# ON HYPERBOLIC INTEGERS

HICHEM GARGOUBI and SAYED KOSSENTINI

*Communicated by Lucian Beznea*

The purpose of this paper is to investigate an $\ell$-ring structure of algebraic integers from an arithmetic point of view. We endow the algebra $\mathbb{D}$ of hyperbolic numbers with its standard $f$-algebra structure [7]. We introduce the ring of hyperbolic integers $\mathcal{Z}_h$ as a sub $f$-ring of the ring $\overline{\mathbb{Z}}^{\mathbb{D}}$ of integers of $\mathbb{D}$. Next, we prove that $\mathcal{Z}_h$ is the unique, up to ring isomorphism, Archimedean $f$-ring of quadratic integers. Our study focuses on arithmetic properties of $\mathcal{Z}_h$ related to its lattice-ordered structure. We show that many basic properties of the ring of integers $\mathbb{Z}$ such as primes, unique factorization theorem and the notions of floor and ceiling functions can be extended to $\mathcal{Z}_h$. A surprising fact is that prime numbers seen as hyperbolic integers are semiprimes. We also obtain some properties of hyperbolic Gaussian integers. As an application, we discuss the Dirichlet divisor problem using hyperbolic intervals.

*AMS 2020 Subject Classification:* 11R04, 06F25.

*Key words:* hyperbolic numbers, hyperbolic Gaussian integers, $f$-ring, $\ell$-group, Riesz space, lattice.

## 1. INTRODUCTION

In order to solve problems concerning certain classes of integers, number theorists of the XIX century are led to study generalizations of the usual arithmetic of the natural numbers in more general settings. In these analogous of $\mathbb{Z}$, concepts like unique factorization into prime elements, Euclidean division and modular arithmetic are developed. One can cite, for instance, Gaussian integers $\mathbb{Z}[i]$ and Kummer's cyclotomic integers $\mathbb{Z}[\exp(2\pi i/n)]$ whose interest came about little by little.

In a process of generalization of the above constructions, Dedekind [3] introduced the notion of ring of integers $\mathcal{O}_K$ of a number field $K$. The ring of integers $\mathbb{Z}$ is the simplest ring of integers. Namely, $\mathbb{Z} = \mathcal{O}_{\mathbb{Q}}$ where $\mathbb{Q}$ is the field of rational numbers. The ring of Gaussian integers $\mathbb{Z}[i]$ is the ring of integers of the number field $\mathbb{Q}(i)$. Also, the ring of cyclotomic integers $\mathbb{Z}[\exp(2\pi i/n)]$ is the ring of integers of the cyclotomic field $\mathbb{Q}(\exp(2\pi i/n))$.

MATH. REPORTS **26(76)** (2024), *2*, 145–170
doi: 10.59277/mrar.2024.26.76.2.145

Another structure of the ring of integers $\mathbb{Z}$ is useful as it corresponds to its order structure. It is clear that one has to take into account the relation between its divisibility and its order properties as the existence of a unique positive gcd and the existence of unique factorization into product of positive primes. In this paper, $\mathbb{Z}$ is seen as an Archimedean $f$-ring. Indeed, the general notion of $f$-algebra is simultaneously a Riesz space (or vector lattice) and an associative real algebra that fulfills certain "positivity" conditions. A typical example of $f$-algebras is the linear space of real valued continuous functions on a topological space. Obviously, the fundamental example of Archimedean $f$-algebras is the field of real numbers.

The purpose of this paper is to answer the following general question: can we extend the order structure of $\mathbb{Z}$ to some of its generalizations in a way compatible with its arithmetic characteristics?

The first part of the answer is given by Theorem 3.1. We prove that there is no analogous of positivity in the ring $\mathcal{O}_K$. Actually, we prove a more general result concerning ring extensions of $\mathbb{Q}$.

This leads us to consider non division extension of real numbers. More precisely, we consider the ring of *hyperbolic numbers*

$$\mathbb{D} = \left\{ z = x + \mathbf{j}y : \quad x, y \in \mathbb{R}, \mathbf{j} \notin \mathbb{R}; \mathbf{j}^2 = 1 \right\}. \tag{1.1}$$

Hyperbolic numbers (also called duplex numbers) are an extension of real numbers defined in the same way as complex numbers $\mathbb{C}$ but with an imaginary unit $\mathbf{j}$ satisfying $\mathbf{j}^2 = 1$ (instead of $\mathbf{i}^2 = -1$). It is clear that $\mathbb{D}$ is not a division algebra. However, it enjoys an important order structure which makes it into the unique (up to an isomorphism) two-dimensional Archimedean $f$-algebra. Therefore, basic notions of real analysis as sign, absolute value, Archimedean and Dedekind completeness are extended to hyperbolic numbers [7]. Note that complex numbers and hyperbolic numbers are the only real commutative Clifford algebras:

$$\mathbb{D} \cong \mathrm{Cl}_{\mathbb{R}}(1,0) \quad \text{and} \quad \mathbb{C} \cong \mathrm{Cl}_{\mathbb{R}}(0,1).$$

The notion of partial order on $\mathbb{D}$ stimulates many authors and leads to interesting applications in different areas of mathematics. Alpay et al. [1] investigated the $\mathbb{D}$-normed bicomplex modulus. In probability theory, it is shown in [2] that Kolmogrov's axioms and Bays' theorems hold in the context of $\mathbb{D}$-valued probabilities. Kumar et al. [15] introduced the notion of $\mathbb{D}$-valued measure on a sigma algebra. As an application to fractal geometry, a concept of Cantor sets in hyperbolic numbers was developed by Balankin et al. [4] and Téllez-Sánchez et al. [24]. Recently, the authors of the present paper used in [8] lattice-theoretical results to go further in the development of the theory of bicomplex zeta function. Further applications are found in [13, 14, 17, 18, 20].

The aim of this paper is to investigate a lattice-ordered ring ($\ell$-ring) of algebraic integers where we are able to generalize many of the basic divisibility and order properties of $\mathbb{Z}$. Our main result (Theorem 3.2) is the existence of a unique, up to ring isomorphism, Archimedean $f$-ring of quadratic integers called the ring of hyperbolic integers denoted $\mathcal{Z}_h$. Namely, $\mathcal{Z}_h$ is the ring of integers of the extension $\mathbb{Q}(\mathbf{j}) = \{\alpha + \mathbf{j}\beta\ ;\ \alpha, \beta \in \mathbb{Q}\}$.

The present paper is organized in the following way: in Section 2, we recall some notions and terminology concerning $\ell$-groups, $f$-rings and $f$-algebras and present basic notions and properties of hyperbolic numbers that are used throughout this article. Section 3 introduces the lattice-ordered ring $\mathcal{Z}_h$ of hyperbolic algebraic integers, and various of its properties are established. We introduce the notions of hyperbolic floor and ceiling functions which generalize that of real numbers. Sections 4 and 5 are devoted to the divisibility in $\mathcal{Z}_h$. Many of the basic concepts of the arithmetic of $\mathbb{Z}$ are extended to $\mathcal{Z}_h$: the Euclidean division, the existence of a unique positive gcd, the existence of a unique factorization into a product of positive primes. In Section 6, we establish some properties of the *hyperbolic Gaussian integers* as a subring of $\mathcal{Z}_h$. As an application, we discuss the Dirichlet divisor problem using hyperbolic intervals.

## 2. PRELIMINARIES

In this section, we recall basic facts which we use throughout this paper.

### 2.1. Basic lattice concepts

We recall some notions and terminology about $\ell$-groups, $f$-rings [5, 23] and $f$-algebras [25].

Let $G$ be a group which is also a partially ordered set. The group operation is denoted additively even if $G$ is not assumed abelian, and so the identity element and the inverse of $a \in G$ are denoted by 0 and $-a$, respectively. $G$ is called a *partially ordered group* if the partial order $\leq$ satisfies: for any $a, b \in G$,

$$a \leq b \Rightarrow a + c \leq b + c \text{ and } c + a \leq c + b \text{ for all } c \in G.$$

In the partially ordered group $G$, an element $a$ is called *positive* if $a \geq 0$. The set $G^+$ of all positive elements is called *the positive cone* of $G$. The partially ordered group $G$ is said to be *Archimedean* if for each nonzero $a$ in $G$ the set $\{na : n \in \mathbb{Z}\}$ has no upper bound in $G$; equivalently, $a, b \in G^+$ and $na \leq b$ for all $n \in \mathbb{N} \Rightarrow a = 0$.

The partially ordered group $G$ is a *lattice-ordered* group (an $\ell$-group) if the partial order is a lattice order (i.e., the supremum $a \vee b$ and the infimum $a \wedge b$ exist in $G$ for all $a, b \in G$). Every element $a$ in an $\ell$-group $G$ can be written as $a = a^+ - a^-$, where $a^+ = a \vee 0$ and $a^- = -a \vee 0$. The absolute value of $a$ is defined as $|a| = a \vee (-a) = a^+ + a^-$. Any Archimedean $\ell$-group is abelian.

A real vectorial space $V$ is said to be a *vector lattice* or *Riesz space* if $V$ as a group is an $\ell$-group satisfying the property: for any $a, b \in V, a \leq b \Rightarrow \alpha a \leq \alpha b$ for all $\alpha \in \mathbb{R}^+$. A ring $R$ is called an *$f$-ring* if $R$ is an $\ell$-group and for any $a, b \in R^+$,

$$ab \in R^+ \text{ and } a \wedge b = 0 \Rightarrow ac \wedge b = ca \wedge b = 0 \text{ for all } c \in R^+.$$

An $f$-ring is Archimedean if its underlying group is Archimedean. For each element $a$ in an $f$-ring, we have $a^+a^- = 0$. Two $f$-rings $R$ and $S$ are called $\ell$-isomorphic if there exists a ring isomorphism $\varphi$ from $R$ to $S$ satisfying $\varphi(a \vee b) = \varphi(a) \vee \varphi(a)$ and $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(a)$ for all $a, b \in R$. An associative real algebra is an $f$-algebra if it is an $f$-ring and its underlying group is a vector lattice.

We give some examples of partially order groups, $\ell$-groups and $f$-rings.

1. The additive group $G = \mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$ is an Archimedean totally $\ell$-group with the usual order between real numbers, and $|x| = x \vee (-x) = \max\{x, -x\}$ for all $x \in G$.

2. Let $(G, P)$ be the partially order group $\mathbb{R} \times \mathbb{R}$ with positive cone $P$.

- If $P = \{(x, y) : x > 0 \text{ or } x = 0 \text{ and } y \geq 0\}$, then $(G, P)$ is a totally ordered group which is not Archimedean since for any $n \in \mathbb{N}$, $n(0, 1) \leq (1, 0)$. The absolute value in $(G, P)$ is given by $|(x, y)| = \max\{(x, y), (-x, -y)\}$.

- If $P = \{(x, y) : x > 0 \text{ and } y > 0 \text{ or } (x, y) = (0, 0)\}$, then $(G, P)$ is an Archimedean partially ordered group but not an $\ell$-group.

- If $P = \{(x, y) : x \geq 0 \text{ and } y \geq 0\}$, then $(G, P)$ is an Archimedean $\ell$-group, and $|(x, y)| = (|x|, |y|)$.

