THE DIAMAGNETIC INEQUALITY FOR THE HEAT SEMIGROUP IN HERMITIAN BUNDLES OVER COMPACT RIEMANNIAN MANIFOLDS

ALEXANDRU MUSTĂȚEA

Communicated by Lucian Beznea

This article presents a geometric-flavoured proof of the diamagnetic inequality for the heat semigroup in a Hermitian bundle based on Chernoff's theorem about the approximation of contraction semigroups in Banach spaces.

AMS 2020 Subject Classification: 58J90, 58J35, 35K08, 46E40, 53B21.

Key words: heat semigroup, diamagnetic inequality, Kato inequality, Hermitian bundle, Riemannian manifold.

If M is a measure space and A and B are operators on $L^2(M)$ that generate the semigroups e^{-tA} and e^{-tB}, B. Simon has given necessary and sufficient conditions in [9] that allow one to"compare" these two semigroups, more precisely that ensure that the "diamagnetic inequality" $|\mathrm{e}^{-tA} f| \leq \mathrm{e}^{-tB} |f|$ holds pointwise almost everywhere for all $f \in L^2(M)$. Extending this work, it is interesting to replace the function f with square-integrable sections in vector bundles over M. Necessary and sufficient conditions have been obtained in order for a similar "diamagnetic inequality" to hold, too; some of these results are mentioned in [7, Appendix B.4]. Unfortunately, these conditions are not easy to verify in practice, therefore, in the present paper, we use a completely new strategy in order to prove that the diamagnetic inequality holds in the important case when A and B are the unique self-adjoint extensions of the connection Laplacean in a Hermitian bundle over a compact Riemannian manifold M and, respectively, of the opposite of the Laplace–Beltrami operator of M.

1. INTRODUCTORY RESULTS

Let M be a closed Riemannian manifold of dimension n, and $E \to M$ a Hermitian vector bundle of complex rank $r < \infty$ over M; the fiber of E over $x \in M$ is denoted E_x. We do not place any other restriction on M or E.

MATH. REPORTS **26(76)** (2024), *2*, 171–182
doi: 10.59277/mrar.2024.26.76.2.171

We denote by $d : M \times M \to [0, \infty)$ the distance induced on M by the Riemannian structure. Let $\operatorname{injrad}(M)$ denote the injectivity radius of M (for details, see [2, p. 118]). For any Banach (or, respectively, Hilbert) space X, its norm is denoted by $\|\cdot\|_X$ (and its Hermitian product, respectively, by $\langle \cdot, - \rangle_X$), and Id_X denotes the identity map. All the Hermitian products used in this text are linear in the first argument. $C(M)$ denotes the space of continuous (and therefore bounded) complex-valued functions on M, endowed with the supremum norm. For bounded linear operators between normed spaces, $\|\cdot\|_{op}$ denotes their operator norm, the spaces being clear from the context. The measure on M obtained using the Riemannian metric is μ. Next, if s is a section of E, the notation $\|s\|$ (without any other index) denotes the function $M \ni x \mapsto \|s(x)\|_{E_x} \in [0, \infty)$. $\Gamma(E)$ is the space of smooth sections in E and $\Gamma^2(E)$ is the space of classes of equivalence of sections in E under equality almost everywhere and having the property that $\|s\| \in L^2(M)$. It is known that $\Gamma(E)$ is dense in $\Gamma^2(E)$.

Furthermore, if ∇ is a Hermitian connection in E (more specifically, $X\langle s, s'\rangle = \langle \nabla_X s, s'\rangle + \langle s, \nabla_X s'\rangle$ for all $X \in \Gamma(TM)$ and $s, s' \in \Gamma(E)$), the connection Laplacean $\nabla^*\nabla : \Gamma(E) \subset \Gamma^2(E) \to \Gamma^2(E)$ is positive-definite and symmetric, therefore it admits a unique densely-defined, positive-definite, self-adjoint extension $H_\nabla : \mathrm{Dom}(H_\nabla) \to \Gamma^2(E)$, for which $\Gamma(E)$ is an essential domain. In the particular case of the trivial bundle $M \times \mathbb{C} \to M$ endowed with the usual Hermitian structure and the connection d (the usual differential), the corresponding operator is H_d.

