ON THE SPACE OF HOMOGENEOUS MODIFIED HARMONIC POLYNOMIALS IN HIGHER DIMENSIONS

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The functions on $\mathbb{R}^{d-1} \times (0, \infty)$ $(d \geq 3)$ that are annihilated by the Laplace–Beltrami operator corresponding to the line-element $\mathrm{d}l^2 = x_d^2(\mathrm{d}x_1^2 + \ldots + \mathrm{d}x_d^2)$ are called *modified harmonic*. In this note, we prove a conjecture of Heinz Leutwiler concerning the space of homogeneous modified harmonic polynomials of a fixed degree.

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1. INTRODUCTION

The "upper half space" $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \,|\, x_d > 0\}$ of $\mathbb{R}^d$ $(d \geq 3)$ equipped with the line-element $\mathrm{d}l^2 = x_d^2(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2)$ becomes a Riemannian manifold, whose Laplace–Beltrami operator is $\frac{1}{x_d^2}\big(\Delta + \frac{d-2}{x_d} \cdot \frac{\partial}{\partial x_d}\big)$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$. The functions u that are annihilated by this operator or, more generally, the solutions of

$$x_d \cdot \Delta u + (d-2) \cdot \frac{\partial u}{\partial x_d} = 0$$

(waiving the restriction $x_d > 0$) are called *modified harmonic functions*. It is straightforward to see that this property passes from u to $\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{d-1}}$.

In [1], Heinz Leutwiler introduces the space $\mathcal{H}_n(\mathbb{R}^d)$ of all homogeneous modified harmonic polynomials of degree n on $\mathbb{R}^d$ (i.e., modified harmonic functions on $\mathbb{R}^d$ which are homogeneous polynomials of degree n) and shows that its dimension equals $\binom{d-2+n}{d-2}$. He further mentions that if u is a modified harmonic function, then so is its *modified Kelvin transform*,

$$M[u](x_1, \ldots, x_d) := \frac{1}{r^{2d-4}} u\left(\frac{x_1}{r^2}, \cdots, \frac{x_d}{r^2}\right),$$

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where $r = \sqrt{x_1^2 + \ldots + x_d^2}$. This can be verified by an elementary, but lengthy computation.

Now, since $u(x_1, \ldots, x_d) := \frac{1}{r^{2d-4}}$ is a modified harmonic function (being the modified Kelvin transform of 1), so are its partial derivatives

$$u_{\alpha_1 \ldots \alpha_{d-1}} := \frac{\partial^n u}{\partial x_1^{\alpha_1} \ldots \partial x_{d-1}^{\alpha_{d-1}}}$$

for $\alpha_1, \ldots, \alpha_{d-1} \in \mathbb{N} \cup \{0\}$, $\alpha_1 + \ldots + \alpha_{d-1} = n$, as well as their modified Kelvin transforms

$$v_{\alpha_1 \ldots \alpha_{d-1}}(x_1, \ldots, x_d) := M[u_{\alpha_1 \ldots \alpha_{d-1}}](x_1, \ldots, x_d)$$

$$(1) \qquad = \frac{1}{r^{2d-4}} u_{\alpha_1 \ldots \alpha_{d-1}} \left(\frac{x_1}{r^2}, \ldots, \frac{x_d}{r^2} \right) = r^{2n+2d-4} \cdot \frac{\partial^n r^{4-2d}}{\partial x_1^{\alpha_1} \ldots \partial x_{d-1}^{\alpha_{d-1}}} ,$$

since $u_{\alpha_1 \ldots \alpha_{d-1}}$ is homogeneous of degree $4 - 2d - n$ (in fact, r^{4-2d} is homogeneous of degree $4 - 2d$, and every partial derivative reduces the degree of homogeneity by 1). Setting $R := r^2$, it follows by induction that $u_{\alpha_1 \ldots \alpha_{d-1}}$ has the form $R^{2-d-n} \cdot P$, where P is a polynomial, whence $v_{\alpha_1 \ldots \alpha_{d-1}}$ is a polynomial too. Altogether, $v_{\alpha_1 \ldots \alpha_{d-1}} \in \mathcal{H}_n(\mathbb{R}^d)$.