3. Let $\mathbb{Z}[\varepsilon] = \{z = x + \varepsilon y : x, y \in \mathbb{Z}, \varepsilon \notin \mathbb{R}; \varepsilon^2 = 0\}$ be the ring of dual Gaussian integers. Let $z \in \mathbb{Z}[\varepsilon]$ belong to the positive cone $P$ of $\mathbb{Z}[\varepsilon]$ if $\mathrm{Re}(z) > 0$ or $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) \geq 0$. Then $(\mathbb{Z}[\varepsilon], P)$ is a totally $f$-ring but not Archimedean, and the absolute value is given by.

$$|z| = \max\{z, -z\} = \begin{cases} z & \text{if } z \geq 0 \\ -z & \text{if } z \leq 0 \end{cases}$$

4. The ring $\mathbb{Z}[i]$ of Gaussian integers cannot be made into an $f$-ring since $i^2 = -1$ (the squares in any $f$-ring are positive). Nevertheless, there is a partial

order on $\mathbb{Z}[i]$ that makes it into an Archimedean $\ell$-group; namely, $z \leq w$ in $\mathbb{Z}[i]$ if and only if $\mathrm{Re}(z) \leq \mathrm{Re}(w)$ and $\mathrm{Im}(z) \leq \mathrm{Im}(w)$. Thus, for any $z \in \mathbb{Z}[i]$, the absolute value is

$$|z| = |\mathrm{Re}(z)| + i|\mathrm{Im}(z)|.$$

It is worth noticing that both $|z|$ and the modulus $\sqrt{z\bar{z}}$ are generalizations of the usual absolute value in the sens that they coincide on real numbers. However, the first one belongs to $\mathbb{Z}[i]$ as an $\ell$-group, and the second is a positive real number which represents the euclidean distance from $z$ to the origin 0 as a lattice point.

### 2.2. **Hyperbolic numbers**

We recall basic properties of hyperbolic numbers equipped by their natural Archimedean $f$-algebra structure (see [7]). Hyperbolic numbers defined by (1.1) are commutative ring with group of units defined by

$$\mathbb{D}_* = \left\{ z \in \mathbb{D} : \|z\|_h \neq 0 \right\},$$

where $\|z\|_h := z\bar{z} = x^2 - y^2$ denotes the *hyperbolic square-norm* of $z = x + \mathbf{j}y$, $(x, y \in \mathbb{R})$ and $\bar{z}$ is the conjugate of $z$ given by exchanging $y \longleftrightarrow -y$. The hyperbolic plane has an important basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ where

$$\mathbf{e}_1 = \frac{1+\mathbf{j}}{2}, \quad \mathbf{e}_2 = \frac{1-\mathbf{j}}{2}.$$

$\mathbf{e}_1$ and $\mathbf{e}_2$ are mutually complementary idempotent zero divisors, i.e.,

$$\mathbf{e}_1^2 = \mathbf{e}_1; \quad \mathbf{e}_2^2 = \mathbf{e}_2; \quad \mathbf{e}_1 + \mathbf{e}_2 = 1; \quad \mathbf{e}_1\mathbf{e}_2 = 0. \tag{2.1}$$

In this basis, each hyperbolic number $z$ can be written as

$$z = \pi_1(z)\mathbf{e}_1 + \pi_2(z)\mathbf{e}_2, \tag{2.2}$$

where the maps $\pi_1, \pi_2 : \mathbb{D} \to \mathbb{R}$ are a pair of surjective ring homomorphisms defined by

$$\pi_1(x + \mathbf{j}y) = x + y \quad \text{and} \quad \pi_2(x + \mathbf{j}y) = x - y.$$

From representation (2.2), called the *spectral decomposition* [22], algebraic operations correspond to coordinate-wise operations, the square norm of $z$ is the product $\pi_1(z)\pi_2(z)$ and its conjugation is given by exchanging $\pi_1(z) \leftrightarrow \pi_2(z)$. Moreover, we can define a partial order $\leq$ on $\mathbb{D}$ that makes it into Archimedean $f$-algebra, where

$$z, w \in \mathbb{D};\ z \leq w \text{ if and only if } \pi_k(z) \leq \pi_k(w), \quad k = 1, 2. \tag{2.3}$$

From this ordering, the lattice operations are

$$z \vee w = \max\{\pi_1(z), \pi_2(z)\}\mathbf{e}_1 + \max\{\pi_1(z), \pi_2(z)\}\mathbf{e}_2,$$
$$z \wedge w = \min\{\pi_1(z), \pi_2(z)\}\mathbf{e}_1 + \min\{\pi_1(z), \pi_2(z)\}\mathbf{e}_2.$$

Moreover, $z \vee w$ and $z \wedge w$ can be expressed as an $I(\mathbb{D})$-combination of $z$ and $w$, where $I(\mathbb{D})$ means the set of all idempotent elements of $\mathbb{D}$. More precisely, this property is formulated in the following result.

PROPOSITION 2.1 (Proposition 3.1 in [7]). *For any $z, w \in \mathbb{D}$ there exist unique $u, v \in I(\mathbb{D})$ satisfying $uv = 0$ and $u + v = 1$ such that*

$$z \vee w = uz + vw \text{ and } z \wedge w = vz + uw.$$

The Riesz space $\mathbb{D}$ is *Dedekind complete*, i.e., every nonempty subset $A$ that is bounded above (resp., below), has a supremum $\sup A$ (resp., infimum $\inf A$), and

$$\sup A = \sup \pi_1(A)\mathbf{e}_1 + \sup \pi_2(A)\mathbf{e}_2, \tag{2.4}$$

$$\inf A = \inf \pi_1(A)\mathbf{e}_1 + \inf \pi_2(A)\mathbf{e}_2. \tag{2.5}$$

In the ring of hyperbolic numbers there is a multiplicative group $\mathfrak{G}$ called *group of signs* given by

$$\mathfrak{G} = \left\{1, -1, \mathbf{j}, -\mathbf{j}\right\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \tag{2.6}$$

THEOREM 2.1 (Theorem 5.1 in [7]). *Let $z \in \mathbb{D}$, then there exists an element $\varepsilon \in \mathfrak{G}$ such that*

$$\varepsilon z \geq 0.$$

*If $\|z\|_h \neq 0$ then $\varepsilon$ is unique, called sign of $z$, denoted* $\operatorname{sgn}(z)$ *and given by*

$$\operatorname{sgn}(z) = \frac{z}{|z|}.$$

The $f$-algebra $\mathbb{D}$ under the norm

$$\|z\|_R := \min\left\{\alpha \in \mathbb{R}^+ : \alpha \geq |z|\right\} = |z| \vee \overline{|z|} \text{ for all } z \in \mathbb{D},$$

is a unital Banach lattice algebra, i.e., the norm $\|.\|_R$ satisfies the properties: (i) $|z| \leq |w|$ implies $\|z\|_R \leq \|w\|_R$; (ii) $\|zw\|_R \leq \|z\|_R \|w\|_R$ and $\|1\|_R = 1$. As $|z| \vee \overline{|z|} = \max\{|\pi_1(z)|, |\pi_2(z)|\}$, then using the standard basis $\{1, \mathbf{j}\}$ an explicit expression of $\|z\|_R$ is given by the formula

$$\|x + \mathbf{j}y\|_R = \max\{|x + y|, |x - y|\} \text{ for all } x, y \in \mathbb{R}.$$

The Banach algebra structure allows us to define the exponential of $z$, for any hyperbolic number $z$, as

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^{\pi_1(z)}\mathbf{e}_1 + e^{\pi_2(z)}\mathbf{e}_2.$$

The hyperbolic exponential function exp is a group homomorphism from the additive group $\mathbb{D}$ to the multiplicative group $\mathbb{D}_*$. But unlike complex numbers, the hyperbolic exponential function is one-to-one with the group $\exp(\mathbb{D}) = \mathbb{D}_*^+ = \{z = z_1\mathbf{e}_1 + z_2\mathbf{e}_2 : z_1, z_2 \in \mathbb{R}_+^*\}$. Therefore, $\exp : \mathbb{D} \longrightarrow \mathbb{D}_*^+$ is a group isomorphism, and this leads to define the hyperbolic logarithm function as the inverse isomorphism $\ln = \exp^{-1}$.

Given $z \in \mathbb{D}_*^+$ and $\alpha \in \mathbb{D}$, we define the hyperbolic exponentiation $z^\alpha$ as

$$z^\alpha := e^{\alpha \ln(z)} = e^{\pi_1(\alpha)\ln(\pi_1(z))}\mathbf{e}_1 + e^{\pi_2(\alpha)\ln(\pi_2(z))}\mathbf{e}_2.$$

We write $z = z_1\mathbf{e}_1 + z_2\mathbf{e}_2$, $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$. Then, from the above formula, we obtain

$$z^\alpha := z_1^{\alpha_1}\mathbf{e}_1 + z_2^{\alpha_2}\mathbf{e}_2. \tag{2.7}$$

Finally, let us mention that we use the following notation: for any $z, w \in \mathbb{D}$, we write

$$z \prec w \quad \text{if and only if} \quad w - z \in \mathbb{D}_*^+.$$

Therefore, if $z, w \in \mathbb{R}$ then $z < w$ in $\mathbb{R}$ if and only if $z \prec w$ in $\mathbb{D}$.

## 3. HYPERBOLIC INTEGERS

### 3.1. Basic definitions and properties

Let $R$ be a ring extension of $\mathbb{Q}$ with degree $n$, i.e., a unital commutative ring in which its underling group is a $\mathbb{Q}$-vector space with dimension $n$. Therefore, each $\alpha \in R$ is an algebraic number (i.e., a root of a polynomial $P \in \mathbb{Z}[X]$) since it satisfies the equation

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0, \; a_i \in \mathbb{Z}.$$

If, in addition, $a_n = 1$, $\alpha$ is said to be an *algebraic integer*. The set $\mathcal{O}_R$ of all algebraic integers of $R$ is a ring [21, Chapter 2] called *the ring of integers of* $R$. In particular, a finite field extension $K$ of $\mathbb{Q}$ is usually refereed as a *number field*. The ring $\mathcal{O}_K$ of its integers is *Notherian* but in general is not a *unique factorization domain* see [10, Chapter 12]. In lattice-ordered rings framework, we obtain the following result.

Theorem 3.1. *Let $R$ be a ring extension of $\mathbb{Q}$ with degree $\neq 1$. If $R$ is an integral domain, then its ring of integers $\mathcal{O}_R$ cannot be made into Archimedean $f$-ring.*

*Proof.* Let $R$ be an integral ring extension of $\mathbb{Q}$ with degree $d > 1$. Suppose that $\mathcal{O}_R$ can be made into Archimedean $f$-ring. Let $x \in \mathcal{O}_R$ then $x^+ = 0$

or $x^- = 0$ since $x^+x^- = 0$ and $\mathcal{O}_R$ is an integral domain. This means that $x \geq 0$ or $x \leq 0$, since $x$ can be written as $x = x^+ - x^-$ with $x^+, x^- \geq 0$. It follows that for any $u, v \in \mathcal{O}_R$, either $u - v \geq 0$ or $u - v \leq 0$, and hence $u \geq v$ or $u \leq v$; that is, $\mathcal{O}_R$ is a totally ordered group. Therefore, from Hölder's theorem [5, Theorem 2.6.3], $\mathcal{O}_R$ is isomorphic to a subgroup of $\mathbb{R}$. This yields a contradiction since $dim_{\mathbb{Q}} R \neq 1$. This completes the proof. $\square$

The aim of this paper is the characterization of all Archimedean $f$-rings of quadratic integers. From Theorem 3.1, these rings must be non integral domains. In fact, we show that, up to an $\ell$-isomorphism, there is only one Archimedean $f$-ring of quadratic integers, namely the ring of integers of the extension $\mathbb{Q}(\mathbf{j}) = \{\alpha + \mathbf{j}\beta \ ; \ \alpha, \beta \in \mathbb{Q}\}$.

