h(z)f^m(z) = 0, (m \in \mathbb{N}), \tag{5}$$

where

$$G(z,f) = \sum_{j=1}^{s} \alpha_j(z) \prod_{i=0}^{k} (f^{(i)}(z+c_i))^{n_{ij}}, (s \in \mathbb{N}^+),$$

$\alpha_j(z)$ are entire functions, c_i are some finite complex numbers and n_{ij} are non-negative integers such that $\gamma = \min\limits_{1\leq j\leq s} \sum_{i=0}^{k} n_{ij} \geq m$ for $0 \leq i \leq k, 1 \leq j \leq s$. Assume that

$$\rho(\alpha_v) = \max_{1\leq j\leq s} \rho(\alpha_j)$$

and $\rho(\alpha_j) < \rho(h), j \neq v$. Under some conditions, we shall prove that the Julia sets of f, its n-th derivatives and its n-th integral primitives of shifts have a large amount of common limiting directions. We denote

$$L(f) = \bigcap_{n\in\mathbb{Z}} \Theta(f^{(n)}(z+\phi)),$$

where ϕ is a finite complex number and $f^{(n)}(z+\phi)$ represents the shift of n-th derivative of $f(z)$ for $n \geq 0$ or the shift of n-th integral primitive of $f(z)$ for $n < 0$. Although $L(f)$ can be degenerate into $E(f)$ when $\phi = 0$, $L(f)$ and $E(f)$ may be quite different for some ϕ. For instance, let $f(z) = e^z$, it follows from Example 1.1 that $E(f) = [0, 2\pi)$ and $L(f) = [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi)$ when $\phi \in (-\infty, -1]$. Now, we show our results as follows.

Theorem 1.2. *Assume $\alpha_v(z)$ is an entire function of finite order and has a finite Borel exceptional value. Let $h(z)$ be an entire function of finite order and $\varepsilon_0 \in (0, \frac{\pi}{4\rho(\alpha_v)})$ such that $\frac{\pi}{\rho(h)} > \frac{\pi}{\rho(\alpha_v)} + 2\varepsilon_0$. For any transcendental entire solution f of* (5)*, we obtain*

$$\mathrm{mes}L(f) \geq \min\left\{\frac{\pi}{\rho(\alpha_v)} - 2\varepsilon_0, \frac{\pi}{\rho(h)} - \frac{\pi}{\rho(\alpha_v)} - 2\varepsilon_0\right\}.$$

An application of Theorem 1.2 is given below.

Example 1.3. The differential-difference equation

$$-2f'(z-1) + e^{z^2+2z-1}(f^2(z))' + 4e^{2z-1}f(z) = 0$$

has an entire solution $f(z)=e^{-z^2}$. Since $\alpha_v(z)=e^{z^2+2z-1}$ and $h(z)=4e^{2z-1}$ satisfy all conditions given in Theorem 1.2, then mes$L(f)\geq \frac{\pi}{2}-2\varepsilon_0$.

Noting that, in Theorem 1.2, one needs $\frac{\pi}{\rho(h)}>\frac{\pi}{\rho(\alpha_v)}+2\varepsilon_0$ which implies $\rho(\alpha_v)>\rho(h)$, the following results consider the other cases without requiring $\rho(\alpha_v)>\rho(h)$.

THEOREM 1.4. *Suppose that $\alpha_v(z)$ has a finite deficient value a and $\mu(\alpha_v)<\infty$. Let $h(z)$ be a transcendental entire function and $\omega\in(0,1]\cap(1-\frac{2}{\pi\mu(\alpha_v)}\arcsin\sqrt{\frac{\delta(a,\alpha_v)}{2}},1]$ such that $T(r,h)\sim\omega\log M(r,h)$ as $r\to\infty$ outside a set of finite logarithmic measure. For any transcendental entire solution f of (5), we have*

$$\mathrm{mes}L(f)\geq\min\Big\{2\pi\omega,\frac{4}{\mu(\alpha_v)}\arcsin\sqrt{\frac{\delta(a,\alpha_v)}{2}}-2\pi(1-\omega)\Big\}=\sigma.$$

As an application of Theorem 1.4, one gives the following example.

Example 1.5. The differential-difference equation

$$ie^{iz}f'(z+\frac{3\pi}{2})-if^2(z+\pi)+\sin zf(z)=0$$

has an entire solution $\cos z$. Here, $\alpha_v(z)=ie^{iz}$ satisfies $\delta(0,\alpha_v)=1$ and $\mu(\alpha_v)=1$, $T(r,\sin z)\sim\frac{2}{\pi}\log M(r,\sin z)$ with $\omega=\frac{2}{\pi}\in(\frac{1}{2},1]$. By Theorem 1.4, we have mes$L(f)\geq 4-\pi$.

THEOREM 1.6. *Let $h(z)$ be a transcendental entire function of finite lower order and $\alpha_j(1\leq j\leq s)$ be some small functions with respect to h. For any non-trivial entire solution f of (5), we obtain*

$$\mathrm{mes}L(f)\geq\min\Big\{2\pi,\frac{\pi}{\mu(h)}\Big\}.$$

Moreover, if $\rho(h)>\rho(\alpha_j)$ for all $1\leq j\leq s$, there exists a closed interval $[\theta_1,\theta_2]\subseteq L(f)$ with $\theta_2-\theta_1\geq\min\{2\pi,\pi/\rho(h)\}$.

