COMPOSITION OPERATORS ON SOBOLEV SPACES AND WEIGHTED MODULI INEQUALITIES

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In this paper, we study connections between composition operators on Sobolev spaces and mappings defined by p-moduli inequalities (p-capacity inequalities). We prove that weighted moduli inequalities lead to composition operators on corresponding Sobolev spaces and conversely, that composition operators on Sobolev spaces imply weighted moduli inequalities.

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1. INTRODUCTION

In this paper, we study connections between composition operators on Sobolev spaces and mappings defined by weighted p-moduli inequalities of curves families or corresponding weighted p-capacity inequalities. In the case $p = n$, the mappings defined by conformal moduli inequalities are quasiconformal mappings [1, 34] and the conformal modules method is one of the basic methods in the geometric theory of quasiconformal mappings [34].

The topological mappings defined by p-capacity inequalities were firstly studied in [5]. In this article [5], the notion of p-distortion of mappings was introduced and the Lipschitz properties of such mappings was proved in the case $n-1 < p < \infty$, $p \neq n$. The topological mappings defined by (p,q)-capacity inequalities were studied in [30] in connections with composition operators on Sobolev spaces. In [30, 40] the weak differentiability of inverse mappings and their Hölder continuity properties were proved in the case $n-1 < q < p < \infty$. Continuous mappings satisfying (p,q)-capacity inequalities were considered in [32]. In this work [32], Liouville type theorems were proved and removability properties of singular sets were considered. In the recent paper [37] (see also [31, 36]), mappings which satisfy weighted (p,q)-capacity inequalities were considered in connection with problems of the geometric theory of composition operators on Sobolev spaces [30, 41, 42].

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Significant contributions to the theory of mappings defined by moduli inequalities belong to the Donetsk geometric mapping theory school. In particular, for mappings satisfying to weighted Poletsky's type inequalities for p-modulus of families of curves, their differentiability almost everywhere and the local integrability of partial derivatives were established [28]. Some problems concerning the local behavior of such mappings and the problem of removability of an isolated singular point were investigated in [6, 7, 27]. In particular, mappings with a distortion of the modulus of order $n-1<p<n$ have no essential singular points, that fundamentally distinguishes them from analytic functions and mappings with bounded distortion, see [6, Theorem 1.1]. Let us note that some results concerning of weighted inequalities with respect to the p-modulus for such classes of mappings were obtained in [29]. It should be noted that the similar theory of mappings was developing independently in the context of directly weighted p-moduli, that can be found in works [2, 3].

Composition operators on Sobolev spaces arise in the geometric analysis of Sobolev spaces [9, 13, 14] and are closely connected with the quasiconformal Reshetnyak problem [38]. In series of works [30, 35, 40, 41, 42] and [10] was founded the geometric theory of composition operators on Sobolev spaces. This theory has significant applications in the spectral theory of elliptic equations, see, for example, [11, 16, 17].

The composition operators on Sobolev spaces are generated by weak (p,q)-quasiconformal mappings [10, 30, 40] and allow characterization in the terms of capacity inequalities. Hence, there is a connection between the geometric theory of composition operators on Sobolev spaces and the theory of mappings defined moduli inequalities. Q-homeomorphisms [22] were considered in connection with composition operators on Sobolev spaces in [25, 26].

In the present work, we give connection between mappings defined by weighted moduli inequalities and weak (p,q)-quasiconformal mappings (mappings that generate composition operators on Sobolev spaces). Namely, we prove the following statement:

Let a homeomorphism $\varphi:\Omega\to\widetilde{\Omega}$ satisfy to the p-modulus inequality

$$M_p\left(\varphi\Gamma\right)\leqslant\int\limits_{\Omega} Q(x)\cdot\rho^p(x)\mathrm{d}x,\ n-1<p<\infty,$$

with a non-negative function $Q\in L_1(\Omega)$. Then φ generates the bounded composition operator

$$\varphi^*: L^1_{p'}(\widetilde{\Omega})\to L^1_1(\Omega),\ p'=p/(p-n+1).$$

The inverse assertion states:

Let a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ generate a bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_{n-1}(\Omega), \qquad n-1 < p < \infty\,.$$

Suppose that the mapping φ satisfies Luzin's N-property. Then

$$M_{p'}\left(\varphi\Gamma\right) \leqslant \int\limits_{\Omega} Q(x) \cdot \rho^{p'}(x) \mathrm{d}x,\ p' = \frac{p}{p-n+1},$$

with a non-negative function $Q \in L_1(\Omega)$.