Theorem 3.2 (Hyperbolic integers). *The ring of integers of* $\mathbb{Q}(\mathbf{j})$ *is given by*

$$\mathcal{Z}_h := \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2,$$

*and it is the unique, up to order and ring isomorphism, Archimedean $f$-ring of quadratic integers called the ring of hyperbolic integers.*

*Proof.* The $\mathbb{Z}$-module $\mathcal{Z}_h = \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2$ is a subring and sublattice (closed under $\vee$ and $\wedge$) of the Archimedean $f$-algebra $\mathbb{D}$. So, it is an Archimedean $f$-ring under the partial order induced from $\mathbb{D}$, and for any $u = n\mathbf{e}_1 + m\mathbf{e}_2$ and $v = p\mathbf{e}_1 + q\mathbf{e}_2$, we have

$$u \vee v = \max\{n, p\}\mathbf{e}_1 + \max\{m, q\}\mathbf{e}_2 \text{ and } u \wedge v = \min\{n, p\}\mathbf{e}_1 + \min\{m, q\}\mathbf{e}_2. \tag{3.1}$$

We prove now that $\mathcal{Z}_h$ is the ring of integers of $\mathbb{Q}(\mathbf{j})$. To do this, we use the decomposition $\mathbb{Q}(\mathbf{j}) = \mathbb{Q}\mathbf{e}_1 + \mathbb{Q}\mathbf{e}_2$ that follows from the identities: $\mathbf{e}_1 + \mathbf{e}_2 = 1$ and $\mathbf{e}_1 - \mathbf{e}_2 = \mathbf{j}$. Let $v = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2 \in \mathbb{Q}\mathbf{e}_1 + \mathbb{Q}\mathbf{e}_2$ then $v \in \mathcal{O}_{\mathbb{Q}(\mathbf{j})}$ if and only if there exists $(a, b) \in \mathbb{Z}^2$ such that $v^2 + av + b = 0$, i.e., from (2.1) $(\alpha^2 + a\alpha + b)\mathbf{e}_1 + (\beta^2 + a\beta + b)\mathbf{e}_2 = 0$. This means that $\alpha$ and $\beta$ are the roots of $x^2 + ax + b$. Then, up to a permutation of the roots one has

$$\alpha = \frac{-a + \sqrt{a^2 - 4b}}{2} = \frac{n}{2} \text{ and } \beta = \frac{-a - \sqrt{a^2 - 4b}}{2} = \frac{m}{2}, \tag{3.2}$$

where $n, m, \alpha + \beta$ and $\alpha\beta$ are integers, so that $\frac{n+m}{2}, \frac{nm}{4} \in \mathbb{Z}$. This holds only if $n$ and $m$ are even, i.e., from (3.2) $\alpha, \beta \in \mathbb{Z}$. Hence $\mathcal{Z}_h = \mathcal{O}_{\mathbb{Q}(\mathbf{j})}$.

Let $\mathcal{O}_R$ be the ring of integers of a quadratic ring extension $R$ of $\mathbb{Q}$. Suppose that $\mathcal{O}_R$ is an Archimeadean $f$-ring. Then, it contains an element $b$ having $u_1 = b^+ \neq 0$ and $u_2 = b^- \neq 0$. Otherwise, $v^+ = 0$ or $v^- = 0$ for every $v \in \mathcal{O}_R$, and so it is a totally ordered Archimedean group which implies (by Hölder's theorem [5, Theorem 2.6.3]), that $\mathcal{O}_R$ is isomorphic to a subgroup of $\mathbb{R}$, which is a contradiction. As $\mathcal{O}_R$ is an $f$-ring, we must have

$u_1u_2 = b^+b^- = 0$ which implies that $\{u_1, u_2\}$ are linearly independent over $\mathbb{Q}$. Indeed, $\alpha u_1 + \beta u_2 = 0$ implies $\alpha u_1^2 = 0$ and $\beta u_2^2 = 0$ and so $\alpha = \beta = 0$ since any unital Archimedean $f$-ring is semiprime (i.e., 0 is the only nilpotent element). Let $\alpha_1, \alpha_2 \in \mathbb{Q}$ be such that $1 = \alpha_1 u_1 + \alpha_2 u_2$. Thus, $(v_1, v_2) := (\alpha_1 u_1, \alpha_2 u_2)$ is a basis of $R$ satisfying the properties

$$1 = v_1 + v_2, \quad v_1 v_2 = 0 \Longrightarrow v_1^2 = v_1, v_2^2 = v_2. \tag{3.3}$$

Let now $v = \alpha v_1 + \beta v_2 \in R = \mathbb{Q}v_1 + \mathbb{Q}v_2$. Then, from (3.3), a similar reasoning to that of $\mathbb{Q}(\mathbf{j}) = \mathbb{Q}\mathbf{e}_1 + \mathbb{Q}\mathbf{e}_2$ shows that $v \in \mathcal{O}_R$ if and only $\alpha, \beta \in \mathbb{Z}$, that is, $\mathcal{O}_R = \mathbb{Z}v_1 + \mathbb{Z}v_2$. We claim that it is an Archimedean $f$-ring under the partial order $(nv_1 + mv_2) \le (pv_1 + qv_2)$ if and only if $n \le p$ and $m \le q$ in $\mathbb{Z}$. It is clear that the positive cone is closed under multiplication. Also, $\mathcal{O}_R$ is an $\ell$-group and for any two elements $u = nv_1 + mv_2$ and $v = pv_1 + qv_2$,

$$u \vee v = \max\{n, p\}v_1 + \max\{m, q\}v_2 \text{ and } u \wedge v = \min\{n, p\}v_1 + \min\{m, q\}v_2. \tag{3.4}$$

The Archimedean property follows from that of $\mathbb{Z}$ with the usual order. For the $f$-ring property, we prove that for any positive elements $u, v$ and $w$ we have $wu \wedge v = 0$ whenever $u \wedge v = 0$. Write $u = n_1v_1 + n_2v_2$, $v = m_1v_1 + m_2v_2$ and $w = p_1v_1 + p_2v_2$ with $n_1, n_2, m_1, m_2, p_1, p_2 \in \mathbb{Z}^+$. Let $c_1 = \min\{p_1, p_2, 1\}$ and $c_2 = \max\{p_1, p_2, 1\}$. Thus, from (3.4), we get $c_1(u \wedge v) \le wu \wedge v \le c_2(u \wedge v)$, and hence $u \wedge v = 0$ implies $wu \wedge v = 0$.

It remains to prove that the $f$-rings $\mathcal{O}_R$ and $\mathcal{Z}_h$ are $\ell$-isomorphic. Define the mapping

$$\varphi : nv_1 + mv_2 \mapsto n\mathbf{e}_1 + m\mathbf{e}_2.$$

Clearly $\varphi$ is bijective and (by (3.1), (3.4)), it preserves $\vee$ and $\wedge$. Moreover, it follows from (2.1) and (3.3) that $\varphi$ is also a ring homomorphism. Therefore, $\varphi$ is an $\ell$-isomorphism between the two $f$-rings. $\square$

*Remark* 3.3. Using notations of [21, Chapter 2] one can define the set $\overline{\mathbb{Z}}^{\mathbb{D}}$ as the *integral closure* of $\mathbb{Z}$ in $\mathbb{D}$, i.e., the ring of algebraic integers of $\mathbb{D}$. From Proposition 2.1, it is clear that $\overline{\mathbb{Z}}^{\mathbb{D}}$ is a sublattice of $\mathbb{D}$, under the induced partial order (2.3), and then an Archimedean $f$-ring. One can easily check that $\overline{\mathbb{Z}}^{\mathbb{D}} = \mathbf{e}_1\overline{\mathbb{Z}}^{\mathbb{R}} \oplus \mathbf{e}_2\overline{\mathbb{Z}}^{\mathbb{R}}$. Indeed, an hyperbolic number $\alpha$ is an algebraic integer if and only if $\alpha_1 = \pi_1(\alpha)$ and $\alpha_2 = \pi_2(\alpha)$ are real algebraic integers. From this point of view, the ring of hyperbolic integers $\mathcal{Z}_h$ can be seen as the smallest (with respect to inclusion) sub $f$-ring of $\overline{\mathbb{Z}}^{\mathbb{D}}$ containing $\{\mathbf{e}_1, \mathbf{e}_2\}$.

As for integers of a quadratic field $\mathbb{Q}(\sqrt{d})$, every hyperbolic integer $a$ is the root of a monic polynomial $P \in \mathbb{Z}[X]$ given by

$$P(X) = X^2 - 2\mathrm{Re}(a)X + \|a\|_h,$$

where $\|a\|_h = a\bar{a}$ and $\mathrm{Re}(a)$ is the real part of $a$. However, $\mathcal{Z}_h$ has zero divisors which are the multiples $n\mathbf{e}$ of $\mathbf{e} \in \{\mathbf{e}_1, \mathbf{e}_2\}$ with $n \in \mathbb{Z} \setminus \{0\}$. For the units of $\mathcal{Z}_h$, we have

Proposition 3.1. *The units of $\mathcal{Z}_h$ coincide with the group of signs of $\mathbb{D}$* (2.6)*; that is*

$$U(\mathcal{Z}_h) = \mathfrak{S} = \Big\{1, -1, \mathbf{j}, -\mathbf{j}\Big\}.$$

*Proof.* The units of $\mathcal{Z}_h$ are characterized by all $v \in \mathcal{Z}_h$ such that $\|v\|_h = \pm 1$ since the square norm $\|.\|_h$ is multiplicative and $v$ satisfies: $v^2 + av + b = 0$, where $a, b \in \mathbb{Z}$ with $a = 2\mathrm{Re}(v)$ and $b = \|v\|_h$. Write $v = n\mathbf{e}_1 + m\mathbf{e}_2$, then $\|v\|_h = nm = \pm 1$ if and only if $(n, m) \in \{\pm(1, 1), \pm(1, -1)\}$. As $\mathbf{e}_1 + \mathbf{e}_2 = 1$ and $\mathbf{e}_1 - \mathbf{e}_2 = \mathbf{j}$, we get $v = \pm 1, \pm \mathbf{j}$. $\square$

In the hyperbolic plane $\mathbb{D} \equiv \mathbb{R}^2$, $\mathcal{Z}_h$ is a "square" *full lattice* [19] with the *fundamental parallelepiped* $P = \{z \in \mathbb{D} : 0 \leq z \prec 1\}$ and minimal elements $\pm\mathbf{e}_1$, $\pm\mathbf{e}_2$ (see Figure 1).