We also give an application of Theorem 1.6 as follows.

Example 1.7. The differential-difference equation

$$f^2(z)f'(z+1)+f^{(k)}(z+2\pi i)-(e^{2z+1}+1)f(z)=0$$

has an entire solution e^z. We note that $h(z)=-e^{2z+1}-1$. It follows from Theorem 1.6 that mes$L(f)\geq\pi$.

The remainder of this paper is organized as follows. In Section 2, some basic notations and auxiliary lemmas in value distribution of Nevanlinna theory are introduced, which are needed for the later proofs. The details for the proofs of our results are showed in Section 3.

2. AUXILIARY LEMMAS

In the following, we first recall the Nevanlinna characteristic for an angle (see [8, 22, 23]). We denote the closure of $\Omega(\alpha,\beta)$ by $\overline{\Omega}(\alpha,\beta)$, where $(\beta-\alpha)\in(0,2\pi]$. Suppose that $g(z)$ is meromorphic in $\overline{\Omega}(\alpha,\beta)$. Define

$$A_{\alpha,\beta}(r,g)=\frac{\omega}{\pi}\int_1^r\Big(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\Big)\{\log^+|g(te^{i\alpha})|+\log^+|g(te^{i\beta})|\}\frac{\mathrm{d}t}{t},$$
$$B_{\alpha,\beta}(r,g)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+|g(re^{i\theta})|\sin\omega(\theta-\alpha)\mathrm{d}\theta,$$
$$C_{\alpha,\beta}(r,g)=2\sum_{1<|b_v|<r}\Big(\frac{1}{|b_v|^\omega}-\frac{|b_v|^\omega}{r^{2\omega}}\Big)\sin\omega(\beta_v-\alpha),$$

where $\omega=\pi/(\beta-\alpha)$ and $b_v=|b_v|e^{i\beta_v}$ denote the poles of $g(z)$ in $\overline{\Omega}(\alpha,\beta)$, each pole occuring with its multiplicity. The Nevanlinna's angular characteristic of $g(z)$ is denoted by $S_{\alpha,\beta}(r,g)$, which is

$$S_{\alpha,\beta}(r,g)=A_{\alpha,\beta}(r,g)+B_{\alpha,\beta}(r,g)+C_{\alpha,\beta}(r,g),$$

and the order of $S_{\alpha,\beta}(r,g)$ is defined as

$$\rho_{\alpha,\beta}(g)=\limsup_{r\to\infty}\frac{\log^+S_{\alpha,\beta}(r,g)}{\log r}.$$

We need some auxiliary lemmas as follows.

Lemma 2.1 ([3]). *If f is a transcendental entire function, then $\mathcal{F}(f)$ has no unbounded multi-connected component.*

Lemma 2.2 ([21, Lemma 2.2]). *Let $f(z)$ be analytic in $\Omega(r_0;\theta_1,\theta_2)$. Suppose that U is a hyperbolic domain and $f(z):\Omega(r_0;\theta_1,\theta_2)\to U$. If there exists a point $a\in\partial U\setminus\{\infty\}$ satisfying $C_U(a)>0$, then there exists a positive constant d such that*

$$|f(z)|=O(|z|^d),z\to\infty,z\in\Omega(r_0;\theta_1+\varepsilon,\theta_2-\varepsilon)$$

for sufficiently small $\varepsilon>0$.

An open set is called hyperbolic if it has at least three boundary points in $\mathbb{C}\cup\{\infty\}$. Let U be a hyperbolic open set in $\mathbb{C}$. For any $a\in\mathbb{C}\setminus U$, we set

$$C_U(a)=\inf\{\lambda_U(z)|z-a|:\forall z\in U\},$$

where $\lambda_U(z)$ is the hyperbolic density on U. It is known that if every component of U is simply connected, then $C_U(a)\geq 1/2$.

LEMMA 2.3 ([23, Theorem 2.5.1]). *Let $f(z)$ be a meromorphic function on $\Omega(\alpha-\varepsilon,\beta+\varepsilon)$ for $\varepsilon>0$ and $0<\alpha<\beta<2\pi$. Then*

$$A_{\alpha,\beta}\Big(r,\frac{f'}{f}\Big)+B_{\alpha,\beta}\Big(r,\frac{f'}{f}\Big)\le K(\log^+ S_{\alpha-\varepsilon,\beta+\varepsilon}(r,f)+\log r+1)$$

for $r>1$ possibly except a set with finite linear measure, and we also have the constant $K>0$.