Since $\operatorname{spec} H_\nabla \subseteq [0, \infty)$, we may deduce from the spectral theorem that the resolvent of H_∇ at every $\lambda < 0$ has the property that

$$\|(H_\nabla - \lambda)^{-1}\|_{op} = \sup \left\{ \left| \frac{1}{\mu - \lambda} \right| \mid \mu \in \operatorname{spec} H_\nabla \right\} \le \sup \left\{ \left| \frac{1}{\mu - \lambda} \right| \mid \mu \in [0, \infty) \right\} = \frac{1}{|\lambda|}$$

so, using the Hille–Yoshida theorem ([3, Corollary 2.22]), we deduce that $-H_\nabla$ generates a strongly-continuous contraction semigroup in $\Gamma^2(E)$ that we denote $(\mathrm{e}^{-tH_\nabla})_{t \ge 0}$.

THEOREM 1.1. *There exists a function $h : (0, \infty) \times M \times M \to [0, \infty)$ called* ***the heat kernel*** *of M with the following properties:*

1. *$h > 0$;*

2. *h is smooth;*

3. *$h(\cdot, -, y)$ satisfies the homogeneous heat equation $(\partial_t - \Delta)u = 0$ for all $y \in M$, where Δ is the Laplace–Beltrami operator on M;*

4. $\lim_{t\to 0}\int_M h(t,x,y)\,f(y)\,\mathrm{d}y = f(x)$ *for all* $x \in M$ *and all* $f \in C(M)$*;*

5. h *is uniquely determined by the above properties;*

6. $h(t,x,y) = h(t,y,x)$ *for all* t, x, y*;*

7. h *enjoys the "convolution" property*

$$\int_M h(u,y,p)\,h(v,p,z)\,\mathrm{d}p = h(u+v,y,z)$$

for all $u, v > 0$ *and* $y, z \in M$*;*

8. $\int_M h(t,x,y)\,\mathrm{d}y = 1$*; for all* $t > 0$ *and* $x \in M$*;*

9. $(\mathrm{e}^{-tH_\mathrm{d}} f)(x) = \int_M h(t,x,y)\,f(y)\,\mathrm{d}y$ *for all* $f \in L^2(M)$ *and almost all* $x \in M$*.*

Proof. All these statements are proved across multiple references: properties (1)-(7) may be obtained by corroborating, for instance, [1, Chapter VIII, Theorem 4] with [5, Theorem 7.13]; property (8) is a consequence of [1, Chapter VIII, Theorem 5], and property (9) is [5, Theorem 9.5]. □

THEOREM 1.2 (Chernoff). *Let* X *be a Banach space and let* $\mathcal{B}(X)$ *be the algebra of bounded operators on* X*. Suppose that* $[0,\infty) \ni t \mapsto Q_t \in \mathcal{B}(X)$ *is a family of bounded operators with* $Q_0 = \mathrm{Id}$ *and that there exists* $a \in \mathbb{R}$ *such that* $\|Q_t\|_{op} \leq \mathrm{e}^{ta}$ *for all* $t \geq 0$*. Let* $C \subseteq X$ *be an essential domain for the generator* Z *of a strongly-continuous* 1*-parameter semigroup* $(T_t)_{t\geq 0}$ *on* X*. If* $\lim_{t\to 0}\frac{1}{t}(Q_t u - u) = Zu$ *for all* $u \in C$*, then* $T_t u = \lim_{k\to\infty}(Q_{\frac{t}{k}})^k u$ *for all* $u \in X$ *and* $t \geq 0$*. Furthermore, the convergence is uniform with respect to* t *from bounded subsets of* $[0,\infty)$*.*

Proof. The uniformity with respect to t is obvious from the proof (see [3, Lemma 3.28]). □