The suggested methods are based on the geometric theory of composition operators on Sobolev spaces and moduli inequalities.

2. SOBOLEV SPACES AND COMPOSITION OPERATORS

2.1. Sobolev spaces

Let Ω be an open subset of $\mathbb{R}^n$. The Sobolev space $W^1_p(\Omega)$, $1 \leqslant p \leqslant \infty$, is defined [23] as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla f \mid L_p(\Omega)\|,$$

where ∇f is the weak gradient of the function f, i. e. $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$.

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leqslant p \leqslant \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_p(\Omega)\| = \|\nabla f \mid L_p(\Omega)\|.$$

In the Sobolev spaces theory, a crucial role belongs to capacity as an outer (capacitary) measure. This measure regulates some natural properties (for example, convergence properties) of corresponding Sobolev spaces [21, 23]. In accordance to this approach, elements of Sobolev spaces $W^1_p(\Omega)$ are equivalence classes up to a set of p-capacity zero [24].

Recall the definition of the capacity [12, 21, 23]. Suppose Ω is an open set in $\mathbb{R}^n$ and $F \subset \Omega$ is a compact set. The p-capacity of F with respect to Ω is defined by

$$\operatorname{cap}_p(F; \Omega) = \inf\{\|\nabla f | L_p(\Omega)\|^p\},$$

where the infimum is taken over all functions $f \in C_0(\Omega) \cap L^1_p(\Omega)$ such that $f \geq 1$ on F and which are called admissible functions for the compact set $F \subset \Omega$. If $U \subset \Omega$ is an open set, we define

$$\operatorname{cap}_p(U; \Omega) = \sup\{\operatorname{cap}_p(F; \Omega)\ :\ F \subset U,\ F \text{ is compact}\}.$$

In the case of an arbitrary set $E \subset \Omega$, we define the inner p-capacity

$$\underline{\operatorname{cap}}_p(E;\Omega) = \sup\{\operatorname{cap}_p(F;\Omega) \,:\, F \subset E \subset \Omega,\ F \text{ is compact}\},$$

and the outer p-capacity

$$\overline{\operatorname{cap}}_p(E;\Omega) = \inf\{\operatorname{cap}_p(U;\Omega) \,:\, E \subset U \subset \Omega,\ U \text{ is open}\}.$$

A set $E \subset \Omega$ is called p-capacity measurable, if $\underline{\operatorname{cap}}_p(E;\Omega) = \overline{\operatorname{cap}}_p(E;\Omega)$. Let $E \subset \Omega$ be a p-capacity measurable set. The value

$$\operatorname{cap}_p(E;\Omega) = \underline{\operatorname{cap}}_p(E;\Omega) = \overline{\operatorname{cap}}_p(E;\Omega)$$

is called the p-capacity measure of the set $E \subset \Omega$. Borel subsets $E \subset \Omega$ are p-capacity measurable [21].