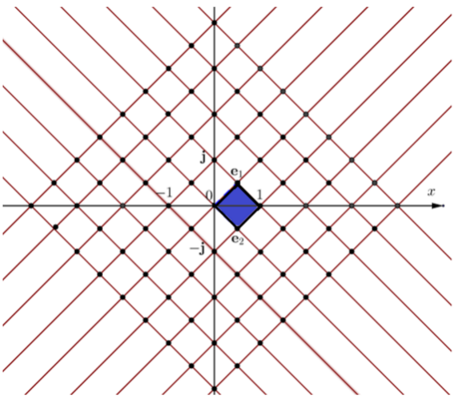


Figure 1 – Hyperbolic integers with fundamental parallelepiped $P$.

Proposition 3.2. *Let $A$ be a nonempty subset of $\mathcal{Z}_h$. Then, the following hold.*

(i) *If $A$ is bounded from above and closed under $\vee$, then it has a largest element.*

(ii) *If $A$ is bounded from below and closed under $\wedge$, then it has a smallest element.*

*Proof.* (i) If $A$ is finite, it is clear that $\max A = \bigvee_{a\in A} a$. Otherwise, $A$ is a countable set which means that $A$ can be viewed as a sequence $(a_n)$. Let $z_n = a_0 \vee \cdots \vee a_n$ for $n = 0, 1, \cdots$. Then, $(z_n)_{n\geq 0}$ is an increasing sequence of $A$ which is bounded above. It follows that $(z_n)$ become constant, i.e., there exists an integer $N \in \mathbb{N}$ such that $z_n = z_N$ for all $n \geq N$. Hence $\max A = z_N$.

(ii) We apply (i) for $-A$ and use the duality formula $\inf A = -\sup(-A)$. □

### 3.2. Ideals of $\mathcal{Z}_h$

In this subsection, we establish some properties involving ideals of $\mathcal{Z}_h$.

PROPOSITION 3.3. *For every ideal $I$ in the ring $\mathcal{Z}_h$ there exists a unique positive element $g_I$ such that $I = g_I\mathcal{Z}_h$. Moreover, $I$ is a sublattice of $\mathcal{Z}_h$.*

*Proof.* Let $I$ be an ideal of the ring $\mathcal{Z}_h$. Since for every $k = 1, 2$ the map $\pi_k$ is a surjective ring homomorphism from $\mathcal{Z}_h$ to $\mathbb{Z}$, then $\pi_k(I)$ is an ideal of the principal ideal domain $\mathbb{Z}$. Therefore, there is a unique positive integer $n_k$ such that $\pi_k(I) = n_k\mathbb{Z}$. Thus, the element $g_I = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$ generates $I$ and it is the only positive one. It follows from Proposition 2.1 that $I$ is a sublattice of $\mathcal{Z}_h$. □

Recall that an $\ell$-subgroup $C$ (i.e., subgroup and sublattice) of an $\ell$-group $G$ is said to be *convex* if $0 \leq a \leq b$ in $G$ and $b \in C$ imply $a \in C$. An *$\ell$-ideal* of an $\ell$-ring $R$ is a convex $\ell$-subgroup of $R$ that is also an ideal of $R$. The following characterizes $\ell$-ideals of the $\ell$-ring $\mathcal{Z}_h$.

PROPOSITION 3.4. *An ideal of $\mathcal{Z}_h$ is an $\ell$-ideal if and only if it is generated by an idempotent element.*

*Proof.* Let $I$ be an ideal of $\mathcal{Z}_h$ with positive generator $g_I$. We see by Proposition 3.3 that $I$ is an $\ell$-subgroup of $\mathcal{Z}_h$. So, $I$ is an $\ell$-ideal if and only if $I$ is convex. Suppose that $g_I$ is an idempotent element, i.e., $g_I \in \{0, 1, \mathbf{e}_1, \mathbf{e}_2\}$. It is obvious that $I$ is convex if $g_I = 0$ or $g_I = 1$. Assume that $g_I = \mathbf{e} \in \{\mathbf{e}_1, \mathbf{e}_2\}$ that means $I = \mathbf{e}\mathcal{Z}_h = \mathbf{e}\mathbb{Z}$. Let $a, b \in \mathcal{Z}_h$ be such that $0 \leq a \leq b$ and $b \in I$. Then $a = \alpha\mathbf{e}$ for some real $\alpha$, since $\mathbf{e}\mathbb{R}$ is an order ideal of $\mathbb{D}$ (see Theorem 3.5 in [7]). We must also have $\alpha \in \mathbb{Z}$, because $a \in \mathcal{Z}_h$. Hence, $a \in I$ and this proves that $I$ is convex. Conversely, assume that $I$ is convex. We have to prove that the generator $g_I$ of $I$ is an idempotent element. We distinguish two cases:

(i) if $\|g_I\|_h = 0$ the case $g_I = 0$ is trivial. Suppose that $g_I = n\mathbf{e}$ for some $\mathbf{e} \in \{\mathbf{e}_1, \mathbf{e}_2\}$ and some integer $n \geq 1$. Then, we have $0 \leq \mathbf{e} \leq n\mathbf{e}$ with $n\mathbf{e} \in I$ which implies that $\mathbf{e} \in I = n\mathbf{e}\mathbb{Z}$. Thus, $nk = 1$ for some $k \in \mathbb{Z}$. This yields that $n = 1$, and hence $g_I = \mathbf{e}$.

(ii) if $\|g_I\|_h \neq 0$, then $0 \leq 1 \leq g_I$ with $g_I \in I$ which means that $1 \in I$, i.e., $I = \mathcal{Z}_h$. Therefore, $g_I = 1$. □

PROPOSITION 3.5. *Let $I$ be an ideal of $\mathcal{Z}_h$ with the positive generator $g_I = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$. Then,*

$$\mathcal{Z}_h/I \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}.$$

*In particular, $\mathcal{Z}_h/I \simeq \mathbb{Z}$ if and only if $I$ is a nontrivial $\ell$-ideal.*

*Proof.* Let us consider next $I$ as an ideal in $\mathcal{Z}_h$ with the positive generator $g_I = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$. One can easily see that the mapping

$$\mathcal{Z}_h/I \ni \dot{a} \mapsto \left(\widehat{\pi_1(a)}, \widetilde{\pi_2(a)}\right) \in \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z},$$

establishes an isomorphism of $\mathcal{Z}_h/I$ with $\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$. In particular, from Proposition 3.4, $\mathcal{Z}_h/I \simeq \mathbb{Z}$ if and only if $I$ is a nontrivial $\ell$-ideal. □

### 3.3. Hyperbolic floor and ceiling functions

Let us consider $z \in \mathbb{D}$. Then the sets $E^+(z) := \{k \in \mathcal{Z}_h : k \leq z\}$ and $E^-(z) := \{k \in \mathcal{Z}_h : k \geq z\}$ are two nonempty sublattices of $\mathcal{Z}_h$. Thus from Proposition 3.2, the notions of floor $\lfloor . \rfloor$ and ceiling $\lceil . \rceil$ functions on real numbers can be extended to the hyperbolic numbers in the following way.

*Definition* 3.4. The functions $\lfloor . \rfloor_{\mathbb{D}}$ and $\lceil . \rceil_{\mathbb{D}}$ from $\mathbb{D}$ to $\mathcal{Z}_h$ defined by

$$\begin{aligned} \lfloor z \rfloor_{\mathbb{D}} &:= \max\Big\{k \in \mathcal{Z}_h : k \leq z\Big\}, \\ \lceil z \rceil_{\mathbb{D}} &:= \min\Big\{k \in \mathcal{Z}_h : k \geq z\Big\} \end{aligned}$$

are called, respectively, hyperbolic floor function and hyperbolic ceiling function.

By (2.4) and (2.5), we derive that

$$\lfloor z \rfloor_{\mathbb{D}} = \lfloor \pi_1(z) \rfloor \mathbf{e}_1 + \lfloor \pi_2(z) \rfloor \mathbf{e}_2 \text{ and } \lceil z \rceil_{\mathbb{D}} = \lceil \pi_1(z) \rceil \mathbf{e}_1 + \lceil \pi_2(z) \rceil \mathbf{e}_2. \tag{3.5}$$

Therefore,

$$z - 1 \prec \lfloor z \rfloor_{\mathbb{D}} \leq z \leq \lceil z \rceil_{\mathbb{D}} \prec z + 1 \quad \text{for all } z \in \mathbb{D}. \tag{3.6}$$

Let us consider a *hyperbolic closed interval* (see [4] and [24]) defined by

$$[\alpha,\beta]_{\mathbb{D}} := \{z \in \mathbb{D} : \alpha \preceq z \preceq \beta\}.$$

Geometrically, $[\alpha,\beta]_{\mathbb{D}}$ is a rectangle (Figure 2) when $\|\alpha-\beta\|_h \neq 0$ called *nondegenerate* interval, otherwise it is a line segment $[\alpha,\beta]$ parallel to one of the two bisector axis. The open interval $(\alpha,\beta)_{\mathbb{D}}$ and half-open intervals $(\alpha,\beta]_{\mathbb{D}}$ and

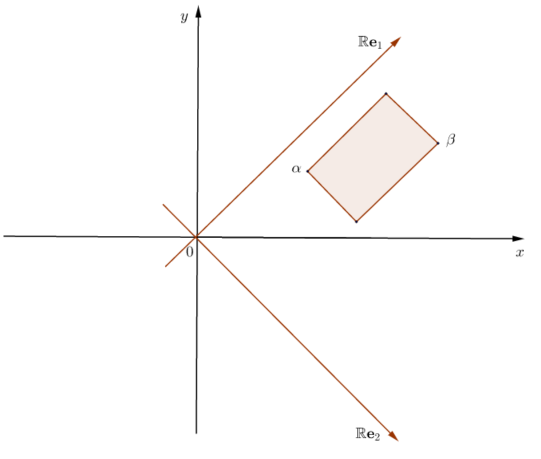


Figure 2 – Nondegenerate hyperbolic closed interval $[\alpha,\beta]_{\mathbb{D}}$.

$[\alpha,\beta)_{\mathbb{D}}$ are defined in a similar way replacing $\preceq$ by $\prec$ in left-right and left/right, respectively. However, all these intervals are empty if $\|\alpha-\beta\|_h = 0$. One has $(\alpha,\beta)_{\mathbb{D}} = [\alpha,\beta]_{\mathbb{D}} \setminus (\partial_\alpha \cup \partial_\beta)$, $(\alpha,\beta]_{\mathbb{D}} = [\alpha,\beta]_{\mathbb{D}} \setminus \partial_\alpha$ and $[\alpha,\beta)_{\mathbb{D}} = [\alpha,\beta]_{\mathbb{D}} \setminus \partial_\beta$ where $\partial_\alpha$ and $\partial_\beta$ are the two edges that meet, respectively, at $\alpha$ and $\beta$. As on real numbers, the functions $\lfloor . \rfloor_{\mathbb{D}}$ and $\lceil . \rceil_{\mathbb{D}}$ allow one to determine the number $N_{\mathcal{Z}_h}(I)$ of hyperbolic integers in a hyperbolic interval $I$ by considering the four types below.