We make use of the parallel transport in E and, as a consequence, we remind that the parallel transport along a smooth curve $c\colon [0,1] \to M$ is, for all $t \in [0,1]$, a linear isometry $PT_{c(0)\to c(t)} : E_{c(0)} \to E_{c(t)}$, the unique solution of the differential equation $\nabla_{\dot{c}(t)} PT_{c(0)\to c(t)} = 0$ subjected to the initial condition $PT_{c(0)\to c(0)} = \mathrm{Id}_{E_{c(0)}}$. If the curve c is contained in a coordinate patch which is also a trivialization domain for E, then in these coordinates, we have that $\nabla = \mathrm{d} + A$, where $A = \sum_{i=1}^n A_i\,\mathrm{d}x^i$ is the local connection 1-form, with $A_i \in \mathrm{Mat}_r(\mathbb{C})$ square matrices of dimension r for all $1 \leq i \leq n$, so that the above differential equation may be rewritten as $\dot{U}(t) = -A(c(t))\,(\dot{c}(t))\,U(t)$, with the initial condition $U(0) = \mathrm{Id}_{\mathbb{C}^r}$, the solution of which may be written as the convergent series

$$(1) \qquad U(1) = \mathrm{Id}_{\mathbb{C}^r} + \sum_{k\geq 1} (-1)^k \int_0^1 \mathrm{d}t_1 \dots \int_0^{t_{k-1}} \mathrm{d}t_k \, A(c(t_1)) \dots A(c(t_k)).$$

In this article, the curve c is always be a minimizing geodesic. Since not all pairs of points may be joined by such geodesics, we need to use a cut-off function. Let $\kappa : [0, \infty) \to [0, 1]$ be a smooth function such that $\kappa|_{[0,\frac{1}{3}]} = 1$ and $\kappa|_{[\frac{1}{2},\infty)} = 0$. Let $\rho \in (0, \mathrm{injrad}(M))$. Defining the desired cut-off function $\chi : M \times M \to [0, 1]$ by $\chi(x, y) = \kappa \left(\frac{d(x,y)^2}{\rho^2} \right)$, we notice that χ is smooth (the square is necessary in order to guarantee the smoothness in the neighbourhood of the pairs with $y = x$). Finally, we define a "truncated parallel transport" $P \in \Gamma(E \boxtimes E^*)$ by

$$P(x, y) = \begin{cases} \chi(x, y) \, PT_{y \to x}, & \text{if } d(x, y) < \rho, \\ 0, & \text{otherwise,} \end{cases}$$

where $PT_{y\to x}$ is the parallel transport along the unique minimizing geodesic defined on $[0, 1]$, which can be shown to be smooth on the open subset of $M \times M$ made of those pairs of points that may be joined uniquely by minimizing geodesics (the geodesic neighbourhood of the diagonal of $M \times M$). The fact that χ is smooth and that its support has been chosen to be contained in the geodesic neighbourhood of the diagonal of $M \times M$ ensures that P is smooth, a property that is necessary later. Let us also notice that $\|P(x, y)\|_{op} \leq \chi(x, y) \leq 1$ for all $x, y \in U$.

2. THE MAIN RESULTS

Using the heat kernel, one may define a semigroup $P_t : C(M) \to C(M)$ by $P_0 f = f$ and

$$(P_t f)(x) = \int_M h(t, x, y) \, f(y) \, \mathrm{d}y$$

for all $f \in C(M)$ and all $t > 0$, as explained in [5, Theorem 7.16]. Let L be the generator of this semigroup; it makes sense, then, to use the intuitive notation e^{-tL} instead of P_t. One knows from the general theory of semigroups in Banach spaces (see [3] for details) that the domain of L is given by

$$\mathrm{Dom}(L) = \left\{ u \in C(M); \lim_{t\to 0} \frac{1}{t} (\mathrm{e}^{-tL} u - u) \in C(M) \right\}.$$

In the following, whenever one applies a differential operator to a map that depends on several arguments, the differential operator carries the argument with respect to which it acts as a lower index; in particular, L_y means the operator L acting with respect to y.

LEMMA 2.1. $\mathrm{Dom}(L)$ *contains the smooth functions* $C^\infty(M)$.

Proof. If $u \in C^\infty(M)$ it is clear that $Lu \in C(M)$. We show that

$$\lim_{t\to 0} \frac{1}{t}(\mathrm{e}^{-tL}u - u) = -Lu$$

in the norm topology of $C(M)$.