LEMMA 2.4 ([11, Lemma 2.2]). *Let $z=re^{i\varsigma}, r>r_0+1$ and $\alpha\le\varsigma\le\beta$, where $0\le\alpha<\beta\le2\pi$, $0<\beta-\alpha\le2\pi$. Suppose that $g(z)$ is analytic in $\overline{\Omega}(r;\alpha,\beta)$ with $\rho_{\alpha,\beta}(g)<\infty$. Choose two real numbers α_1 and β_1 satisfying $\alpha<\alpha_1<\beta_1<\beta$. Then, for every $\varepsilon_j\in(0,\frac{\beta_j-\alpha_j}{2})(j=1,2,\cdots,n-1)$ outside a set of linear measure zero, where $n\ge2$ is an integer, and*

$$\alpha_j=\alpha+\sum_{s=1}^{j-1}\varepsilon_s,\quad \beta_j=\beta-\sum_{s=1}^{j-1}\varepsilon_s\quad (j=2,3,\cdots,n-1),$$

there exist $K>0$ and $M>0$ only depending on $g(z)$, $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{n-1}$ and $\Omega(\alpha_{n-1},\beta_{n-1})$ not depending on z such that

$$\Big|\frac{g'(z)}{g(z)}\Big|\le Kr^M\left(\sin k(\varsigma-\alpha)\right)^{-2}$$

and

$$\Big|\frac{g^{(n)}(z)}{g(z)}\Big|\le Kr^M\Big(\sin k(\varsigma-\alpha)\prod_{j=1}^{n-1}\sin k_j(\varsigma-\alpha_j)\Big)^{-2}$$

for all $z\in\Omega(\alpha_{n-1},\beta_{n-1})$ outside an R-set, where we have $k=\pi/(\beta-\alpha)$ and $k_j=\pi/(\beta_j-\alpha_j)$ with $j=1,2,\cdots,n-1$.

Remark 2.5. An R-set in $\mathbb{C}$ is a countable union of discs whose radii have finite sum (see [12]). Obviously, the union of two R-set is again an R-set. The set of angles θ for which the ray $re^{i\theta}$ meets infinitely many discs of a given R-set has linear measure zero.

LEMMA 2.6 ([13]). *Let $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$, where $n\in\mathbb{N}^+$ and $a_n=b_ne^{i\theta_n}, b_n>0,\theta_n\in[0,2\pi)$. For any given $\varepsilon\in(0,\pi/4n)$, we introduce $2n$ open angles*

$$S_j=\Big\{z\in\mathbb{C}:-\frac{\theta_n}{n}+(2j-1)\frac{\pi}{2n}+\varepsilon<\arg z<-\frac{\theta_n}{n}+(2j+1)\frac{\pi}{2n}-\varepsilon\Big\},$$

where $j=0,1,\cdots,2n-1$. Then there exists a positive number $R=R(\varepsilon)$ such that for $|z|=r>R$,

$$Re\{P(z)\}>b_n(1-\varepsilon)\sin(n\varepsilon)r^n,$$

if $z \in S_j$ *when* j *is even; while*

$$Re\{P(z)\} < -b_n(1-\varepsilon)\sin(n\varepsilon)r^n,$$

if $z \in S_j$ *when* j *is odd.*

LEMMA 2.7 ([19]). *If* $g(z)$ *is an entire function with finite positive order, then there exists an angular domain* $\Omega(\alpha, \beta)$ *with* $\beta - \alpha \geq \pi/\rho(g)$ *such that*

$$\limsup_{r\to\infty} \frac{\log^+ \log^+ |g(re^{i\theta})|}{\log r} = \rho(g)$$

for any $\theta \in (\alpha, \beta)$.

LEMMA 2.8 ([1, 7]). *Let* $f(z)$ *be a transcendental meromorphic function of finite lower order* μ, *and* f *has a deficient value* a. *Let* $\Lambda(r)$ *be a positive function with* $\Lambda(r) = o(T(r, f))$ *as* $r \to \infty$. *Then, for any fixed sequence of Pólya peaks* $\{r_n\}$ *of order* μ, *we have*

$$\liminf_{r_n\to\infty} \mathrm{mes} D_\Lambda(r_n, a) \geq \min\Big\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(a, f)}{2}}\Big\},$$

where

$$D_\Lambda(r, \infty) = \big\{\theta \in [0, 2\pi) : |f(re^{i\theta})| > e^{\Lambda(r)}\big\}$$

and

$$D_\Lambda(r, a) = \big\{\theta \in [0, 2\pi) : |f(re^{i\theta}) - a| < e^{-\Lambda(r)}\big\}, a \in \mathbb{C}.$$

Remark 2.9 ([23]). The definition of Pólya peaks for $T(r, f)$ was first posed by Edrei (see [6]). A sequence of increasing and unbound positive number $\{r_n\}$ is called a sequence of Pólya peaks $\{r_n\}$ of order σ for $T(r, f)$, for sufficiently small $\varepsilon_n, \varepsilon_n'$, $K > 0$ and $t \in [r_n', r_n'']$, satisfies

$$(1) r_n' \to \infty, r_n/r_n' \to \infty, r_n''/r_n \to \infty,$$
$$(2) \liminf_{n\to\infty} \log T(r_n, f)/ \log r_n \geq \sigma,$$
$$(3) T(t, f) < (1 + \varepsilon_n)(t/r_n)^\sigma T(r_n, f),$$
$$(4) T(r, f)/t^{\sigma-\varepsilon_n'} \leq KT(r_n, f)/(r_n)^{\sigma-\varepsilon_n'},$$

outside a finite logarithmic measure.