Proposition 3.6. *Let $\alpha,\beta \in \mathbb{D}$ be such that $\alpha \preceq \beta$ then*

$$\begin{aligned}
N_{\mathcal{Z}_h}\left([\alpha,\beta]_{\mathbb{D}}\right) &= \left\| \lfloor \beta \rfloor_{\mathbb{D}} - \lceil \alpha \rceil_{\mathbb{D}} + 1 \right\|_h, \\
N_{\mathcal{Z}_h}\left([\alpha,\beta)_{\mathbb{D}}\right) &= \left\| \lceil \beta \rceil_{\mathbb{D}} - \lceil \alpha \rceil_{\mathbb{D}} \right\|_h, \\
N_{\mathcal{Z}_h}\left((\alpha,\beta]_{\mathbb{D}}\right) &= \left\| \lfloor \beta \rfloor_{\mathbb{D}} - \lfloor \alpha \rfloor_{\mathbb{D}} \right\|_h, \\
N_{\mathcal{Z}_h}\left((\alpha,\beta)_{\mathbb{D}}\right) &= \left\| \lceil \beta \rceil_{\mathbb{D}} - \lfloor \alpha \rfloor_{\mathbb{D}} - 1 \right\|_h.
\end{aligned}$$

*Proof.* Let us denote by $I_1 = [\alpha,\beta]_{\mathbb{D}}$, $I_2 = [\alpha,\beta)_{\mathbb{D}}$, $I_3 = (\alpha,\beta]_{\mathbb{D}}$ and $I_4 = (\alpha,\beta)_{\mathbb{D}}$. Then the sets $\mathcal{Z}_h \cap I_k$ are bijectively mapped onto $\mathbb{Z}^2 \cap \varphi(I_k)$ via the map from $\mathbb{D}$ to $\mathbb{R}^2$ defined by $\varphi(z) = (\pi_1(z), \pi_2(z))$. Thus, for $k = 1, \dots, 4$, $N_{\mathcal{Z}_h}(I_k) = \#\mathbb{Z}^2 \cap R_k$ where $R_k = \varphi(I_k)$ are the rectangles

$$R_1 = [\pi_1(\alpha), \pi_1(\beta)] \times [\pi_2(\alpha), \pi_2(\beta)], \quad R_2 = [\pi_1(\alpha), \pi_1(\beta)) \times [\pi_2(\alpha), \pi_2(\beta)),$$
$$R_3 = (\pi_1(\alpha), \pi_1(\beta)] \times (\pi_2(\alpha), \pi_2(\beta)], \quad R_4 = (\pi_1(\alpha), \pi_1(\beta)) \times (\pi_2(\alpha), \pi_2(\beta)).$$

Therefore,

$$\begin{aligned}
N_{\mathcal{Z}_h}(I_1) &= (\lfloor \pi_1(\beta) \rfloor - \lceil \pi_1(\alpha) \rceil + 1)(\lfloor \pi_2(\beta) \rfloor - \lceil \pi_2(\alpha) \rceil + 1), \\
N_{\mathcal{Z}_h}(I_2) &= (\lceil \pi_1(\beta) \rceil - \lceil \pi_1(\alpha) \rceil)(\lceil \pi_2(\beta) \rceil - \lceil \pi_2(\alpha) \rceil), \\
N_{\mathcal{Z}_h}(I_3) &= (\lfloor \pi_1(\beta) \rfloor - \lfloor \pi_1(\alpha) \rfloor)(\lfloor \pi_2(\beta) \rfloor - \lfloor \pi_2(\alpha) \rfloor), \\
N_{\mathcal{Z}_h}(I_4) &= (\lceil \pi_1(\beta) \rceil - \lfloor \pi_1(\alpha) \rfloor - 1)(\lceil \pi_2(\beta) \rceil - \lfloor \pi_2(\alpha) \rfloor - 1).
\end{aligned}$$

Finally, the results follow from (3.5) and the propriety $\|z\|_h = \pi_1(z)\pi_2(z)$. □

## 4. DIVISIBILITY

### 4.1. First properties

Divisibility in $\mathcal{Z}_h$ is defined naturally: we say $b$ *divides* $a$, or $a$ is a *multiple* of $b$ (and write $b|a$) if $a = bc$ for some $c \in \mathcal{Z}_h$. In this case, we call $b$ a *divisor* of $a$.

Proposition 4.1. *For $a, b \in \mathcal{Z}_h$ we have*

(i) *$a|b$ in $\mathcal{Z}_h$ implies $\|a\|_h \,\Big|\, \|b\|_h \in \mathbb{Z}$;*

(ii) *$a|b$ and $\|b\|_h \neq 0$ implies $|a| \leq |b|$;*

(iii) *$a|b$ and $b|a$ if and only if $|a| = |b|$.*

*Proof.* (i) If $a|b$ then $b = ac$ which implies, by multiplicativity of $\|.\|_h$, that $\|b\|_h = \|a\|_h \|c\|_h$. Hence, $\|a\|_h \,\Big|\, \|b\|_h \in \mathbb{Z}$.
(ii) If $a|b$ and $\|b\|_h \neq 0$, then $b = ca$ for some $c \in \mathcal{Z}_h$ with $\|c\|_h \neq 0$ that means $|c| \geq 1$. Therefore,

$$|b| - |a| = (|c| - 1)|a| \geq 0.$$

(iii) $a|b$ and $b|a$ if and only if $a = \varepsilon b$ for some unit $\varepsilon$, i.e., (by Proposition 5.1 in [7]) if and only if $|a| = |b|$. □

### 4.2. **Hyperbolic euclidean division and congruence**

THEOREM 4.1. *Let $a, b \in \mathcal{Z}_h$ with $\|b\|_h \neq 0$, then there exist unique $q, r \in \mathcal{Z}_h$ such that*

$$a = bq + r, \quad 0 \leq r \prec |b|.$$

*The hyperbolic integers $q$ and $r$ are called, respectively, the quotient and the remainder of the division of $a$ by $b$.*

*Proof.* We consider first the uniqueness. Assume that

$$a = bq_1 + r_1 = bq_2 + r_2, \quad 0 \leq r_1, r_2 \prec |b|.$$

Then,

$$0 \leq |q_1 - q_2| = \frac{|r_1 - r_2|}{|b|} \prec 1.$$

Hence, $q_1 = q_2$ which implies $r_1 = r_2$.

Consider now the existence. Put

$$q = \varepsilon \left\lfloor \frac{a}{|b|} \right\rfloor_{\mathbb{D}} \text{ and } r = a - bq,$$

where $\varepsilon = \operatorname{sgn}(b) = \frac{|b|}{b}$ and $\lfloor . \rfloor_{\mathbb{D}}$ is the hyperbolic floor function (3.4). Then, we have $q, r \in \mathcal{Z}_h$ and $a = bq + r$. It remains to prove that $0 \leq r \prec |b|$. From (3.6) one has

$$\frac{a}{|b|} - 1 \prec \left\lfloor \frac{a}{|b|} \right\rfloor_{\mathbb{D}} \leq \frac{a}{|b|}. \tag{4.1}$$

Multiply (4.1) by $-\varepsilon b = -|b|$ and use $r = a - bq$ to get the desired inequality. □

As for integers, congruences in $\mathcal{Z}_h$ are defined using divisibility.

*Definition* 4.2. Let $a, b, v \in \mathcal{Z}_h$. We further write $a \equiv b \bmod v$ if and only if $v|(a - b)$.

Since congruence modulo 0 means equality and $a|b$ if and only if $|a|\Big|b$, we usually assume the modulus is a nonzero positive element of $\mathcal{Z}_h$.

PROPOSITION 4.2. *For $a, b, c, v \in \mathcal{Z}_h$ one has*

(i) $a \equiv b \bmod v$ *and* $c \equiv d \bmod v$ *imply* $a + c \equiv b + d \bmod v$ *and* $ac \equiv bd \bmod v$;

(ii) *if also* $a, b, v \in \mathbb{Z}$ *then* $a \equiv b \bmod v$ *in* $\mathcal{Z}_h$ *if and only if* $a \equiv b \bmod v$ *in* $\mathbb{Z}$;

(iii) $a \equiv b \bmod v$ *in* $\mathcal{Z}_h$ *if and only if* $\pi_k(a) \equiv \pi_k(b) \bmod \pi_k(v)$ *in* $\mathbb{Z}$ *for* $k = 1, 2$;

(iv) $a \equiv b \bmod v$ *if and only if* $\bar{a} \equiv \bar{b} \bmod \bar{v}$.

*Proof.* The proof is straightforward. $\square$

Let $v \in \mathcal{Z}_h$, $v \succ 0$. Then (by Proposition 3.5), the number of the residue classes modulo $v$ is its square norm $\|v\|_h$. For instance, the four binary classes are the set

$$\mathcal{Z}_h/2\mathcal{Z}_h = \{\hat{0}, \hat{1}, \widehat{\mathbf{e}_1}, \widehat{\mathbf{e}_2}\}. \tag{4.2}$$

## 4.3. **Positive gcd and positive lcm**

According to Proposition 3.3, for every $a, b \in \mathcal{Z}_h$ the ideals $a\mathcal{Z}_h + b\mathcal{Z}_h$ and $a\mathcal{Z}_h \cap b\mathcal{Z}_h$ are generated by a unique positive element. This justifies the following result.

THEOREM 4.3. *Every* $a, b \in \mathcal{Z}_h$ *have a unique positive greatest common divisor* $\gcd_{\mathcal{Z}_h}(a, b)$ *and a unique positive latest common multiple* $\mathrm{lcm}_{\mathcal{Z}_h}(a, b)$. *Moreover,*

$$\begin{aligned} \gcd_{\mathcal{Z}_h}(a, b) &= \gcd(\pi_1(a), \pi_1(b))\mathbf{e}_1 + \gcd(\pi_2(a), \pi_2(b))\mathbf{e}_2\,, \\ \mathrm{lcm}_{\mathcal{Z}_h}(a, b) &= \mathrm{lcm}(\pi_1(a), \pi_1(b))\mathbf{e}_1 + \mathrm{lcm}(\pi_2(a), \pi_2(b))\mathbf{e}_2\,. \end{aligned}$$

As an immediate consequence of Theorem 4.3, we have the following properties of $\gcd_{\mathcal{Z}_h}$ and $\mathrm{lcm}_{\mathcal{Z}_h}$ which are extension of the corresponding ones in $\mathbb{Z}$.

PROPOSITION 4.3. *For* $a, b \in \mathcal{Z}_h$ *we have*

(i) $\gcd_{\mathcal{Z}_h}(|a|, |b|) = \gcd_{\mathcal{Z}_h}(a, b)$ *and* $\mathrm{lcm}_{\mathcal{Z}_h}(|a|, |b|) = \mathrm{lcm}_{\mathcal{Z}_h}(a, b)$;

(ii) $\gcd_{\mathcal{Z}_h}(a, b)\,\mathrm{lcm}_{\mathcal{Z}_h}(a, b) = |ab|$;

(iii) $\overline{\gcd_{\mathcal{Z}_h}(a, b)} = \gcd_{\mathcal{Z}_h}(\bar{a}, \bar{b})$ *and* $\overline{\mathrm{lcm}_{\mathcal{Z}_h}(a, b)} = \mathrm{lcm}_{\mathcal{Z}_h}(\bar{a}, \bar{b})$;

(iv) $\gcd_{\mathcal{Z}_h}(a \vee b, a \wedge b) = \gcd_{\mathcal{Z}_h}(a, b)$ *and* $\mathrm{lcm}_{\mathcal{Z}_h}(a \vee b, a \wedge b) = \mathrm{lcm}_{\mathcal{Z}_h}(a, b)$.