To begin with, let us show that $[0,\infty) \ni t \mapsto (\mathrm{e}^{-tL}u)(x) \in \mathbb{C}$ is smooth for all $x \in M$. Since h is smooth, the function $(t,x) \mapsto h(t,x,y)\,u(y)$ is smooth for all $y \in M$; being smooth in y, it is also integrable with respect to it. Moreover, since h satisfies the heat equation, we obtain that $\partial_t(h(t,x,y)\,u(y)) = [-L_y h(t,x,y)]\,u(y)$; since M is compact, the latter function is bounded on it, therefore using the dominated convergence theorem, we may differentiate with respect to $t \in (0,\infty)$ under the integral and obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-tL}u)(x) = \int_M -L_y h(t,x,y)\,u(y)\,\mathrm{d}y = \int_M h(t,x,y)\,(-Lu)(y)\,\mathrm{d}y.$$

This argument may be iterated indefinitely, so $(0,\infty) \ni t \mapsto (\mathrm{e}^{-tL}u)(x) \in \mathbb{C}$ is smooth for all $x \in M$. Passing to the limit when $t \to 0$ also gets us the smoothness at 0.

It is easy to see that

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t}(\mathrm{e}^{-tL}u - u)(x) &= \lim_{t\to 0} \partial_t \int_M h(t,x,y)\,u(y)\,\mathrm{d}y \\ &= \lim_{t\to 0} \int_M h(t,x,y)\,(-Lu)(y)\,\mathrm{d}y = (-Lu)(x)\end{aligned}$$

for all $x \in M$.

To illustrate, we consider the function $F_u : [0,\infty) \to C(M)$ given by $F_u(t) = \mathrm{e}^{-tL}u - u + tLu$. We have that $F_u(0)(x) = 0$ and $F_u'(0)(x) = 0$ for all $x \in M$, whence it follows that

$$\begin{aligned}\|F_u(t)\|_{C(M)} &= \sup_{x\in M} |F_u(t)(x)| = \sup_{x\in M} \left| \int_0^t (t-\tau) F_u''(t)(x)\,\mathrm{d}\tau \right| \\ &\le \frac{t^2}{2} \sup_{x\in M} |F_u''(t)(x)| \le \frac{t^2}{2} \int_M h(t,x,y)\,|L^2u|(y)\,\mathrm{d}y \\ &\le \frac{t^2}{2} \|L^2u\|_{C(M)},\end{aligned}$$

which shows that $\lim_{t\to 0}\|F_u(t)\|_{C(M)} = 0$, which means that $\lim_{t\to 0} \frac{1}{t}(\mathrm{e}^{-tL}u - u) = -Lu$ in the norm topology of $C(M)$, as desired, whence $u \in \mathrm{Dom}\,L$ as claimed. $\square$

We have come now to the main technical result of this article. Its main underlying intuition is that the heat semigroup in E may be approximated, at small time, by the product between the heat semigroup acting on functions and the parallel transport in E; semi-formally,

$$(\mathrm{e}^{-tH_\nabla}\sigma)(x) \approx \mathrm{e}^{-tH_\mathrm{d}}[P(x,\cdot)\sigma](x)$$

for small $t \geq 0$. This intuition is made rigorous by the application of Chernoff's theorem to the family of operators $(Q_t)_{t\geq 0}$ defined by

$$(Q_t\sigma)(x) = \int_M h(t,x,y)\, P(x,y)\, \sigma(y)\, \mathrm{d}y$$

and $Q_0\sigma = \sigma$ for all $\sigma \in \Gamma^2(E)$ and all $t > 0$. One notices immediately that, using the contractivity property of $\mathrm{e}^{-tH_\mathrm{d}}$,

$$\begin{aligned}\|Q_t\sigma\|^2_{\Gamma^2(E)} &\leq \int_M \mathrm{d}x\Big(\int_M h(t,x,y)\, \|\sigma(y)\|_{E_y}\mathrm{d}y\Big)^2 \\ &= \big\|\mathrm{e}^{-tH_\mathrm{d}}\|\sigma\|\big\|^2_{L^2(M)} \leq \|\|\sigma\|\|^2_{L^2(M)} = \|\sigma\|^2_{\Gamma^2(E)}\end{aligned}$$

so Q_t is a bounded linear operator in $\Gamma^2(E)$ with $\|Q_t\|_{op}\leq 1$.