The mapping $\varphi : \Omega \to \mathbb{R}^n$ belongs to the Sobolev space $W^1_{1,\mathrm{loc}}(\Omega)$, if its coordinate functions belongs to $W^1_{1,\mathrm{loc}}(\Omega)$. In this case, the formal Jacobi matrix $D\varphi(x)$ and its determinant (Jacobian) $J(x,\varphi)$ are well defined at almost all points $x \in \Omega$. The norm $|D\varphi(x)|$ is the operator norm of $D\varphi(x)$,

Let us recall the change of variable formula in the Lebesgue integral [4, 20]. Suppose a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ is such that there exists a collection of closed sets $A_k \subset A_{k+1} \subset \Omega$, $k = 1, 2, \ldots$, for which restrictions $\varphi|_{A_k}$ are Lipschitz mappings on the sets A_k and

$$\left|\Omega \setminus \bigcup_{k=1}^{\infty} A_k\right| = 0.$$

Then there exists a Borel set $S \subset \Omega$, $|S| = 0$, such that on the set $\Omega \setminus S$ the homeomorphism φ has the Luzin N-property (the image of a set of measure zero has measure zero) and the change of variable formula

$$\int\limits_E f \circ \varphi(x) |J(x,\varphi)|\ \mathrm{d}x = \int\limits_{\widetilde{\Omega}\setminus\varphi(S)} f(y)\ \mathrm{d}y \tag{1}$$

holds for every measurable set $E \subset \Omega$ and every non-negative measurable function $f : \widetilde{\Omega} \to \mathbb{R}$.

Note, that Sobolev homeomorphisms of the class $W^1_{1,\mathrm{loc}}(\Omega)$ satisfy the conditions of the change of variable formula [20] and, therefore, for Sobolev homeomorphisms the change of variable formula (1) holds.

If the mapping φ possesses the Luzin N-property, then $|\varphi(S)| = 0$ and the second integral can be rewritten as the integral on $\widetilde{\Omega}$. Note, that Sobolev homeomorphisms of the class $L^1_p(\Omega)$, $p \geqslant n$, possess the Luzin N-property [39].

2.2. **Composition operators and regularity of inverse mappings**

Let Ω and $\widetilde{\Omega}$ be domains in the Euclidean space $\mathbb{R}^n$. Then a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; 1 \leqslant q \leqslant p \leqslant \infty,$$

by the composition rule $\varphi^*(f) = f \circ \varphi$, if for any function $f \in L^1_p(\widetilde{\Omega})$, the composition $\varphi^*(f) \in L^1_q(\Omega)$ defined quasi-everywhere in Ω and there exists a constant $K_{p,q}(\varphi; \Omega) < \infty$ such that

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leqslant K_{p,q}(\varphi; \Omega) \|f \mid L^1_p(\widetilde{\Omega})\|.$$

Recall that the p-dilatation [5] of a Sobolev mapping $\varphi : \Omega \to \widetilde{\Omega}$ at a point $x \in \Omega$ defined as

$$K_p(x) = \inf\{k(x) : |D\varphi(x)| \leqslant k(x)|J(x, \varphi)|^{\frac{1}{p}}\}.$$

The following theorem gives a characterization of composition operators in terms of integral characteristics of mappings of finite distortion. Recall that a weakly differentiable mapping $\varphi : \Omega \to \mathbb{R}^n$ is the mapping of finite distortion if $D\varphi(x) = 0$ for almost all x from $Z = \{x \in \Omega : J(x, \varphi) = 0\}$ [39].

THEOREM 2.1. *The homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; 1 \leqslant q \leqslant p \leqslant \infty,$$

if and only if $\varphi \in W^1_{q,\mathrm{loc}}(\Omega)$, has finite distortion, and

$$K_{p,q}(\varphi; \Omega) := \|K_p \mid L_\kappa(\Omega)\| < \infty, \;\; 1/q - 1/p = 1/\kappa \;\; (\kappa = \infty, \;\; \textit{if}\; p = q).$$

The norm of the operator φ^* is estimated as $\|\varphi^*\| \leqslant K_{p,q}(\varphi; \Omega)$.

This theorem, in the case $p = q = n$, was proved in [38], the case $p = q > n$ was proved in [35] (see, also [10]). The general case $q \leq p < \infty$ was proved in [30] (see, also [40]) and the limit case $p = \infty$ was considered in [15].

Let us recall, that homeomorphisms $\varphi : \Omega \to \widetilde{\Omega}$ which satisfy conditions of Theorem 2.1 *are called weak (p, q)-quasiconformal mappings* [10, 40].