*Remark* 4.4. In view of Proposition 4.1, the *quazi-order* on $\mathcal{Z}_h^+$ defined by

$$a \leq b \text{ if and only if } a|b,$$

is a partial order, and under such order, $\mathcal{Z}_h^+$ is a lattice ordered multiplicative monoid with $a \wedge b = \gcd_{\mathcal{Z}_h}(a, b)$ and $a \vee b = \mathrm{lcm}_{\mathcal{Z}_h}(a, b)$ for all $a, b \in \mathcal{Z}_h^+$.

## 5. PRIMES AND IRREDUCIBLES IN $\mathcal{Z}_h$

In this section, we characterize all prime and irreducible elements of $\mathcal{Z}_h$. We also extend the unique factorization theorem of $\mathbb{Z}$ to $\mathcal{Z}_h$. The set of all prime numbers: $2, 3, 5, 7, 11, \cdots$ is denoted by $\mathbb{P}$. For basic notions and terminology about prime and irreducible elements we refer to [16, Chapter II].

### 5.1. Characterization

THEOREM 5.1 (Hyperbolic primes). *The following statements are satisfied.*

(i) *Prime elements of $\mathcal{Z}_h$ are $u = \varepsilon v$ where $\varepsilon$ is a unit and*

$$v \in \{\mathbf{e}_1, \mathbf{e}_2, p\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + p\mathbf{e}_2 : p \in \mathbb{P}\}.$$

(ii) *Irreducible elements of $\mathcal{Z}_h$ are $u = \varepsilon v$ where $\varepsilon$ is a unit and $v$ is in*

$$\mathcal{P}_h := \{p\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + p\mathbf{e}_2 : p \in \mathbb{P}\}$$

*where $\mathcal{P}_h$ is defined as the set of hyperbolic prime numbers (or hyperbolic primes),*

*Proof.* Let $v = n\mathbf{e}_1 + m\mathbf{e}_2$ be a nonzero and nonunit positive element of the ring $\mathcal{Z}_h$.
(i) By Proposition 3.5, $v$ is prime if and only if $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ is an integral domain, i.e, if and only if one of $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$ is an integral domain and the other one is zero. It follows that $(n, m) \in \{(1, 0), (0, 1), (p, 1), (1, p) : p \in \mathbb{P}\}$. Which means that

$$v \in \{\mathbf{e}_1, \mathbf{e}_2, p\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + p\mathbf{e}_2 : p \in \mathbb{P}\}.$$

(ii) If $v$ is irreducible, then $v\mathcal{Z}_h$ is a maximal ideal, since by Proposition 3.3, every ideal in the ring $\mathcal{Z}_h$ is principal. Therefore, $v\mathcal{Z}_h$ is a prime ideal which means that $v$ is prime. So from (i), either $v = \mathbf{e}_1$ or $v = \mathbf{e}_2$ or $v = p\mathbf{e}_1 + \mathbf{e}_2$ or $v = \mathbf{e}_1 + p\mathbf{e}_2$ for some prime number $p$. We prove that each nonzero-divisor prime element $v$ is irreducible, since the atoms $\mathbf{e}_1$ and $\mathbf{e}_2$ are not. Let $a, b \in \mathcal{Z}_h$ be such that $v = ab$. Then, taking the norm $\|.\|_h$, we obtain $p = \|a\|_h \|b\|_h$. Since $p$ is irreducible in $\mathbb{Z}$ it follows that $\|a\|_h = \pm 1$ or $\|b\|_h = \pm 1$. Hence, either $a \in \mathfrak{G}$ or $b \in \mathfrak{G}$. $\square$

*Remark* 5.2. (i) Theorem (5.1) shows that hyperbolic primes are positive nonzero-divisor prime elements of $\mathcal{Z}_h$ and they are the hyperbolic integers of the form $p\mathbf{e}_1 + \mathbf{e}_2$ or $\mathbf{e}_1 + p\mathbf{e}_2$ with $p \in \mathbb{P}$. Write $p = p\mathbf{e}_1 + p\mathbf{e}_2$, then using the hyperbolic exponentiation (2.7), we have $p\mathbf{e}_1 + \mathbf{e}_2 = p^{\mathbf{e}_1}$ and $\mathbf{e}_1 + p\mathbf{e}_2 = p^{\mathbf{e}_2}$. Thus, hyperbolic primes are the set

$$\mathcal{P}_h := \{p^{\mathbf{e}} : (p, \mathbf{e}) \in \mathbb{P} \times \{\mathbf{e}_1, \mathbf{e}_2\}\}.$$

(ii) It is well known that Gaussian primes are, up to units, prime numbers $p$ with $p \equiv 3 \bmod 4$ or Gaussian integers $z = a + \mathbf{i}b$ with the norm $z\bar{z} = a^2 + b^2$, being a prime number. Similarly, (nonzero-divisor) hyperbolic primes are, up to units, hyperbolic integers $v$ with square norm $\|v\|_h = v\bar{v} = p$, where $p$ is a prime number. However, for each prime number $p$ one has the decomposition $p = p^{\mathbf{e}_1} p^{\mathbf{e}_2}$. It is quite remarkable to see that $p$ viewed as a hyperbolic integer is in fact "semiprime".

### 5.2. **Unique factorization theorem**

The fundamental theorem of arithmetic states that every nonzero integers $n$ can be written uniquely in the form

$$n = \varepsilon \prod_{p \in \mathbb{P}} p^{v_n(p)},$$

where $\varepsilon$ is a unit ($\varepsilon = \operatorname{sgn}(n)$) and $v_n : \mathbb{P} \longrightarrow \mathbb{N}$ with $v_n(p) \neq 0$ for a finite number of $p$.

Using the hyperbolic exponentiation (2.7), the following statements show that this property can be generalized to hyperbolic integers.

Theorem 5.3. *Every $a \in \mathcal{Z}_h$ with $\|a\|_h \neq 0$ can be written uniquely in the form*

$$a = \varepsilon \prod_{p \in \mathbb{P}} p^{\mathbf{v}_a(p)},$$

*where $\varepsilon$ is a unit and $\mathbf{v}_a : \mathbb{P} \longrightarrow \mathcal{Z}_h^+$ with $\mathbf{v}_a(p) \neq 0$ for a finite number of $p$.*

*Proof.* Let $a \in \mathcal{Z}_h$ with $\|a\|_h \neq 0$. By Theorem 2.1, there is a unique unit $\varepsilon \in \mathfrak{G}$ such that $a = \varepsilon|a|$. Let $n_1, n_2 \in \mathbb{N}$ be such that $|a| = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$. Then, $n_1, n_2 \neq 0$. By the fundamental theorem of arithmetic, for every $n \in \mathbb{N}$ there is a unique application $\mu_n : \mathbb{P} \longrightarrow \mathbb{N}$ with $\mu_a(n) = 0$ for almost all $p$ such that

$$n = \prod_{p \in \mathbb{P}} p^{\mu_n(p)}.$$

Let $\mathbf{v}_a : \mathbb{P} \longrightarrow \mathcal{Z}_h^+$ be the function defined by $\mathbf{v}_a(p) = \mu_{n_1}(p)\mathbf{e}_1 + \mu_{n_2}(p)\mathbf{e}_2$. Therefore,

$$|a| = \prod_{p \in \mathbb{P}} p^{\mathbf{v}_a(p)}. \quad \square$$

Theorem 5.4 (Unique factorization theorem). *Every $a \in \mathcal{Z}_h$ with $\|a\|_h \neq 0$ can be written uniquely in the form*

$$a = \varepsilon \prod_{u \in \mathcal{P}_h} u^{v_a(u)},$$

*where $\varepsilon$ is a unit and $v_a : \mathcal{P}_h \longrightarrow \mathbb{N}$ with $v_a(u) = 0$ for almost all $u$.*

*Proof.* The proof follows immediately from Theorem 5.3 by observing that hyperbolic primes are $p^{\mathbf{e}_1}$ and $p^{\mathbf{e}_2}$ with $p \in \mathbb{P}$ (Theorem 5.1). □

## 6. HYPERBOLIC GAUSSIAN INTEGERS

By analogy to complex numbers, the *hyperbolic Gaussian integers* or, more simply, the *h-Gaussian integers* (also called split Gaussian integers [11]) are the set

$$\mathcal{G}_h = \mathbb{Z}[\mathbf{j}] := \Big\{ x + \mathbf{j}y : x, y \in \mathbb{Z} \Big\}.$$

We see next that it is a subring of $\mathcal{Z}_h$ with zero divisors that are the multiples $n(1 \pm \mathbf{j})$, $n \in \mathbb{Z} \setminus \{0\}$, and units that are $1, -1, \mathbf{j}$, and $-\mathbf{j}$. From the four binary classes (4.2) of hyperbolic integers, we have the following characterization of $h$-Gaussian integers.

Theorem 6.1. *Let $a \in \mathcal{Z}_h$, then $a \in \mathcal{G}_h$ if and only if either $a \equiv 0 \bmod 2$ or $a \equiv 1 \bmod 2$.*

*Proof.* Let $a = n\mathbf{e}_1 + m\mathbf{e}_2 = \left(\frac{n+m}{2}\right) + \mathbf{j}\left(\frac{n-m}{2}\right) \in \mathcal{Z}_h$ with $n, m \in \mathbb{Z}$. So, $a \in \mathcal{G}_h$ if and only if $n \equiv m \bmod 2$, i.e., if and only if either $a \equiv 0 \bmod 2$ or $a \equiv 1 \bmod 2$. □

In view of Theorem 2.1 and by the units of $\mathcal{G}_h$ being the set $\mathfrak{S}$, one can see that $\mathcal{G}_h$ is closed under absolute value. But it is not an $\ell$-subgroup of $\mathcal{Z}_h$, since $0 \vee \mathbf{j} = \mathbf{e}_1 \notin \mathcal{G}_h$. However, the next result gives under which condition the supremum of two incomparable (with respect to the order induced by $\mathcal{Z}_h$) $h$-Gaussian integers exists in $\mathcal{G}_h$.