THEOREM 2.2. $\lim_{k\to\infty} Q^k_{\frac{t}{k}} = \mathrm{e}^{-tH_\nabla}$ *strongly in* $\Gamma^2(E)$, *uniformly with respect to* t *from bounded subsets of* $[0,\infty)$.

Proof. The proof consists in checking the hypotheses of Chernoff's theorem. $Q_0\sigma = \sigma$ trivially, by construction. We have also shown that $\|Q_t\|_{op}\leq 1$. It remains to check the last (and most difficult) hypothesis, namely that $\lim_{t\to 0}\frac{1}{t}(Q_t\sigma - \sigma) = -H_\nabla\,\sigma$ in the norm topology of $\Gamma^2(E)$ for all $\sigma \in \Gamma(E)$. We first show that this convergence holds almost everywhere, and in a second step that it is valid in the norm topology of $\Gamma^2(E)$.

Since we have seen that the domain of L contains $C^\infty(M)$, it follows that the domain of $L \otimes \mathrm{Id}_{E_x}$, the generator of the heat semigroup $t \mapsto \mathrm{e}^{-tL\otimes\mathrm{Id}_{E_x}}$ acting on the Banach space $C(M,E_x) \simeq C(M) \otimes E_x$ of the continuous maps defined on M with values in the fiber E_x, contains the smooth maps from M to E_x.

In order to show the smoothness of $Q_t\sigma$ with respect to t, let us notice that we may write

$$(Q_t\sigma)(x) = \{\mathrm{e}^{-tL\otimes\mathrm{Id}_{E_x}}[P(x,\cdot)\,\sigma(\cdot)]\}(x).$$

Indeed, since σ is smooth, and $P(x,\cdot)$ is smooth by construction, the product $P(x,\cdot)\,\sigma(\cdot)$ is a smooth map from M to the fiber E_x, therefore, belongs to the domain of $L \otimes \mathrm{Id}_{E_x}$. It follows from the general theory of C_0-semigroups of operators that the map

$$[0,\infty) \ni t \mapsto \mathrm{e}^{-tL\otimes\mathrm{Id}_{E_x}}[P(x,\cdot)\,\sigma(\cdot)] \in C(M,E_x)$$

is differentiable. Even more so, then, the map

$$[0,\infty) \ni t \mapsto \{\mathrm{e}^{-tL\otimes \mathrm{Id}_{E_x}}[P(x,\cdot)\,\sigma(\cdot)]\}(x) = \langle \delta_x, \mathrm{e}^{-tL\otimes \mathrm{Id}_{E_x}}[P(x,\cdot)\,\sigma(\cdot)]\rangle \in E_x$$

is differentiable, because δ_x (the Dirac measure concentrated at x) belongs to the dual of $C(M)$, being a finite Borel regular measure. Since $L \otimes \mathrm{Id}_{E_x}$ is a differential operator and $P(x,\cdot)\,\sigma(\cdot)$ is a smooth map from M to E_x, one may repeat this argument arbitrarily many times, showing that $(t,x) \mapsto (Q_t\sigma)(x)$ is smooth.

Using the definition of Q_t, we have that

$$\begin{aligned}\lim_{t\to 0} \frac{Q_t\sigma - \sigma}{t}(x) &= \frac{\partial}{\partial t}\Big|_{t=0} (Q_t\sigma)(x) = -[(L\otimes \mathrm{Id}_{E_x})\,P(x,\cdot)\,\sigma(\cdot)](x) \\ &= [\Delta_y(PT_{y\to x}\sigma)](x),\end{aligned}$$