LEMMA 2.10. *Suppose that* $h(z)$ *and* $\alpha_v(z)$ *are entire functions of finite positive order. Take* $\varepsilon_0 \in (0, \frac{\pi}{4\rho(\alpha_v)})$ *such that* $\frac{\pi}{\rho(h)} > \frac{\pi}{\rho(\alpha_v)} + 2\varepsilon_0$. *If* $\alpha_v(z)$ *has a finite Borel exceptional value* a, *there exists a infinite sequence* $\{r_n\}_{n=1}^{\infty}$ *such that*

$$\mathrm{mes} E_0 \geq \min\Big\{\frac{\pi}{\rho(\alpha_v)} - 2\varepsilon_0, \frac{\pi}{\rho(h)} - \frac{\pi}{\rho(\alpha_v)} - 2\varepsilon_0\Big\},$$

where

$$E_0 = \left\{\theta \in [0, 2\pi) \;\middle|\; \begin{array}{r} |\alpha_v(r_n e^{i\theta}) - a| < \exp(-K r_n^{\rho(\alpha_v)}) \\ |h(r_n e^{i\theta})| \geq \exp(r_n^{\rho(h)-\eta}) \end{array} \right\} \tag{6}$$

for any $\eta > 0$ *and a positive constant* K.

Proof. By the decomposition theorem, we deduce

$$\alpha_v(z) = \beta(z)e^{P(z)} + a,$$

where $\beta(z)$ is an entire function with $\rho(\beta) < \rho(\alpha_v)$ and $P(z)$ is a polynomial of degree $\rho = \rho(\alpha_v)$. We may further assume that

$$P(z) = b_\rho z^\rho + b_{\rho-1}z^{\rho-1} + \cdots + b_0, \quad \rho \in \mathbb{N}^+.$$

Take $\varepsilon_0 \in (0, \frac{\pi}{4\rho})$, it follows from Lemma 2.6 that there exist 2ρ open angles S_j such that

$$Re\{P(z)\} < -K_0 r^\rho, \ z \in \cup_{j=1}^{\rho} S_{2j-1}, \tag{7}$$

where K_0 is a positive number. For $0 \le j \le 2\rho - 1$, we set $L_j = \{\arg z | z \in S_j\}$. By Lemma 2.6, one further knows

$$\text{mes}L_0 = \cdots = \text{mes}L_{2\rho-1} = \frac{\pi}{\rho} - 2\varepsilon_0, \tag{8}$$

and

$$\text{Dist}(L_j, L_{j+2}) = \frac{\pi}{\rho} + 2\varepsilon_0. \tag{9}$$

For $0 \le j_1 \ne j_2 \le 2\rho - 1$,

$$L_{j_1} \cap L_{j_2} = \emptyset. \tag{10}$$

Since $\rho(\beta) < \rho$, there exists a $\varepsilon_1 \in (0, \frac{\rho-\rho(\beta)}{2})$ such that for sufficiently large r,

$$|\beta(z)| \le M(r,\beta) < \exp(r^{\rho(\beta)+\varepsilon_1}).$$

Furthermore, one deduces from (7) that

$$\begin{aligned} |\alpha_v(z) - a| &= |\beta(z)| \exp(Re\{P(z)\}) \\ &\le \exp\left(r^{\rho(\beta)+\varepsilon_1} - K_0 r^\rho\right) < \exp\left(-K r^\rho\right) \end{aligned}$$

for $z \in \bigcup_{j=1}^{\rho} S_{2j-1}$, where K is a positive number.

On the other hand, Lemma 2.7 gives that for any $\eta > 0$, there exist an angular domain $\Omega(\theta_1, \theta_2)$ with $\theta_2 - \theta_1 \ge \pi/\rho(h)$ and a sequence $\{r_n\}_{n=1}^\infty$ with $r_n \to \infty$ such that

$$|h(r_n e^{i\theta})| \ge \exp(r_n^{\rho(h)-\eta})$$

for any $\theta \in (\theta_1, \theta_2)$.

Let E_0 be defined as in (6), one can deduce that E_0 is non-empty and

$$\text{mes}E_0 \ge \min\left\{\frac{\pi}{\rho(\alpha_v)} - 2\varepsilon_0, \frac{\pi}{\rho(h)} - \frac{\pi}{\rho(\alpha_v)} - 2\varepsilon_0\right\}$$

from (8), (9), (10) and the discussion above. This completes the proof. □

3. PROOFS OF MAIN THEOREMS

The proof of Theorem 1.2. Let

$$\tau = \min\left\{\frac{\pi}{\rho(\alpha_v)} - 2\varepsilon_0, \frac{\pi}{\rho(h)} - \frac{\pi}{\rho(\alpha_v)} - 2\varepsilon_0\right\}.$$

We proceed by contradiction. Suppose that $\mathrm{mes}L(f) < \tau$. As we know, $L(f)$ is a non-empty closed set in $[0, 2\pi)$ and $\Phi = (0, 2\pi) \setminus L(f)$ is open. So, Φ can be covered by at most countably many open intervals and we further choose finitely m open intervals $I_l = (a_l, b_l), (l = 1, 2, ..., m)$ in Φ such that

$$\mathrm{mes}(\Phi \setminus \bigcup_{l=1}^{m} I_l) < \frac{\tau - \mathrm{mes}L(f)}{4}.$$

For any $\theta \in I_l$, one knows $\theta \notin L(f)$, which implies that there exist $\zeta_\theta, n_\theta \in \mathbb{Z}$, only depending on θ, such that $(\theta - \zeta_\theta, \theta + \zeta_\theta) \subset I_l$ and for sufficiently large r,

$$\Omega(r; \theta - \zeta_\theta, \theta + \zeta_\theta) \cap \mathcal{J}(f^{(n_\theta)}(z + \phi)) = \emptyset.$$