H. Leutwiler conjectured that the $\binom{d-2+n}{d-2}$ polynomials $v_{\alpha_1 \ldots \alpha_{d-1}} \in \mathcal{H}_n(\mathbb{R}^d)$ are linearly independent (and therefore, form a basis of $\mathcal{H}_n(\mathbb{R}^d)$) (see [1]). The purpose of this article is to prove this conjecture. To this end, we follow the same reasoning as in our earlier paper [3], where we had proven the older partial conjecture of Leutwiler (see [2]), which concerned the four-dimensional case ($d = 4$). We remark that the general proof given here also covers the case of the lowest significant dimension $d = 3$.

Finally, we close this introduction by listing the polynomials $v_{\alpha_1 \ldots \alpha_{d-1}}$ for $\alpha_1 + \ldots + \alpha_{d-1} \leq 2$:
$v_{0 \ldots 0} = 1$;
$v_{0 \ldots 010 \ldots 0} = (4 - 2d)x_i$ (the index 1 is at the i-th position);
$v_{0 \ldots 020 \ldots 0} = (4 - 2d)r^2 + (4 - 2d)(2 - 2d)x_i^2$ (the index 2 is at the i-th position);
$v_{0 \ldots 010 \ldots 010 \ldots 0} = (4 - 2d)(2 - 2d)x_i x_j$ (the two indices 1 are at the positions i and j).

2. PROOF OF LEUTWILER'S CONJECTURE

We introduce the new variables $X_1 := x_1^2$, $X_2 := x_2^2, \ldots, X_d := x_d^2$. Then, $R = r^2 = X_1 + X_2 + \cdots + X_d$. Furthermore, we relate every function f of the variables $X_1, \ldots, X_d$ to the function

$$g(x_1, \ldots, x_d) := f\,(X_1, \ldots, X_d)|_{X_1 = x_1^2, \ldots, X_d = x_d^2} \ ,$$

where we assume $x_1, \ldots, x_d \geq 0$. The following relations take place for the partial derivatives of f and g:

$$\begin{aligned}\frac{\partial g}{\partial x_i}(x_1, \ldots, x_d) &= \frac{\partial f}{\partial X_i}(X_1, \ldots, X_d)\Big|_{X_1=x_1^2, \ldots, X_d=x_d^2} \cdot 2x_i \\ &= \left[\frac{\partial f}{\partial X_i}(X_1, \ldots, X_d) \cdot 2\sqrt{X_i}\right]\Big|_{X_1=x_1^2, \ldots, X_d=x_d^2}\end{aligned}$$

for $1 \leq i \leq d$, which we shall express in the shorter form

$$\frac{\partial g}{\partial x_i} = 2\sqrt{X_i} \cdot \frac{\partial f}{\partial X_i}.$$

Under this convention, which we shall always use in the sequel, it further holds for $1 \leq i, j \leq d$, $i \neq j$:

$$\frac{\partial^2 g}{\partial x_i^2} = 2 \cdot \frac{\partial f}{\partial X_i} + 4X_i \cdot \frac{\partial^2 f}{\partial X_i^2}, \qquad \frac{\partial^2 g}{\partial x_i \partial x_j} = 4\sqrt{X_i X_j} \cdot \frac{\partial^2 f}{\partial X_i \partial X_j},$$

$$\frac{\partial^3 g}{\partial x_i^3} = 12\sqrt{X_i} \cdot \frac{\partial^2 f}{\partial X_i^2} + 8X_i\sqrt{X_i} \cdot \frac{\partial^3 f}{\partial X_i^3} \quad \text{etc.}$$

For the proof of the conjecture, we need the next three lemmas.

Lemma 1. *Let the notation be as above.*

1. *For $\alpha \in 2\mathbb{N} \cup \{0\}$ and $i \in \{1, \ldots, d\}$ it holds:*

$$\frac{\partial^\alpha g}{\partial x_i^\alpha} = \sum_{j=0}^{\frac{\alpha}{2}} c_{i,\alpha,j} X_i^j \cdot \frac{\partial^{\frac{\alpha}{2}+j} f}{\partial X_i^{\frac{\alpha}{2}+j}}$$

with certain $c_{i,\alpha,j} \in \mathbb{N}$.