for all $x \in M$, where the last line is justified by the fact that $\chi(x,\cdot) = 1$ near x, and $y \mapsto P(x,y)\sigma(y)$ is smooth, so we may replace L by $-\Delta$. Since Δ is a local operator, we may choose around every $x \in M$ some domain U_x of normal coordinates centered at x which, at the same time, is also a local trivialization domain for E. Let $\nabla = \mathrm{d} + A$ in this trivialization, with $A = \sum_{i=1}^n A_i\,\mathrm{d}x^i$ being the local connection 1-form. In order to simplify the notation, we identify the points in this coordinate domain with their images under the inverse $\exp_x^{-1}$ of the Riemannian exponential map at x. With all these preparations, the unique minimizing geodesic defined on $[0,1]$ which joins y to x becomes the line segment $[0,1] \ni u \mapsto x + (1-u)(y-x) \in U_x$, and the parallel transport is given by the series (1)

$$\begin{aligned}(PT_{y\to x}\sigma)(y) = \sigma(y) + \int_0^1 [A(x+(1-u)(y-x))(y-x)]\,\sigma(y)\,\mathrm{d}u + \int_0^1\int_0^u [A(x \\ + (1-u)(y-x))(y-x)\,A(x \\ + (1-v)(y-x))(y-x)]\,\sigma(y)\,\mathrm{d}v\,\mathrm{d}u + R\,,\end{aligned}$$

where R collects all the monomials of degree at least 3 in the components of the vector $y-x$. Since Δ_y is a differential operator of order 2, each of the terms in $\Delta_y R$ is of degree at least 1 in the components of $y-x$, so the evaluation of $\Delta_y R$ at $y = x$ is 0, hence

$$\begin{aligned}[\Delta_y(PT_{y\to x}\sigma)](x) = (\Delta\sigma)(x) + \Delta_y \int_0^1 A(x+(1-u)(y-x))(y-x)\,\sigma(y)\,\mathrm{d}u\Big|_{y=x} \\ + \Delta_y \int_0^1\int_0^u A(x+(1-u)(y-x))(y-x)\,A(x \\ + (1-v)(y-x))(y-x)\,\sigma(y)\,\mathrm{d}v\,\mathrm{d}u\Big|_{y=x}.\end{aligned}$$

Moreover, if f is an arbitrary smooth function, then in normal coordinates centered at x, we may write $(\Delta f)(x) = \sum_{i=1}^n (\partial_i^2 f)(x)$, so that

$$[\Delta_y(PT_{y\to x}\sigma)](x) = \sum_{i=1}^n (\partial_i^2\sigma)(x) + \sum_{i=1}^n \partial_{y_i}^2 \int_0^1 A(x + (1-u)(y-x))(y-x)\,\sigma(y)\,\mathrm{d}u\Big|_{y=x} + \sum_{i=1}^n \partial_{y_i}^2 \int_0^1 \int_0^u A(x+(1-u)(y-x))(y-x)\,A(x + (1-v)(y-x))(y-x)\,\sigma(y)\,\mathrm{d}v\,\mathrm{d}u\Big|_{y=x}.$$

The second term is a sum of monomials of degree 2 in the components of $y-x$, therefore its only non-vanishing terms after the application of ∂_i^2 and the evaluation at $y=x$ are

$$\begin{aligned}
&\sum_{i=1}^n \partial_{y_i}^2 \int_0^1 A(x+(1-u)(y-x))\,(y-x)\,\sigma(y)\,\mathrm{d}u\Big|_{y=x}\\
&= 2\sum_{i=1}^n \int_0^1 \partial_{y_i}A(x+(1-u)(y-x))\,\partial_{y_i}(y-x)\,\sigma(y)\,\mathrm{d}u\Big|_{y=x}\\
&+ 2\sum_{i=1}^n \int_0^1 A(x+(1-u)(y-x))\,\partial_{y_i}(y-x)\,\partial_{y_i}\sigma(y)\,\mathrm{d}u\Big|_{y=x}\\
&= 2\sum_{i=1}^n \int_0^1 (1-u)(\partial_{y_i}A_i)(x+(1-u)(y-x))\,\sigma(y)\,\mathrm{d}u\Big|_{y=x}\\
&+ 2\sum_{i=1}^n \int_0^1 A_i(x+(1-u)(y-x))\,\partial_{y_i}\sigma(y)\,\mathrm{d}u\Big|_{y=x}\\
&= \sum_{i=1}^n (\partial_{y_i}A_i)(x)\,\sigma(x) + 2\sum_{i=1}^n A_i(x)\,(\partial_{y_i}\sigma)(x).
\end{aligned}$$