In the case of weak (p, q)-quasiconformal mappings, the following composition duality property was introduced in [30] (the detailed proof can be found in [18]):

THEOREM 2.2. *Let the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ generate a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \;\; n - 1 < q \leqslant p < \infty.$$

Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ *generates a bounded composition operator*

$$\left(\varphi^{-1}\right)^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}),$$

where $p' = p/(p-n+1)$, $q' = q/(q-n+1)$. *In the case* $n=2$, *this theorem is correct for* $1 \leqslant q \leqslant p < \infty$, *the inverse assertion is also correct and* $p'' = p'/(p'-1) = p$, $q'' = q'/(q'-1) = q$.

The limit case of this theorem $p = \infty$ and $q = n-1$ was considered in [15] in the frameworks of the weak inverse theorem for Sobolev spaces.

In the present work, we consider the case $q = n-1 < p < \infty$.

THEOREM 2.3. *Let the homeomorphism* $\varphi : \Omega \to \widetilde{\Omega}$ *between two domains* Ω *and* $\widetilde{\Omega}$ *possess the Luzin* N*-property and generate a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_{n-1}(\Omega), \;\; n-1 < p < \infty.$$

Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ *generates a bounded composition operator*

$$\left(\varphi^{-1}\right)^* : L^1_{\infty}(\Omega) \to L^1_{p'}(\widetilde{\Omega}),$$

where $p' = p/(p-n+1)$. *In the case* $n=2$ *the inverse assertion is also correct and* $p'' = p'/(p'-1) = p$.

Proof. Since $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_{n-1}(\Omega), \;\; n-1 < p < \infty,$$

then by [30] the mapping $\varphi \in W^1_{n-1,\mathrm{loc}}(\Omega)$ and has a finite distortion. Because φ possesses the Luzin N-property, then the inverse mapping belongs to $W^1_{1,\mathrm{loc}}(\widetilde{\Omega})$ and is a mapping of finite distortion [15]. Hence

$$|D\varphi^{-1}(y)| \leq \frac{|D\varphi(x)|^{n-1}}{|J(x,\varphi)|},$$

for almost all $x \in \Omega \setminus (S \cup Z)$, $y = \varphi(x) \in \widetilde{\Omega} \setminus \varphi(S \cup Z)$, and $|D\varphi^{-1}(y)| = 0$ for almost all $y \in \varphi(S)$, where $Z = \{x \in \Omega : J(x,\varphi) = 0\}$ and S is the singular set in the change of variables formula (1). Because the measure of S is zero and the mapping φ has N-property then the measure of $\varphi(S)$ is also zero.

Therefore,

$$\int\limits_{\widetilde{\Omega}} |D\varphi^{-1}(y)|^{p'}\ \mathrm{d}y = \int\limits_{\widetilde{\Omega}\setminus\varphi(S\cup Z)} |D\varphi^{-1}(y)|^{p'}\mathrm{d}y \leq \int\limits_{\widetilde{\Omega}\setminus\varphi(S\cup Z)} \left(\frac{|D\varphi(\varphi^{-1}(y))|^{n-1}}{|J(\varphi^{-1}(y),\varphi)|}\right)^{p'}\mathrm{d}y$$

$$= \int\limits_{\Omega\setminus(S\cup Z)} \left(\frac{|D\varphi(x)|^{n-1}}{|J(x,\varphi)|}\right)^{p'} |J(x,\varphi)|\ \mathrm{d}x$$

$$= \int\limits_{\Omega} \left(\frac{|D\varphi(x)|^{p}}{|J(x,\varphi)|}\right)^{\frac{n-1}{p-(n-1)}}\ \mathrm{d}x < \infty,$$

by Theorem 2.1. Hence [15], $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates a bounded composition operator

$$\left(\varphi^{-1}\right)^{*} : L^1_\infty(\Omega) \to L^1_{p'}(\widetilde{\Omega}),$$

where $p' = p/(p-n+1)$. $\square$

3. COMPOSITION OPERATORS AND MODULI INEQUALITIES

3.1. Modulus and capacity

Let Γ be a family of curves in $\mathbb{R}^n$. Denote by $\operatorname{adm}(\Gamma)$ the set of Borel functions (admissible functions) $\rho : \mathbb{R}^n \to [0,\infty]$ such that the inequality

$$\int\limits_{\gamma} \rho\ \mathrm{d}s \geqslant 1$$

holds for locally rectifiable curves $\gamma \in \Gamma$.