Proposition 6.1. *Let $a, b \in \mathcal{G}_h$ be incomparable, then*

$$a \vee b \in \mathcal{G}_h \text{ if and only if } a \equiv b \bmod 2.$$

*Proof.* Let $a, b \in \mathcal{G}_h$ be incomparable. Then, from Proposition 2.1, we can write

$$a \vee b = ua + vb,$$

for some $u, v \in \{\mathbf{e}_1, \mathbf{e}_2\}$ with $u + v = 1$. Therefore, Theorem 6.1 implies that $a \vee b \in \mathcal{G}_h$ if and only if $a \equiv b \bmod 2$. □

Proposition 6.2. *Every $a \in \mathcal{G}_h$ with $\|a\|_h \neq 0$ can be written uniquely in the form*

$$a = \varepsilon_a 2^\nu \prod_{p \neq 2} p^{\mathbf{v}_a(p)},$$

*where $\varepsilon_a$ is a unit, $\mathbf{v}_a : \mathbb{P} \longrightarrow \mathcal{Z}_h^+$ with $\mathbf{v}_a(p) = 0$ for almost all $p$ and $\nu \in \mathcal{Z}_h^+$ is such that $\nu = 0$ if $a \equiv 1 \bmod 2$ and $\nu \succ 0$ if $a \equiv 0 \bmod 2$.*

*Proof.* From Theorem 5.3, $a$ can be uniquely expressed in the form

$$a = \varepsilon_a \prod_p p^{\mathbf{v}_a(p)} = \varepsilon_a 2^\nu \prod_{p \neq 2} p^{\mathbf{v}_a(p)},$$

where $\varepsilon_a$ is a unit and $\mathbb{P} \xrightarrow{\mathbf{v}_a} \mathcal{Z}_h^+$ with $\mathbf{v}_a(p) = 0$ for almost all $p$ and any $\nu = \mathbf{v}_a(2) \in \mathcal{Z}_h^+$. Therefore, $a \equiv 2^\nu \bmod 2$ since, $p^{\mathbf{v}_a(p)} \equiv 1 \bmod 2$ for $p \neq 2$. It follows that $\nu = 0$ if $a \equiv 1 \bmod 2$ and $\nu \succ 0$ if $a \equiv 0 \bmod 2$. $\square$

## 7. DIRICHLET DIVISOR PROBLEM

The Dirichlet divisor problem, arises from estimating $D(\rho) := \sum_{n \leq \rho} d(n)$, where $d(n)$ is the number of positive divisors of $n$. A well-known result is $D(\rho) = \rho \ln \rho + (2\gamma - 1)\rho + \Delta(\rho)$, where $\gamma$ is Euler's constant and $\Delta(\rho)$ is the error term. The Dirichlet divisor problem asks for the correct order of magnitude of $\Delta(\rho)$ as $\rho \longrightarrow \infty$ (see e.g., [12, Chapter 5]). From a geometrical point of view $D(\rho)$ is equal to the number of lattice points in the first quadrant under the hyperbola $xy = \rho$. Thus, this is equivalent to determine the number of hyperbolic integers $a \succ 0$ with $\|a\|_h \leq \rho$, i.e.,

$$D(\rho) = \#\mathcal{Z}_h \cap \mathcal{D}^+(\rho), \quad \mathcal{D}(\rho) := \{z \in \mathbb{D}_* : \|z\|_h \leq \rho\}. \tag{7.1}$$

Define

$$\mathcal{D}_\star(\rho) = \mathcal{D}(\rho) \cap [-\rho, \rho]_{\mathbb{D}}\,.$$

Geometrically, $\mathcal{D}_\star(\rho)$ is the square $[-\rho, \rho]_{\mathbb{D}}$ if $\rho \leq 1$, and $\mathcal{D}_\star(\rho) \subsetneq [-\rho, \rho]_{\mathbb{D}}$ if $\rho > 1$ as represented in Figure 3.

Proposition 7.1. *We have $D(\rho) = \#\mathcal{Z}_h \cap \mathcal{D}_\star^+(\rho)$.*

*Proof.* It suffices, from (7.1), to prove that $\mathcal{Z}_h \cap \mathcal{D}^+(\rho) = \mathcal{Z}_h \cap \mathcal{D}_\star^+(\rho)$. Then, suppose that

$$\mathcal{Z}_h \cap ([-\rho, \rho]_{\mathbb{D}}^c \cap \mathcal{D}^+(\rho)) \neq \emptyset. \tag{7.2}$$

Observing that $[-\rho, \rho]_{\mathbb{D}}$ is the closed ball $\overline{B_R}(0, \rho)$ in $(\mathbb{D}, \|.\|_R)$ where $\|.\|_R$ is the lattice norm (2.2), then equation (7.2) yields that there exists a hyperbolic integer $a = n\mathbf{e}_1 + m\mathbf{e}_2 \succ 0$ such that $\|a\|_R = \max\{n, m\} > \rho$ and $\|a\|_h = nm \leq \rho$.

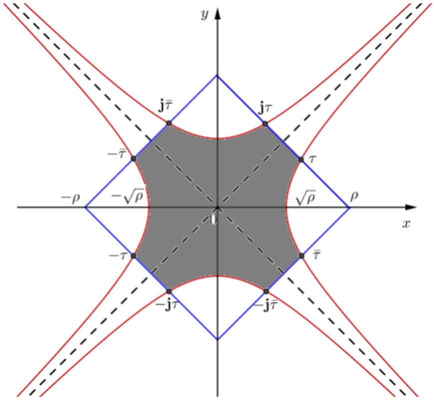


Figure 3 – A representation of $\mathcal{D}_\star(\rho)$ for $\rho > 1$ with $\tau = \sqrt{\rho}e^{\frac{1}{2}\mathbf{j}\ln\rho}$.

So, from the identity $nm = \max\{n,m\}\min\{n,m\}$ one has that $\rho \geq nm > \rho$, which is a contraction. Hence,

$$\mathcal{Z}_h \cap \mathcal{D}^+(\rho) = \mathcal{Z}_h \cap \mathcal{D}^+_\star(\rho). \qquad \square$$

Let $n$ be an integer $\geq 2$. Define $\xi_k$, $\lambda_k$, $\mu_k$ and $\eta_k$ such that

$$\begin{aligned} \xi_k &= \sqrt{\rho}e^{\frac{\mathbf{j}k}{2n}\ln\rho}, \ \lambda_k = \xi_k \wedge \sqrt{\rho}, \quad \text{for } k = 0, \cdots, n; \\ \eta_k &= \xi_k \vee \xi_{k-1}, \ \mu_k = \xi_k \wedge \xi_{k-1}, \quad \text{for } k = 1, \cdots, n. \end{aligned}$$

Thus, and referring to Figure 4, Proposition 7.1 yields that

$$D_n^-(\rho) \leq D(\rho) \leq D_n^+(\rho), \tag{7.3}$$

where

$$D_n^-(\rho) = N_{\mathcal{Z}_h}\left((0,\sqrt{\rho}]_{\mathbb{D}}\right) + 2N_{\mathcal{Z}_h}\left((\alpha,\tau]_{\mathbb{D}}\right) + 2\sum_{k=2}^{k=n} N_{\mathcal{Z}_h}\left((\lambda_k,\xi_{k-1}]_{\mathbb{D}}\right), \tag{7.4}$$

$$D_n^+(\rho) = N_{\mathcal{Z}_h}\left((0,\sqrt{\rho}]_{\mathbb{D}}\right) + 2N_{\mathcal{Z}_h}\left((\alpha,\tau]_{\mathbb{D}}\right) + 2\sum_{k=1}^{k=n} N_{\mathcal{Z}_h}\left((\lambda_k,\eta_k]_{\mathbb{D}}\right). \tag{7.5}$$

Let $\Delta_n^-(\rho)$ and $\Delta_n^+(\rho)$ be such that

$$\begin{aligned} \Delta_n^-(\rho) &= D_n^-(\rho) - (\rho\ln\rho + (2\gamma-1)\rho), \\ \Delta_n^+(\rho) &= D_n^+(\rho) - (\rho\ln\rho + (2\gamma-1)\rho). \end{aligned}$$

Next, let $\delta(\rho)$ be a function defined for $\rho > 1$ by $\delta(\rho) = 0$ if $\rho \notin \mathbb{N}$ and $\delta(\rho) = d(\rho) + \chi_{\mathbb{N}}(\sqrt{\rho})$, otherwise.

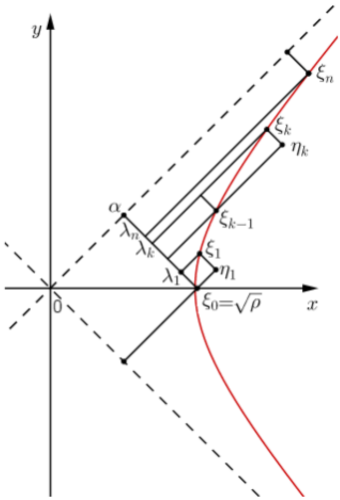


Figure 4 – .

Proposition 7.2. *One has* $\limsup_{n\to\infty}(\Delta_n^+(\rho)-\Delta_n^-(\rho))\leq\delta(\rho)$.

*Proof.* We have

$$\Delta_n^+(\rho)-\Delta_n^-(\rho)=D_n^+(\rho)-D_n^-(\rho)=2\sum_{k=1}^{k=n}N_{\mathcal{Z}_h}\left((\mu_k,\eta_k]_{\mathbb{D}}\right).$$

We further denote by $\mathcal{J}=\mathcal{Z}_h\cap(\mathcal{D}_\star(\rho)\setminus\gamma_\rho^*)$ where $\gamma_\rho^*$ is the image of $\gamma_\rho(t)=\sqrt{\rho}e^{\mathbf{j}t}$ defined in $\left[0,\frac{1}{2}\ln\rho\right]$. Since $\mathcal{J}$ is a nonempty finite set, one obtains $d_{\gamma_\rho^*}=\min\limits_{h\in\mathcal{J}}d(h,\gamma_\rho^*)=\min\limits_{h\in\mathcal{J}}\inf\limits_{\xi\in\gamma_\rho^*}\|h-\xi\|>0$. Let $N$ be an integer such that

$$2\sqrt{\rho}\sinh\left(\frac{\ln\rho}{4N}\right)\sqrt{\cosh(\ln\rho)}<d_{\gamma_\rho^*}.$$

Then, for every $n\geq N$ and for every $k=1,\cdots,n$ we have

$$\begin{aligned}\operatorname{diam}[\mu_k,\eta_k] &= \|\xi_k-\xi_{k-1}\|\\ &\leq 2\sqrt{\rho}\sinh\left(\frac{\ln\rho}{4n}\right)\sqrt{\cosh(\ln\rho)}\\ &\leq 2\sqrt{\rho}\sinh\left(\frac{\ln\rho}{4N}\right)\sqrt{\cosh(\ln\rho)}\\ &< d_{\gamma_\rho^*}\end{aligned}$$

Thus, if $\mathcal{Z}_h\cap(\mu_k,\eta_k]\neq\emptyset$ then for every $h\in\mathcal{Z}_h\cap(\mu_k,\eta_k]$, we have

$$d(h,\gamma_k^*)\leq\operatorname{diam}[\mu_k,\eta_k]<d_{\gamma_\rho^*}$$

where $\gamma_k^* = \gamma_\rho\left(\left[\frac{(k-1)}{2}\ln\rho, \frac{k}{2}\ln\rho\right]\right)$. Therefore, $\mathcal{Z}_h \cap (\mu_k, \eta_k]_{\mathbb{D}} \subset \mathcal{Z}_h \cap \gamma_\rho^*$. Hence

$$\bigcup_{k=1}^{n} \mathcal{Z}_h \cap (\mu_k, \eta_k]_{\mathbb{D}} \subset \mathcal{Z}_h \cap \gamma_\rho^*.$$

It follows from the inclusion above that

$$\Delta_n^+(\rho) - \Delta_n^-(\rho) \le 2\#\mathcal{Z}_h \cap \gamma_\rho^* \quad \text{for all } n \ge N.$$

Therefore,

$$\limsup_{n\to\infty}(\Delta_n^+(\rho) - \Delta_n^-(\rho)) \le \sup_{n\ge N}(\Delta_n^+(\rho) - \Delta_n^-(\rho)) \le 2\#\mathcal{Z}_h \cap \gamma_\rho^*.$$

Thus, it follows from $\mathcal{Z}_h \cap \gamma_\rho^* = \{h \in \mathcal{Z}_h^+, \mathrm{Im}(h) \ge 0 : \|h\|_h = \rho\}$ that

$$2\#\mathcal{Z}_h \cap \gamma_\rho^* = 2\#\{(x,y) \in \mathbb{Z}^2, 0 \ \le x \le y : xy = \rho\} = \delta(\rho).$$