With the same argument as above, the third term (which is of degree 4 in the components of $y-x$) is

$$\sum_{i=1}^n \int_0^1 A(x+(1-u)(y-x))\,\partial_{y_i}(y-x) \cdot \int_0^u A(x+(1-v)(y-x))\,\partial_{y_i}(y-x)\,\sigma(y)\,\mathrm{d}v\,\mathrm{d}u\Big|_{y=x}$$

$$= 2 \sum_{i=1}^{n} \int_0^1 \int_0^u A_i(x + (1-u)(y-x)) \, A_i(x + (1-v)(y-x)) \, \sigma(y) \, \mathrm{d}v \, \mathrm{d}u \Big|_{y=x}$$

$$= \sum_{i=1}^{n} A_i(x)^2 \sigma(x).$$

Collecting the results obtained so far, we conclude that

$$\text{(2)} \qquad \lim_{t \to 0} \frac{1}{t} (Q_t \sigma - \sigma)(x)$$
$$= \sum_{i=1}^{n} [(\partial_i^2 \sigma)(x) + (\partial_i A_i)(x) \sigma(x) + 2 A_i(x) (\partial_i \sigma)(x) + A_i(x)^2 \sigma(x)]$$

for almost all $x \in M$.

On the other hand, using the formula

$$\nabla^* \nabla = - \sum_{i,j=1}^{n} g^{ij} \nabla_{\partial_i} \nabla_{\partial_j} + \sum_{i,j,k=1}^{n} g^{ij} \Gamma_{ij}^k \nabla_{\partial_k},$$

the fact that $\nabla_{\partial_i} = \partial_i + A_i$ for all $1 \le i \le n$, and remembering that $g^{ij}(x) = 1$ and $\Gamma_{ij}^k(x) = 0$ for all $1 \le i, j, k \le n$ (because we are working in normal coordinates centered at x), we may write that

$$\text{(3)} \qquad (-\nabla^* \nabla \sigma)(x) =$$
$$= \sum_{i,j=1}^{n} \delta^{ij} [(\partial_i + A_i) \, (\partial_j + A_j) \, \sigma](x) - \sum_{i,j,k=1}^{n} \delta^{ij} \cdot 0 \cdot [(\partial_k + A_k) \, \sigma](x)$$
$$= \sum_{i=1}^{n} [(\partial_i^2 \sigma)(x) + (\partial_i A_i)(x) \, \sigma(x) + 2 A_i(x) \, (\partial_i \sigma)(x) + A_i(x)^2 \sigma(x)].$$

Comparing formulae 2 and 3, we obtain that $\lim_{t \to 0} \frac{1}{t} (Q_t \sigma - \sigma) = -H_\nabla \sigma$ for all $\sigma \in \Gamma(E)$, pointwise. We need to check now that this convergence holds $\Gamma^2(E)$. The map $F_\sigma : [0, \infty) \to \Gamma^2(E)$ given by $F_\sigma(t) = Q_t \sigma - \sigma + t(H_\nabla \sigma)$ is smooth with respect to t, with the same argument as the one in Lemma 2.1. Since we have just shown that $F_\sigma(0) = F_\sigma'(0) = 0$, it follows that

$$\|F_\sigma(t)\|_{\Gamma^2(E)} \le \int_0^t (t - \tau) \, \|F_\sigma''(\tau)\|_{\Gamma^2(E)} \, \mathrm{d}\tau \le \frac{t^2}{2} \sup_{\tau \in [0,t]} \|F_\sigma''(\tau)\|_{\Gamma^2(E)}$$
$$= \frac{t^2}{2} \sup_{\tau \in [0,t]} \|\partial_\tau^2 \, Q_\tau \sigma\|_{\Gamma^2(E)} = \frac{t^2}{2} \sup_{\tau \in [0,t]} \Big(\int_M \|\partial_\tau^2 (Q_\tau \sigma)(x)\|_{E_x}^2 \, \mathrm{d}x \Big)^{\frac{1}{2}}$$
$$= \frac{t^2}{2} \sup_{\tau \in [0,t]} \Big[\int_M \Big\| \int_M \partial_\tau^2 h(\tau, x, y) \, P(x, y) \, \sigma(y) \, \mathrm{d}y \Big\|_{E_x}^2 \mathrm{d}x \Big]^{\frac{1}{2}}$$