Let Γ be a family of curves in $\overline{\mathbb{R}^n}$, where $\overline{\mathbb{R}^n}$ is a one point compactification of the Euclidean space $\mathbb{R}^n$. The quantity

$$M_p(\Gamma) = \inf \int\limits_{\mathbb{R}^n} \rho^p\ \mathrm{d}x$$

is called the p-module of the family of curves Γ [22]. The infimum is taken over all admissible functions $\rho \in \operatorname{adm}(\Gamma)$.

Let Ω be a bounded domain in $\mathbb{R}^n$ and F_0, F_1 be disjoint non-empty compact sets in the closure of Ω. Let $M_p(\Gamma(F_0,F_1;\Omega))$ denote the moduli of a family of curves which connect F_0 and F_1 in Ω. Then [22]

$$M_p(\Gamma(F_0,F_1;\Omega)) = \operatorname{cap}_p(F_0,F_1;\Omega)\,, \tag{2}$$

where $\operatorname{cap}_p(F_0,F_1;\Omega)$ is the p-capacity of the condenser $(F_0,F_1;\Omega)$ [23].

Suppose that a homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ satisfies the moduli inequality

$$M_p(\varphi\Gamma) \leqslant \int\limits_{\Omega} Q(x) \cdot \rho^p(x) \mathrm{d}x \tag{3}$$

with a non-negative measurable function Q for every family Γ of rectifiable curves in Ω and every admissible function ρ for Γ. Such homeomorphisms are called Q-homeomorphisms.

The next section is devoted to connections between Q-homeomorphisms and the composition operators in the case $Q \in L_1(\Omega)$.

3.2. Composition operators and Q-homeomorphisms

Firstly, we define two dilatation functions for Sobolev mappings of finite distortion $\varphi : \Omega \to \widetilde{\Omega}$.
The outer p-dilatation is the following quantity

$$K_p^O(x, \varphi) = \begin{cases} \frac{|D\varphi(x)|^p}{|J(x,\varphi)|}, & J(x,\varphi) \neq 0, \\ 0, & J(x,\varphi) = 0. \end{cases}$$

The inner p-dilatation is the following quantity

$$K_p^I(x, \varphi) = \begin{cases} \frac{|J(x,\varphi)|}{l(D\varphi(x))^p}, & J(x,\varphi) \neq 0, \\ 0, & J(x,\varphi) = 0, \end{cases}$$

where $l(D\varphi(x)) = \min\limits_{h=1} |D\varphi(x) \cdot h|$ for almost all $x \in \Omega$.

THEOREM 3.1. *Let the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ satisfy the moduli inequality*

$$M_p(\varphi\Gamma) \leqslant \int\limits_{\Omega} Q(x) \cdot \rho^p(x) \mathrm{d}x, \ n-1 < p < \infty, \tag{4}$$

with a non-negative function $Q \in L_1(\Omega)$. Then φ generates the bounded composition operator

$$\varphi^* : L^1_{p'}(\widetilde{\Omega}) \to L^1_1(\Omega), \ p' = p/(p-n+1).$$

Proof. Because φ satisfies the moduli inequality with $Q \in L_1(\Omega)$, then by [28] the mapping $\varphi \in \mathrm{ACL}(\Omega)$, has finite distortion and the inequality

$$|D\varphi(x)|^p \leqslant C(n,p) |J(x,\varphi)|^{p-n+1} Q^{n-1}(x) \tag{5}$$

holds for almost all $x \in \Omega$. Hence

$$\left(\frac{|D\varphi(x)|^{\frac{p}{p-n+1}}}{|J(x,\varphi)|}\right)^{\frac{p-n+1}{n-1}} \leqslant C(n,p)Q(x) \text{ for almost all } x \in \Omega.$$