2. *For $\alpha \in 2(\mathbb{N} \cup \{0\}) + 1$ and $i \in \{1, \ldots, d\}$ it holds:*

$$\frac{\partial^\alpha g}{\partial x_i^\alpha} = \sum_{j=0}^{\frac{\alpha-1}{2}} c_{i,\alpha,j+\frac{1}{2}} X_i^{j+\frac{1}{2}} \cdot \frac{\partial^{\frac{\alpha+1}{2}+j} f}{\partial X_i^{\frac{\alpha+1}{2}+j}}$$

with certain $c_{i,\alpha,j+\frac{1}{2}} \in \mathbb{N}$.

Proof. 1. We only have to verify the inductive step from α to $\alpha + 2$. Let A denote the right side of the assertion for an even α. Then,

$$\frac{\partial A}{\partial x_i} = \sum_{j=0}^{\frac{\alpha}{2}} 2c_{i,\alpha,j}\left(jX_i^{j-\frac{1}{2}} \cdot \frac{\partial^{\frac{\alpha}{2}+j} f}{\partial X_i^{\frac{\alpha}{2}+j}} + X_i^{j+\frac{1}{2}} \cdot \frac{\partial^{\frac{\alpha}{2}+j+1} f}{\partial X_i^{\frac{\alpha}{2}+j+1}}\right),$$

$$\frac{\partial^2 A}{\partial x_i^2} = \sum_{j=0}^{\frac{\alpha}{2}} 4c_{i,\alpha,j}\left[j\left(j-\frac{1}{2}\right)X_i^{j-1}\cdot\frac{\partial^{\frac{\alpha}{2}+j}f}{\partial X_i^{\frac{\alpha}{2}+j}} + \left(2j+\frac{1}{2}\right)X_i^j\cdot\frac{\partial^{\frac{\alpha}{2}+j+1}f}{\partial X_i^{\frac{\alpha}{2}+j+1}} + X_i^{j+1}\cdot\frac{\partial^{\frac{\alpha}{2}+j+2}f}{\partial X_i^{\frac{\alpha}{2}+j+2}}\right]$$

$$= \sum_{j=0}^{\frac{\alpha}{2}+1}\left[2(4j+1)c_{i,\alpha,j} + 4c_{i,\alpha,j-1}\right]X_i^j\cdot\frac{\partial^{\frac{\alpha}{2}+j+1}f}{\partial X_i^{\frac{\alpha}{2}+j+1}} + \sum_{j=0}^{\frac{\alpha}{2}-1} 4(j+1)\left(j+\frac{1}{2}\right)c_{i,\alpha,j+1}\cdot X_i^j\cdot\frac{\partial^{\frac{\alpha}{2}+j+1}f}{\partial X_i^{\frac{\alpha}{2}+j+1}}$$

$$= \sum_{j=0}^{\frac{\alpha+2}{2}}\left[4c_{i,\alpha,j-1} + 2(4j+1)c_{i,\alpha,j} + 2(j+1)(2j+1)c_{i,\alpha,j+1}\right]X_i^j\cdot\frac{\partial^{\frac{\alpha+2}{2}+j}f}{\partial X_i^{\frac{\alpha+2}{2}+j}},$$

where we have set $c_{i,\alpha,-1} = c_{i,\alpha,\frac{\alpha+2}{2}} = c_{i,\alpha,\frac{\alpha+4}{2}} = 0$. This completes the proof for even α, as it is inductively clear that the square brackets do not vanish.

2. If α is odd, then $\alpha - 1$ is even, so by what has just been proven,

$$\frac{\partial^\alpha g}{\partial x_i^\alpha} = \sum_{j=0}^{\frac{\alpha-1}{2}} 2c_{i,\alpha-1,j}\left(jX_i^{j-\frac{1}{2}}\cdot\frac{\partial^{\frac{\alpha-1}{2}+j}f}{\partial X_i^{\frac{\alpha-1}{2}+j}} + X_i^{j+\frac{1}{2}}\cdot\frac{\partial^{\frac{\alpha-1}{2}+j+1}f}{\partial X_i^{\frac{\alpha-1}{2}+j+1}}\right)$$

$$= \sum_{j=0}^{\frac{\alpha-1}{2}}\left[2c_{i,\alpha-1,j} + 2(j+1)c_{i,\alpha-1,j+1}\right]X_i^{j+\frac{1}{2}}\cdot\frac{\partial^{\frac{\alpha+1}{2}+j}f}{\partial X_i^{\frac{\alpha+1}{2}+j}}.$$

Obviously, the square brackets do not vanish in this case either. □

Remark 1. Any confusion in the coefficients regarding the two cases of the lemma is avoided by the fact that the third index in the coefficients is an integer only in case 1.