In other words, there exist a corresponding r_θ and an unbounded Fatou component U_θ of $\mathcal{F}(f^{(n_\theta)}(z + \phi))$ such that $\Omega(r_\theta; \theta - \zeta_\theta, \theta + \zeta_\theta) \subset U_\theta$. Next, by Lemma 2.1, $\mathcal{F}(f^{(n_\theta)}(z + \phi))$ has no unbounded multi-connected component. So, we can take an unbounded and connected closed section $\Gamma_\theta \subset \partial U_\theta$ such that $\mathbb{C} \setminus \Gamma_\theta$ is simply connected. By the definition of hyperbolic domains, one knows $\mathbb{C} \setminus \Gamma_\theta$ is hyperbolic and open. Take $a \in \Gamma_\theta \setminus \{\infty\}$, $C_{\mathbb{C}\setminus\Gamma_\theta}(a) \geq \frac{1}{2}$. Based on the fact that the mapping

$$f^{(n_\theta)}(z + \phi) : \Omega(r_\theta; \theta - \zeta_\theta, \theta + \zeta_\theta) \to \mathbb{C} \setminus \Gamma_\theta$$

is analytic, by Lemma 2.2 there exists a positive constant d_θ such that for sufficiently small $\varepsilon_\theta > 0$,

$$\left|f^{(n_\theta)}(z + \phi)\right| = O(|z|^{d_\theta}) \quad \text{as} \quad |z| \to \infty, \tag{11}$$

where $z \in \Omega(r_\theta; \theta - \zeta_\theta + \varepsilon_\theta, \theta + \zeta_\theta - \varepsilon_\theta)$.

Let's divide it into two cases:

- If $n_\theta > 0$, then

$$\left|f^{(n_\theta - 1)}(z + \phi)\right| \leq \int_0^z \left|f^{(n_\theta)}(\xi + \phi)\right| |d\xi| + O(1) = O(|z|^{d_\theta + 1}),$$

  where the integral path is the segment of a straight line from 0 to z. Similar to the above procedure, repeating the discussion n_θ times, one can deduce that

$$|f(z + \phi)| \leq \int_0^z \left|f'(\xi + \phi)\right| |d\xi| + O(1) = O(|z|^{d_\theta + n_\theta})$$

holds for $z \in \Omega(r_\theta; \theta - \zeta_\theta + \varepsilon_\theta, \theta + \zeta_\theta - \varepsilon_\theta)$. It ensures that

$$S_{\theta-\zeta_\theta+\varepsilon_\theta, \theta+\zeta_\theta-\varepsilon_\theta}(r, f(z+\phi)) = O(\log r).$$

- If $n_\theta < 0$, then (11) gives

$$S_{\theta-\zeta_\theta+\varepsilon_\theta, \theta+\zeta_\theta-\varepsilon_\theta}(r, f^{(n_\theta)}(z+\phi)) = O(\log r), \tag{12}$$

and $\rho_{\theta-\zeta_\theta+\varepsilon_\theta, \theta+\zeta_\theta-\varepsilon_\theta}(f^{(n_\theta)}(z+\phi)) = 0$. Then, by Lemma 2.4, there exist two constants M', K' such that

$$\left|\frac{f^{(n_\theta+1)}(z+\phi)}{f^{(n_\theta)}(z+\phi)}\right| \leq K' r^{M'}$$

for all $z \in \Omega(r_\theta; \theta - \zeta_\theta + \varepsilon_\theta + \frac{\varepsilon}{|n_\theta|}, \theta + \zeta_\theta - \varepsilon_\theta - \frac{\varepsilon}{|n_\theta|})$ outside an R-set, where ε is small constant. Furthermore, it follows from Lemma 2.3 that

$$S_{\theta-\zeta_\theta+\varepsilon_\theta+\frac{\varepsilon}{|n_\theta|}, \theta+\zeta_\theta-\varepsilon_\theta-\frac{\varepsilon}{|n_\theta|}}\left(r, \frac{f^{(n_\theta+1)}(z+\phi)}{f^{(n_\theta)}(z+\phi)}\right) = O(\log r).$$

Together with (12), one has

$$S_{\theta-\zeta_\theta+\varepsilon_\theta+\frac{\varepsilon}{|n_\theta|}, \theta+\zeta_\theta-\varepsilon_\theta-\frac{\varepsilon}{|n_\theta|}}(r, f^{(n_\theta+1)}(z+\phi)) = O(\log r).$$

Similarly, repeating the discussion $|n_\theta|$ times, one can deduce that

$$S_{\theta-\zeta_\theta+\varepsilon_\theta+\varepsilon, \theta+\zeta_\theta-\varepsilon_\theta-\varepsilon}(r, f(z+\phi)) = O(\log r).$$

By shrinking the angular domain $\Omega(a_l, b_l)$ appropriately, there exists sufficiently small ς such that

$$S_{a_l+\varsigma, b_l-\varsigma}(r, f) = O(\log r), \tag{13}$$

i.e., $\rho_{a_l+\varsigma, b_l-\varsigma}(f) = 0$ for each $l = 1, 2, \cdots, m$. Again, by appropriately shrinking the angular domain $\Omega(r; a_l + \varsigma, b_l - \varsigma)$ for $l = 1, 2, \cdots, m$, we have that for sufficiently large r, $z + c_0, \cdots, z + c_k \in \Omega(r; a_l + \varsigma, b_l - \varsigma)$. So, by Lemma 2.4,

$$\left|\frac{f^{(i)}(z+c_i)}{f(z+c_i)}\right| \leq K_i r^{M_i}, (i = 1, 2, \cdots, k) \tag{14}$$

for all $z \in \cup_{l=1}^{m}(\Omega(r; a_l + 2\varsigma, b_l - 2\varsigma))$ outside an R-set, where we have that $K_i, M_i (i = 1, 2, \cdots, k)$ are some positive constants.