Since $Q \in L_1(\Omega)$, we have

$$\int\limits_{\Omega} \left(\frac{|D\varphi(x)|^{\frac{p}{p-n+1}}}{|J(x,\varphi)|}\right)^{\frac{p-n+1}{n-1}} \mathrm{d}x \leqslant C(n,p) \int\limits_{\Omega} Q(x)\ \mathrm{d}x < \infty.$$

Then, by Theorem 2.1, the mapping φ generates the bounded composition operator $\varphi^* : L^1_{p'}(\widetilde{\Omega}) \to L^1_{q'}(\Omega)$, where $p' = p/(p-n+1)$ and the number $q' = 1$ is the solution of the equation

$$\frac{q'}{p'-q'} = \frac{p-n+1}{n-1}. \qquad \square$$

In [8], the integrability of Jacobians of open discrete mappings with controlled p-modulus was considered. As a consequence of Theorem 3.1, we obtain that homeomorphisms which satisfy the weighted p-moduli inequality (4) posses measure distortion properties as weak $(p',1)$-quasiconformal mappings [40, 41].

Now, using the composition duality property in the case of planar domains $\Omega, \widetilde{\Omega} \subset \mathbb{R}^2$, we obtain:

Theorem 3.2. *Let the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$, $\Omega, \widetilde{\Omega} \subset \mathbb{R}^2$, satisfy the moduli inequality*

$$M_p(\varphi\Gamma) \leqslant \int\limits_{\Omega} Q(x) \cdot \rho^p(x) \mathrm{d}x,\ 1 \leqslant p < \infty,$$

with a non-negative function $Q \in L_1(\Omega)$. Then, the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates the bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{\infty}(\Omega) \to L^1_p(\widetilde{\Omega}).$$

In particular, $\varphi^{-1} \in L^1_p(\widetilde{\Omega})$.

Now, we prove the inverse property.

Theorem 3.3. *Let the homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ generate the bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_{n-1}(\Omega), \qquad n-1 < p < \infty\,.$$

Suppose that the mapping φ satisfies Luzin's N-property. Then φ is a Q-homeomorphism with respect to p'-modulus with $Q(x) = K^I_{p'}(x,\varphi) \in L_1(\Omega)$, where $p' = \frac{p}{p-n+1}$.

Proof. On the first step, we prove that under conditions of the theorem the inner distortion function $K^I_{p'}(x,\varphi)\in L_1(\Omega)$. Since φ generates the bounded composition operator

$$\varphi^*: L^1_p(\widetilde{\Omega})\to L^1_{n-1}(\Omega), \qquad n-1<p<\infty\,.$$

then, by Theorem 2.3, the inverse mapping $\varphi^{-1}:\widetilde{\Omega}\to\Omega$ generates a bounded composition operator

$$\left(\varphi^{-1}\right)^*: L^1_\infty(\Omega)\to L^1_{p'}(\widetilde{\Omega}),\ p'=\frac{p}{p-n+1}$$

and belongs to the Sobolev space $L^1_{p'}(\widetilde{\Omega})$ [15].

Because the inverse mapping is a mapping of finite distortion, then by [19]

$$\int\limits_{\Omega}\frac{|J(x,\varphi)|}{l(D\varphi(x))^{p'}}\,\mathrm{d}x=\int\limits_{\Omega\setminus Z}\frac{|J(x,\varphi)|}{l(D\varphi(x))^{p'}}\,\mathrm{d}x=\int\limits_{\Omega\setminus Z}|D\varphi^{-1}(\varphi(x))|^{p'}|J(x,\varphi)|\,\mathrm{d}x$$
$$=\int\limits_{\widetilde{\Omega}}|D\varphi^{-1}(y)|^{p'}\,\mathrm{d}y<\infty.$$