LEMMA 2. *The functions* $(X_1, \ldots, X_{d-1}) \mapsto X_1^{i_1} \ldots X_{d-1}^{i_{d-1}}$, *where we have that* $i_1, \ldots, i_{d-1}$ *run through* $\frac{1}{2}\mathbb{N} \cup \{0\}$, *are linearly independent.*

Proof. After the substitution $X_1 = x_1^2, \ldots, X_{d-1} = x_{d-1}^2$, these functions become the monomials $x_1^{2i_1} \ldots x_{d-1}^{2i_{d-1}}$, which obviously are linearly independent. □

LEMMA 3. *Let* $k \in \mathbb{N}$ *and* $h(X_1, \ldots, X_d) = \frac{1}{(X_1 + \ldots + X_d)^k}$. *Then, for every* $l \in \mathbb{N} \cup \{0\}$ *it holds that*

$$\frac{\partial^l h}{\partial X_1^l} = \ldots = \frac{\partial^l h}{\partial X_d^l} = \frac{(-1)^l \cdot (k)_l}{(X_1 + \ldots + X_d)^{k+l}},$$

where $(k)_l := \prod_{i=0}^{l-1}(k+i)$ *is the Pochhammer symbol.*

Proof. The claim follows easily by induction. $\square$

We now start with the actual proof of the conjecture.

For the function f defined by $f(X_1, \ldots, X_d) = \frac{1}{(X_1 + \cdots + X_d)^{d-2}}$, the last lemma gives for $\alpha_1, \alpha_2, \ldots, \alpha_{d-1} \in \mathbb{N} \cup \{0\}$:

$$\frac{\partial^{\alpha_1} f(X_1, \ldots, X_d)}{\partial X_1^{\alpha_1}} = \frac{(-1)^{\alpha_1} \cdot (d-2)_{\alpha_1}}{(X_1 + \cdots + X_d)^{d-2+\alpha_1}} = \frac{(-1)^{\alpha_1} \cdot (d + \alpha_1 - 3)!}{(d-3)!\,(X_1 + \cdots + X_d)^{d+\alpha_1-2}},$$

$$\frac{\partial^{\alpha_1+\alpha_2} f(X_1, \ldots, X_d)}{\partial X_1^{\alpha_1} \partial X_2^{\alpha_2}} = \frac{(-1)^{\alpha_1} \cdot (d + \alpha_1 - 3)! \cdot (-1)^{\alpha_2} \cdot (d + \alpha_1 - 2)_{\alpha_2}}{(d-3)! \cdot (X_1 + \cdots + X_d)^{d+\alpha_1-2+\alpha_2}}$$
$$= \frac{(-1)^{\alpha_1+\alpha_2} \cdot (d - 3 + \alpha_1 + \alpha_2)!}{(d-3)! \cdot (X_1 + \cdots + X_d)^{d-2+\alpha_1+\alpha_2}}, \ \ldots,$$

(2)
$$\frac{\partial^{\alpha_1+\ldots+\alpha_{d-1}} f(X_1, \ldots, X_d)}{\partial X_1^{\alpha_1} \partial X_2^{\alpha_2} \ldots \partial X_{d-1}^{\alpha_{d-1}}} = \frac{(-1)^{\alpha_1+\cdots+\alpha_{d-1}} \cdot (d - 3 + \alpha_1 + \cdots + \alpha_{d-1})!}{(d-3)! \cdot (X_1 + \cdots + X_d)^{d-2+\alpha_1+\cdots+\alpha_{d-1}}}.$$

Since $f(X_1, \ldots, X_d)|_{X_1=x_1^2, \ldots, X_d=x_d^2} = r^{4-2d}$ for $r = \sqrt{x_1^2 + \ldots + x_d^2}$, the conjecture is proven if we show that the functions

$$\frac{\partial^n}{\partial x_1^{\alpha_1} \ldots \partial x_{d-1}^{\alpha_{d-1}}} \left[f(X_1, \ldots, X_d)|_{X_1=x_1^2, \ldots, X_d=x_d^2} \right] = \frac{v_{\alpha_1 \ldots \alpha_{d-1}}(x_1, \ldots, x_d)}{r^{2d+2n-4}}$$