In addition, (5) can be rewritten as follows:

$$\begin{aligned} |h(z)| \leq & \sum_{j=1, j\neq v}^{s} \left|\alpha_j(z) \prod_{i=0}^{k} \left(\frac{f^{(i)}(z+c_i)}{f(z)}\right)^{n_{ij}} f(z)^{n_{0j}+\cdots+n_{kj}-m}\right| \\ & + \left|\alpha_v(z) \prod_{i=0}^{k} \left(\frac{f^{(i)}(z+c_i)}{f(z)}\right)^{n_{iv}} f(z)^{n_{0v}+\cdots+n_{kv}-m}\right|. \end{aligned}$$

On the other hand, by Lemma 2.10,

$$\mathrm{mes}E_0 \geq \min\left\{\frac{\pi}{\rho(\alpha_v)} - 2\varepsilon_0, \frac{\pi}{\rho(h)} - \frac{\pi}{\rho(\alpha_v)} - 2\varepsilon_0\right\} = \tau.$$

By the definition of Φ and $\cup_{l=1}^m I_l \subset \Phi$, we have

$$\begin{aligned}\mathrm{mes}(E_0 \cap (\cup_{l=1}^m I_l)) &= \mathrm{mes}(\Phi \cap E_0) - \mathrm{mes}\left((\Phi \setminus \cup_{l=1}^m I_l) \cap E_0\right)\\ &\geq \mathrm{mes}[E_0 \backslash (L(f) \cap E_0)] - \mathrm{mes}(\Phi \setminus \cup_{l=1}^m I_l)\\ &\geq \mathrm{mes}E_0 - \mathrm{mes}L(f) - \mathrm{mes}(\Phi \setminus \cup_{l=1}^m I_l)\\ &\geq \frac{3(\tau - \mathrm{mes}L(f))}{4} > 0.\end{aligned}$$

Then there exists an open interval $(a_0, b_0) \subset \cup_{l=1}^m (a_l + 2\varsigma, b_l - 2\varsigma)$ such that $E_0 \cap (a_0, b_0)$ is non-empty.

Noting that $\sum_{i=0}^k n_{ij} \geq m$ for all $1 \leq j \leq s$ and

$$\left|\frac{f^{(i)}(z+c_i)}{f(z)}\right| \leq \left|\frac{f^{(i)}(z+c_i)}{f(z+c_i)}\right|\left|\frac{f(z+c_i)}{f(z)}\right|, \quad |\alpha_v(z)| \leq |\alpha_v(z) - a| + |a|. \tag{15}$$

From (13), (14), (15) and Lemma 2.10 implies that for any $\theta \in E_0 \cap (a_0, b_0)$ and $\varepsilon \in (0, \frac{\rho(h)-\rho(\alpha_j)}{3})$, there exists a sequence $\{r_n\}(\to \infty)$ such that

$$\exp(r_n^{\rho(h)-\varepsilon}) \leq |h(r_n e^{i\theta})| \leq K r_n^M \left(\sum_{j=1, j\neq v}^{s} \exp(r_n^{\rho(\alpha_j)+\varepsilon}) + \exp(-K r_n^{\rho(\alpha_v)})\right),$$

where K and M are constants. It is impossible for $\rho(\alpha_j) < \rho(h), j \neq v$. We thus complete the proof of Theorem 1.2. $\square$

The proof of Theorem 1.4. Let $\sigma_0 = \min\left\{2\pi, \frac{4}{\mu(\alpha_v)} \arcsin\sqrt{\frac{\delta(a,\alpha_v)}{2}}\right\}$, we deduce that

$$\sigma = \min\left\{2\pi\omega, \frac{4}{\mu(\alpha_v)} \arcsin\sqrt{\frac{\delta(a,\alpha_v)}{2}} - 2\pi(1-\omega)\right\} = \sigma_0 - 2\pi(1-\omega),$$

where $\omega \in (0,1] \cap (1 - \frac{2}{\pi\mu(\alpha_v)} \arcsin\sqrt{\frac{\delta(a,\alpha_v)}{2}}, 1]$.