Now, by the definition of the Q-homeomorphism with respect to p'-modulus, we have to show that for every family Γ of curves in Ω and every $\rho\in\mathrm{adm}(\Gamma)$

$$M_{p'}(\varphi\Gamma)\leqslant\int\limits_{\Omega}K^I_{p'}(x,\varphi)\rho^{p'}(x)\,\mathrm{d}x.$$

First, note that by Theorem 2.1, $\varphi\in W^1_{n-1,\mathrm{loc}}(\Omega)$. Also, $\varphi^{-1}\in W^1_{p',\mathrm{loc}}(\widetilde{\Omega})$ by Theorem 2.2. It implies, that $\varphi^{-1}\in\mathrm{ACL}^{p'}_{\mathrm{loc}}(\widetilde{\Omega})$, is differentiable a.e. (see [33, Lemma 3]).

By Fuglede's theorem (see [34], p. 95), if $\widetilde{\Gamma}$ is the family of all curves $\gamma\in\varphi\Gamma$ for which φ^{-1} is absolutely continuous on all closed subcurves of γ, then $M_{p'}(\varphi\Gamma)=M_{p'}(\widetilde{\Gamma})$. Then, for given $\rho\in\mathrm{adm}(\Gamma)$, one consider

$$\widetilde{\rho}(y)=\begin{cases}\rho(\varphi^{-1}(y))|D\varphi^{-1}(y)|, & y\in\widetilde{\Omega},\\ 0, & \text{otherwise.}\end{cases}$$

Then, for $\widetilde{\gamma}\in\widetilde{\Gamma}$

$$\int\limits_{\widetilde{\gamma}}\widetilde{\rho}\,\mathrm{d}s\geqslant\int\limits_{\varphi^{-1}\circ\widetilde{\gamma}}\rho\,\mathrm{d}s\geqslant 1,$$

and consequently $\widetilde{\rho}\in\mathrm{adm}(\widetilde{\Gamma})$.

We denote by Z_0 the set of all points $y \in \widetilde{\Omega}$, where $J_{\varphi^{-1}}(y) = 0$. By change of variable formula (see [4, Theorem 3.2.5]), we obtain that

$$
\begin{aligned}
M_{p'}(\varphi\Gamma) &= M_{p'}(\widetilde{\Gamma}) \leqslant \int\limits_{\widetilde{\Omega}} \widetilde{\rho}^{p'}\,\mathrm{d}y \\
&= \int\limits_{\widetilde{\Omega}} \rho^{p'}(\varphi^{-1}(y))|D\varphi^{-1}(y)|^{p'}\,\mathrm{d}y = \int\limits_{\widetilde{\Omega}\setminus Z_0} \frac{\rho^{p'}(\varphi^{-1}(y))}{l(D\varphi(\varphi^{-1}(y)))^{p'}}\,\mathrm{d}y \\
&= \int\limits_{\widetilde{\Omega}} \rho^{p'}(\varphi^{-1}(y))K^{I}_{p'}(\varphi^{-1}(y),\varphi)J_{\varphi^{-1}}(y)\,\mathrm{d}y \leqslant \int\limits_{\Omega} K^{I}_{p'}(x,\varphi)\rho^{p'}(x)\,\mathrm{d}x
\end{aligned}
$$

which completes the proof. □

In the planar case $\Omega, \widetilde{\Omega} \subset \mathbb{R}^2$, we have the following theorem:

THEOREM 3.4. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphism of the domains $\Omega, \widetilde{\Omega} \subset \mathbb{R}^2$. Then φ satisfies the moduli inequality*

$$
M_p\left(\varphi\Gamma\right) \leqslant \int\limits_{\Omega} Q(x)\cdot\rho^p(x)\mathrm{d}x,\ 1<p<\infty, \tag{6}
$$

with a non-negative function $Q \in L_1(\Omega)$, if and only if φ generates the bounded composition operator

$$
\varphi^{*} : L^1_{p'}(\widetilde{\Omega}) \to L^1_1(\Omega),\ p' = p/(p-1).
$$

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