(see (1)) for $\alpha_1, \ldots, \alpha_{d-1} \in \mathbb{N} \cup \{0\}$, $\alpha_1 + \ldots + \alpha_{d-1} = n$, are linearly independent. By *reductio ad absurdum*, we assume that there exists a linear combination

(3)
$$\sum_{\substack{\alpha_1+\cdots+\alpha_{d-1}=n \\ \alpha_1, \ldots, \alpha_{d-1} \geq 0}} C_{\alpha_1, \ldots, \alpha_{d-1}} \cdot \frac{\partial^n}{\partial x_1^{\alpha_1} \ldots \partial x_{d-1}^{\alpha_{d-1}}} \left[f(X_1, \ldots, X_d)|_{X_1=x_1^2, \ldots, X_d=x_d^2} \right] = 0,$$

where not all $C_{\alpha_1, \ldots, \alpha_{d-1}}$ vanish.

Next, let $\widehat{\alpha_1}$ be the biggest value of α_1 such that $C_{\alpha_1, \ldots, \alpha_{d-1}} \neq 0$ for certain $\alpha_2, \ldots, \alpha_{d-1}$. Let then $\widehat{\alpha_2}$ be the biggest value of α_2 as to $C_{\widehat{\alpha_1}, \alpha_2, \alpha_3, \ldots, \alpha_{d-1}} \neq 0$ for certain $\alpha_3, \ldots, \alpha_{d-1}$. Continuing inductively, let eventually $\widehat{\alpha_{d-2}}$ be the biggest value of α_{d-2} for which $C_{\widehat{\alpha_1},\widehat{\alpha_2},\ldots,\widehat{\alpha_{d-3}},\alpha_{d-2},\alpha_{d-1}} \neq 0$ for certain α_{d-1}. Obviously, there is only one such value of α_{d-1}, namely $\widehat{\alpha_{d-1}} := n - \widehat{\alpha_1} - \widehat{\alpha_2} - \ldots - \widehat{\alpha_{d-2}}$.

According to Lemma 1, the term with the highest order monomial $X_1^{j_1} \ldots X_{d-1}^{j_{d-1}}$ in $\dfrac{\partial^n}{\partial x_1^{\alpha_1} \ldots \partial x_{d-1}^{\alpha_{d-1}}} \left[f(X_1, \ldots, X_d)|_{X_1 = x_1^2, \ldots, X_d = x_d^2} \right]$ is

$$c_{1,\alpha_1,\frac{\alpha_1}{2}} \cdot \ldots \cdot c_{d-1,\alpha_{d-1},\frac{\alpha_{d-1}}{2}} \cdot X_1^{\frac{\alpha_1}{2}} \ldots X_{d-1}^{\frac{\alpha_{d-1}}{2}} \cdot \frac{\partial^{\alpha_1 + \ldots + \alpha_{d-1}} f(X_1, \ldots, X_d)}{\partial X_1^{\alpha_1} \ldots \partial X_{d-1}^{\alpha_{d-1}}}.$$

Therefore, after setting $X_d = 1 - X_1 - \ldots - X_{d-1}$ (restricting $x_1, \ldots, x_{d-1}$ to $\left[0, \frac{1}{\sqrt{d-1}}\right]$, that is, $X_1, \ldots, X_{d-1}$ to $\left[0, \frac{1}{d-1}\right]$, which does not affect Lemma 2) and taking (2) into account, the product $X_1^{\frac{\widehat{\alpha_1}}{2}} X_2^{\frac{\widehat{\alpha_2}}{2}} \ldots X_{d-1}^{\frac{\widehat{\alpha_{d-1}}}{2}}$ appears only once in (3), and its coefficient is

$$C_{\widehat{\alpha_1},\ldots,\widehat{\alpha_{d-1}}} \cdot c_{1,\widehat{\alpha_1},\frac{\widehat{\alpha_1}}{2}} \cdot c_{2,\widehat{\alpha_2},\frac{\widehat{\alpha_2}}{2}} \cdot \ldots \cdot c_{d-1,\widehat{\alpha_{d-1}},\frac{\widehat{\alpha_{d-1}}}{2}} \cdot \frac{(-1)^n \cdot (d-3+n)!}{(d-3)!} \neq 0\,,$$

which contradicts Lemma 2. At this point, the proof is completed.

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