Assume that $\mathrm{mes}L(f) < \sigma$, i.e., $\sigma - \mathrm{mes}L(f) > 0$, and

$$\sigma_0 - \mathrm{mes}L(f) > 2\pi(1-\omega).$$

For any given $\kappa \in \left(0, 1 - \frac{2\pi(1-\omega)}{\sigma_0 - \mathrm{mes}L(f)}\right)$, we set

$$\chi_\kappa(r) = \{\theta \in [0, 2\pi) : \log^+ |h(re^{i\theta})| \leq \kappa \log^+ M(r,h)\}. \tag{16}$$

By the definition of proximity function $m(r,h)$, one has

$$m(r,h) = \frac{1}{2\pi}\int_{\chi_\kappa(r)} \log^+ |h(re^{i\theta})| d\theta + \frac{1}{2\pi}\int_{[0,2\pi)\backslash\chi_\kappa(r)} \log^+ |h(re^{i\theta})| \mathrm{d}\theta$$

$$\leq \frac{\kappa \mathrm{mes}(\chi_\kappa(r))}{2\pi}\log M(r,h) + \left(1 - \frac{\mathrm{mes}(\chi_\kappa(r))}{2\pi}\right)\log M(r,h).$$

Since $T(r,h) \sim \omega \log M(r,h)$ outside a set F of finite logarithmic measure, for sufficiently large $r \notin F$, we have

$$\mathrm{mes}(\chi_\kappa(r)) \leq \frac{2\pi(1-\omega)}{1-\kappa} < \sigma_0 - \mathrm{mes}L(f).$$

Set $\varrho = \sigma_0 - \mathrm{mes}L(f) - \mathrm{mes}(\chi_\kappa(r)) > 0$ and $\Phi = (0, 2\pi) \setminus L(f)$. As it was shown in the proof of Theorem 1.2, one can choose finite many open intervals $I_l = (a_l, b_l), (l = 1, 2, ..., m)$ in Φ such that $\mathrm{mes}(\Phi \setminus \bigcup_{l=1}^m I_l) < \frac{\varrho}{4}$. Similarly, one also gets that

$$S_{a_l+\varsigma, b_l-\varsigma}(r,f) = O(\log r)$$

for $l = 1, 2, \cdots, m$, and

$$\left|\frac{f^{(i)}(z+c_i)}{f(z+c_i)}\right| \leq K_i r^{M_i} \quad (i = 1, 2, \cdots, k),$$

holds for $z \in \cup_{l=1}^m (\Omega(r; a_l + 2\varsigma, b_l - 2\varsigma))$ outside an R-set.

However, let $\Lambda(r)$ be any positive function with $\Lambda(r) = o(T(r, \alpha_v))$. Applying Lemma 2.8 for $\alpha_v(z)$,

$$\liminf_{r_n \to \infty} \mathrm{mes}D_\Lambda(r_n, a) \geq \min\left\{2\pi, \frac{4}{\mu(\alpha_v)} \arcsin\sqrt{\frac{\delta(a, \alpha_v)}{2}}\right\} = \sigma_0$$

holds for any fixed sequence of pólya peaks $\{r_n\}$, where $D_\Lambda(r_n, a) = \{\theta \in [0, 2\pi) : |\alpha_v(r_n e^{i\theta}) - a| < e^{-\Lambda(r_n)}\}$. For positive function $\Lambda(r)$,

$$|\alpha_v(r_n e^{i\theta}) - a| < e^{-\Lambda(r_n)} < 1. \tag{17}$$

For sufficiently large n, one has

$$\mathrm{mes}D_0(r_n, a) > \sigma_0 - \frac{\varrho}{2}, \tag{18}$$

where $D_0(r_n, a) = \{\theta \in [0, 2\pi) : |\alpha_v(r_n e^{i\theta}) - a| < 1\}$.

From (18) and $\mathrm{mes}(\Phi \setminus \cup_{l=1}^m I_l) < \frac{\varrho}{4}$, we have

$$\begin{aligned}
\mathrm{mes}((\cup_{l=1}^m I_l) \cap D_0(r_n, a)) &= \mathrm{mes}(\Phi \cap D_0(r_n, a)) - \mathrm{mes}\left((\Phi \setminus \cup_{l=1}^m I_l) \cap D_0(r_n, a)\right) \\
&\geq \mathrm{mes}D_0(r_n, a) - \mathrm{mes}L(f) - \mathrm{mes}(\Phi \setminus \cup_{l=1}^m I_l)) \\
&\geq \sigma_0 - \mathrm{mes}L(f) - \frac{3\varrho}{4} \qquad (19) \\
&\geq \frac{\varrho}{4} + \mathrm{mes}(\chi_\kappa(r_n)) > 0.
\end{aligned}$$

Based on the above discussion, there exists an open interval $I_0 \subset \cup_{l=1}^m I_l$ such that $(I_0 \cap D_0(r_n, a)) \setminus \chi_\kappa(r_n)$ is non-empty.

For any $\theta \in (I_0 \cap D_0(r_n, a))\backslash\chi_\kappa(r_n)$, it follows from (16) and (17) that

$$|\alpha_v(r_n e^{i\theta})| < |a| + 1, \qquad |h(r_n e^{i\theta})| > [M(r_n, h)]^\kappa$$

holds for sufficiently larger $r_n (\notin F)$. As showed in the proof of Theorem 1.2, one has (13), (14) and (15). We thus get from (5) that for $r_n \notin F$,

$$[M(r_n, h)]^\kappa < |h(r_n e^{i\theta})| \leq K r_n^M \Big(\sum_{j=1, j\neq v}^{s} |\alpha_j(z)| + |a| + 1 \Big), r_n \to \infty,$$

where K, M are constants. This is impossible for transcendental entire function $h(z)$ and $\rho(h) > \rho(\alpha_j) (j \neq v)$. This completed the proof of Theorem 1.4. $\square$

The proof of Theorem 1.6. From (5), it is not difficult to verify that all non-trivial solutions of (5) are transcendental. In the following, we assume that

$$\mathrm{mes} L(f) < \min \left\{ 2\pi, \frac{\pi}{\mu(h)} \right\} = \tau_1.$$

Let $\Phi = (0, 2\pi) \setminus L(f)$. We also choose finitely many open intervals $I_l = (a_l, b_l) \subset \Phi, (l = 1, 2, ..., m)$ such that $\mathrm{mes}(\Phi \setminus \bigcup_{l=1}^m I_l) < \frac{\tau_1 - \mathrm{mes} L(f)}{4}$.