This completes the proof. $\square$

Proposition 7.3. *We have* $0 \le \Delta(\rho) - \Delta^-(\rho) \le \delta(\rho)$ *where*

$$\begin{aligned}
\Delta^-(\rho) &= \psi(\rho) + 2\liminf_{n\to\infty}\left(\sum_{k=2}^{k=n}\left(\lfloor\rho^{\frac{1}{2}+\frac{(k-1)}{2n}}\rfloor - \lfloor\rho^{\frac{1}{2}}\rfloor\right)\left(\lfloor\rho^{\frac{1}{2}-\frac{(k-1)}{2n}}\rfloor - \lfloor\rho^{\frac{1}{2}-\frac{k}{2n}}\rfloor\right)\right),\\
\psi(\rho) &= \lfloor\rho^{\frac{1}{2}}\rfloor^2 + 2\left(\lfloor\rho\rfloor - \lfloor\rho^{\frac{1}{2}}\rfloor\right) - (\rho\ln\rho + (2\gamma-1)\rho).
\end{aligned}$$

*Proof.* We have

$$0 \le \Delta(\rho) - \Delta_n^-(\rho) \le D_n^+(\rho) - D_n^-(\rho).$$

Put $\Delta^-(\rho) = \liminf_{n\to\infty}\Delta_n^-(\rho)$. Thus, from Proposition 7.2 one has

$$0 \le \limsup_{n\to\infty}(\Delta(\rho) - \Delta_n^-(\rho)) = \Delta(\rho) - \Delta^-(\rho) \le \delta(\rho).$$

From the above inequality, we have

$$\begin{aligned}
\Delta_n^-(\rho) &= \psi(\rho) + 2\sum_{k=2}^{k=n} N_{\mathcal{Z}_h}\left((\lambda_k, \xi_{k-1}]_{\mathbb{D}}\right),\\
\psi(\rho) &= N_{\mathcal{Z}_h}\left((0,\sqrt{\rho}]_{\mathbb{D}}\right) + 2N_{\mathcal{Z}_h}\left((\alpha,\tau]_{\mathbb{D}}\right) - (\rho\ln\rho - (2\gamma-1)\rho).
\end{aligned}$$

Straightforward calculations give

$$\tau = \rho\mathbf{e}_1 + \mathbf{e}_2, \quad \alpha = \rho^{\frac{1}{2}}\mathbf{e}_1,$$

$$\lambda_k = \rho^{\frac{1}{2}}\mathbf{e}_1 + \rho^{\frac{n-k}{2n}}\mathbf{e}_2, \quad \xi_k = \rho^{\frac{n+k}{2n}}\mathbf{e}_1 + \rho^{\frac{n-k}{2n}}\mathbf{e}_2, \quad \eta_k = \rho^{\frac{n+k}{2n}}\mathbf{e}_1 + \rho^{\frac{n-(k-1)}{2n}}\mathbf{e}_2.$$

Therefore, from Proposition 3.6, we have

$$\begin{aligned} N_{\mathcal{Z}_h}\left((0,\sqrt{\rho}]_{\mathbb{D}}\right) &= \lfloor \rho^{\frac{1}{2}} \rfloor^2, \\ N_{\mathcal{Z}_h}\left((\alpha,\tau]_{\mathbb{D}}\right) &= \lfloor \rho \rfloor - \lfloor \rho^{\frac{1}{2}} \rfloor, \\ N_{\mathcal{Z}_h}\left((\lambda_k,\xi_{k-1}]_{\mathbb{D}}\right) &= \left(\lfloor \rho^{\frac{n+(k-1)}{2n}} \rfloor - \lfloor \rho^{\frac{1}{2}} \rfloor\right)\left(\lfloor \rho^{\frac{n-(k-1)}{2n}} \rfloor - \lfloor \rho^{\frac{n-k}{2n}} \rfloor\right). \end{aligned}$$

Hence

$$\begin{aligned} \Delta_n^-(\rho) &= \psi(\rho) + 2\sum_{k=2}^{k=n}\left(\lfloor \rho^{\frac{1}{2}+\frac{(k-1)}{2n}} \rfloor - \lfloor \rho^{\frac{1}{2}} \rfloor\right)\left(\lfloor \rho^{\frac{1}{2}-\frac{(k-1)}{2n}} \rfloor - \lfloor \rho^{\frac{1}{2}-\frac{k}{2n}} \rfloor\right), \\ \psi(\rho) &= \lfloor \rho^{\frac{1}{2}} \rfloor^2 + 2\left(\lfloor \rho \rfloor - \lfloor \rho^{\frac{1}{2}} \rfloor\right) - (\rho\ln\rho - (2\gamma-1)\rho). \end{aligned}$$

Finally,

$$\Delta^-(\rho) = \psi(\rho) + 2\liminf_{n\to\infty}\left(\sum_{k=2}^{k=n}\left(\lfloor \rho^{\frac{1}{2}+\frac{(k-1)}{2n}} \rfloor - \lfloor \rho^{\frac{1}{2}} \rfloor\right)\left(\lfloor \rho^{\frac{1}{2}-\frac{(k-1)}{2n}} \rfloor - \lfloor \rho^{\frac{1}{2}-\frac{k}{2n}} \rfloor\right)\right).$$

□

THEOREM 7.1. *For every real $\epsilon > 0$, we have*

$$\Delta(\rho) = \psi(\rho) + 2\liminf_{n\to\infty}\left(\sum_{k=2}^{k=n}\left(\lfloor \rho^{\frac{1}{2}+\frac{(k-1)}{2n}} \rfloor - \lfloor \rho^{\frac{1}{2}} \rfloor\right)\left(\lfloor \rho^{\frac{1}{2}-\frac{(k-1)}{2n}} \rfloor - \lfloor \rho^{\frac{1}{2}-\frac{k}{2n}} \rfloor\right)\right) + O(\rho^{\epsilon}).$$

*Proof.* From Proposition 7.3, $\Delta(\rho)$ is given by $\Delta^-(\rho)$ and the error is the order of $\delta(\rho)$. Thus, the proof follows from the definition of $\delta$ by observing that $d(n) = O(n^{\epsilon})$ for every $\epsilon > 0$. □

**Acknowledgments.** We are grateful to Mounir Bezzarga for enlightening discussions and fruitful comments.

REFERENCES

[1] D. Alpay, M.E. Luna-Elizarrarás, M. Shapiro, and D.C. Struppa, *Basics of Functional Analysis with Bicomplex Scalars, and Bicomplex Schur Analysis.* SpringerBriefs Math., Springer Science & Business Media, 2014.

[2] D. Alpay, M.E. Luna-Elizarrarás, and M. Shapiro, *Klomogrov's axioms for probabilities with values in hyperbolic numbers.* Adv. Appl. Clifford Algebr. **27** (2017), *2*, 913–929.

[3] R. Dedekind, *Theory of Algebraic Integers.* (A translation of *Über die theorie der ganzen algebraischen zahlen*, 1932). Cambridge Univ. Press, 1996.

[4] A.S. Balankin, J. Bory-Reyes, M.E. Luna-Elizarrarás, and M. Shapiro, *Cantor-type sets in hyperbolic numbers.* Fractals **24** (2016), *4*, article no. 1650051.

[5] A. Bigard, K. Keimel, and S. Wolfenstein, *Groupes et anneaux réticulés.* Lecture Notes in Math. **608**, Springer, Berlin, 1977.

[6] F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti, and E. Zampatti, *The Mathematics of Minkowski Space-Time.* Front. Math., Birkhäuser, Basel, 2008.

[7] H. Gargoubi and S. Kossentini, *$f$-algebra structure on hyperbolic numbers.* Adv. Appl. Clifford Algebr. **26** (2016), *4*, 1211–1233.

[8] H. Gargoubi and S. Kossentini, *Bicomplex numbers as a normal complexified $f$-algebra.* Commun. Math. **30** (2022), *1*.

[9] R. Guy, *Unsolved Problems in Number Theory* (Vol. 1). Probl. Books in Math., Springer Science & Business Media, New York, 2004.

[10] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory,* Second Edition. Springer Science & Business Media, New York, 1982.

[11] J.C. Cifuente, J.E. Strapasson, A.C. Corrêa, and P.M. Kitani, *The ring of integers in the canonical structures of the planes.* Preprint, arXiv: 0707.0700, 2007.

[12] E. Krätzel, *Lattice Points.* Mathematics and its Applications **33**. Springer Science & Business Media, 1989.

[13] R. Kumar, K. Singh, H. Saini, and S. Kumar, *Bicomplex weighted Hardy spaces and bicomplex $C^*$-algebra.* Adv. Appl. Clifford Algebr. **26** (2016), *1*, 217–235.

[14] R. Kumar and H. Saini, *Topological bicomplex modules.* Adv. Appl. Clifford Algebr. **26** (2016), *4*, 1249–1270.

[15] R. Kumar, K. Sharma, R. Tundup, and S. Wazir, *Orlicz Spaces with Bicomplex Scalars.* Preprint, arXiv:1401.7112v2, 2017.

[16] S. Lang, *Algebra.* Undergrad. Texts Math. **211**, Springer, New York, 2005.

[17] M.E. Luna-Elizarrarás, C.O. Perez-Ragalado, and M. Shapiro, *On linear functionals and Hahn–Banach theorems for hyperbolic and bicomplex modules.* Adv. Appl. Clifford Algebr. **24** (2014), 1105–1129.

[18] M.E. Luna-Elizarrarás, C.O. Perez-Ragalado, and M. Shapiro, *On the Laurent series for bicomplex holomorphic functions.* Complex Var. Elliptic Equ. **62** (2017), *9*, 1–21.

[19] J. Martinet, *Perfect Lattices in Euclidean Spaces.* Die Grundlehren der mathematischen Wissenschaften **327**, Springer, Berlin, 2003.

[20] D. Rochon and S. Trembla, *Bicomplex quantum mechanics II. The Hilbert space.* Adv. Appl. Clifford Algebr. **16** (2006), *2*, 135–157.

[21] P. Samuel, *Théorie algébrique des nombres.* Collection methodes, Hermann & Cie, Paris, 1967.

[22] G. Sobczyk, *The hyperbolic number plane.* College Math. J. **26** (1995), *4*, 268–280.

[23] S. Steinberg, *Lattice-Ordered Rings and Modules.* Springer, Dordrecht, 2010.

[24] G.Y. Téllez-Sánchez and J. Bory-Reyes, *More about Cantor-like sets in hyperbolic numbers.* Fractals **25** (2017), *5*, article no. 1750046.

[25] A.C. Zaanen, *Riesz Spaces II.* North-Holland Mathematical Library **30**, North-Holland, Amsterdam, 1983.

*Received 10 March 2022*

*Hichem Gargoubi*
*University of Tunis*
*I.P.E.I.T., Department of Mathematics*
*2 Street Jawaher Lel Nehru, Monfleury, 1008 Tunis, Tunisia*
hichem.gargoubi@ipeit.rnu.tn

*Sayed Kossentini*
*University of Tunis El Manar*
*Faculty of Sciences of Tunis*
*Department of Mathematics*
*Campus University 2092, Tunis, Tunisia*
kossentinisayed@hotmail.fr