Let $\Lambda(r) = \max\limits_{1\leq j\leq s} \left\{ \sqrt{\log r}, \sqrt{T(r, \alpha_j)} \right\} \cdot \sqrt{T(r, h)}$ and $T(r, \alpha_j) = S(r, h)$. Since $h(z)$ is transcendental, one has for $j = 1, 2, \cdots, s$

$$\log r = o(\Lambda(r)), T(r, \alpha_j) = o(\Lambda(r)). \tag{20}$$

Applying Lemma 2.8 for entire function $h(z)$, then for any fixed sequence of pólya peaks $\{r_n\}$ and sufficiently large n, we have

$$\mathrm{mes} D_\Lambda(r_n, \infty) > \tau_1 - \frac{\tau_1 - \mathrm{mes} L(f)}{2},$$

where $D_\Lambda(r_n, \infty) = \left\{ \theta \in [0, 2\pi) : |h(r_n e^{i\theta})| > e^{\Lambda(r_n)} \right\}$.

As showed in (19), we get $\mathrm{mes}\left(D_\Lambda(r_n, \infty) \cap (\cup_{l=1}^m I_l) \right) \geq \frac{\tau_1 - \mathrm{mes} L(f)}{4}$. Then there exists an open interval $I_0 \subset \bigcup_{l=1}^m I_l$ such that

$$\mathrm{mes}(D_\Lambda(r_n, \infty) \cap I_0) \geq \frac{\tau_1 - \mathrm{mes} L(f)}{4m} > 0.$$

According to the definition of $D_\Lambda(r_n, \infty)$, one has

$$\int_{D_\Lambda(r_n, \infty) \cap I_0} \log^+ |h(r_n e^{i\theta})| \mathrm{d}\theta \geq \frac{\tau_1 - \mathrm{mes} L(f)}{4m} \Lambda(r_n). \tag{21}$$

Furthermore, it can be concluded from (5) that

$$|h(z)| \leq \sum_{j=1}^{s} \left| \alpha_j(z) \prod_{i=0}^{k} \left(\frac{f^{(i)}(z + c_i)}{f(z)} \right)^{n_{ij}} f(z)^{\sum_{i=0}^k n_{ij} - m} \right|. \tag{22}$$

It follows from the proof of Theorem 1.2 that Equations (13), (14) and (15) are verified. Together with (21), (22), we obtain

$$\begin{aligned}\frac{\tau_1 - \mathrm{mes}L(f)}{4m}\Lambda(r_n) &\le \int_{D_\Lambda(r_n,\infty)\cap I_0} \log^+|h(r_n e^{i\theta})|\mathrm{d}\theta \\ &\le \sum_{j=1}^{s} m(r_n,\alpha_j) + O(\log r_n),\end{aligned}$$

a contradiction is derived from (20). Hence,

$$\mathrm{mes}L(f) \ge \min\Big\{2\pi, \frac{\pi}{\mu(h)}\Big\}.$$

Moreover, we further assume that $\rho(h) > \rho(\alpha_j)$ for $1 \le j \le s$, and we will show that there exists a closed interval in $L(f)$ with the measure at least $\min\{2\pi, \pi/\rho(h)\}$. By Lemma 2.7, there exists an interval (θ_1, θ_2) with $\theta_2 - \theta_1 \ge \min\{2\pi, \pi/\rho(h)\}$ such that

$$\limsup_{r\to\infty} \frac{\log^+\log^+|h(re^{i\theta})|}{\log r} = \rho(h). \tag{23}$$

If $[\theta_1, \theta_2] \not\subseteq L(f)$, there exists an open subinterval $(\vartheta_1, \vartheta_2) \subset (\theta_1, \theta_2) \setminus L(f)$ such that $\Omega(r; \vartheta_1, \vartheta_2) \subset \mathcal{F}(f^{(n)}(z+\phi))$ for some $n \in \mathbb{Z}$ and sufficiently large r. Similarly, one concludes from (22) that

$$\log^+|h(re^{i\theta})| \le \sum_{j=1}^{s} \log^+|\alpha_j(re^{i\theta})| + O(\log r) \le r^{\rho(\alpha_j)+\varepsilon} \le r^{\rho(h)-\varepsilon},$$

where ε can be selected such that $0 < \varepsilon < \min\limits_{1\le j\le s}\left\{\frac{\vartheta_2-\vartheta_1}{4}, \frac{\rho(h)-\rho(\alpha_j)}{3}\right\}$, which contradicts (23). That completes the proof of Theorem 1.6. $\square$

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