$$\leq \frac{t^2}{2} \sup_{\tau \in [0,t]} \Big[\int_M \Big\| \int_M h(\tau, x, y)\, (\Delta_y^2 \otimes \mathrm{Id}_{E_x})[P(x,y)\, \sigma(y)]\, \mathrm{d}y \Big\|_{E_x}^2 \mathrm{d}x \Big]^{\frac{1}{2}}$$

$$\leq \frac{t^2}{2} \sup_{\tau \in [0,t]} \Big[\int_M \Big(\int_M h(\tau, x, y)\, \|(\Delta_y^2 \otimes \mathrm{Id}_{E_x})[P(x,y)\, \sigma(y)]\|_{E_x}\, \mathrm{d}y \Big)^2 \mathrm{d}x \Big]^{\frac{1}{2}}$$

$$\leq \frac{t^2}{2} \sup_{\tau \in [0,t]} \sqrt{\int_M \Big(\int_M h(\tau, x, y)\, \|(\Delta_y^2 \otimes \mathrm{Id}_{E_x})[P(x,y)\, \sigma(y)]\|_{E_x}\, \mathrm{d}y \Big)^2 \mathrm{d}x}.$$

The function

$$M \times M \ni (x,y) \mapsto \|(\Delta_y^2 \otimes \mathrm{Id}_{E_x})[P(x,y)\, \sigma(y)]\|_{E_x} \in [0, \infty)$$

is obviously continuous, therefore it is bounded by some $C > 0$ (which depends on σ and χ, of course). On the other hand, $\int_M h(\tau, x, y)\, \mathrm{d}y = 1$ (this, in particular, makes the supremum $\sup_{\tau \in [0,t]}$ disappear). These facts corroborated with the inequality shown above imply that

$$\|F_\sigma(t)\|_{\Gamma^2(E)} \leq C \sqrt{\mu(M)}\, \frac{t^2}{2}.$$

This means that

$$0 \leq \lim_{t \to 0} \Big\| \frac{Q_t \sigma - \sigma}{t} - (-H_\nabla \sigma) \Big\|_{\Gamma^2(E)} \leq \lim_{t \to 0} C \sqrt{\mu(M)}\, \frac{t}{2} = 0,$$

so the last hypothesis in Chernoff's theorem is checked.

Following, we may now apply Chernoff's theorem, which gives us that $\mathrm{e}^{-tH_\nabla} \sigma = \lim_{k \to \infty} (Q_{\frac{t}{k}})^k \sigma$ for all $\sigma \in \Gamma^2(E)$, as claimed. □

We have shown so far that Q_t is a good approximation of e^{-tH_∇}; a useful consequence of this technical result is the **diamagnetic inequality for the heat semigroup** in E.

THEOREM 2.3.

$$\|(\mathrm{e}^{-tH_\nabla}\, \sigma)(x)\|_{E_x} \leq (\mathrm{e}^{-tH_\mathrm{d}}\, \|\sigma\|)(x)$$

for all $\sigma \in \Gamma^2(E)$ *and almost all* $x \in M$.

Proof. Let $\sigma \in \Gamma^2(E)$. We begin with the inequality

$$\Big\| \Big(Q_{\frac{t}{k}}^k \sigma \Big)(x) \Big\|_{E_x} = \Big\| \int_M h\Big(\frac{t}{k}, x, y_1\Big)\, P(x, y_1) \Big(Q_{\frac{t}{k}}^{k-1} \sigma \Big)(y_1)\, \mathrm{d}y_1 \Big\|_{E_x}$$
$$\leq \int_M h\Big(\frac{t}{k}, x, y_1\Big) \Big\| \Big(Q_{\frac{t}{k}}^{k-1} \sigma \Big)(y_1) \Big\|_{E_{y_1}} \mathrm{d}y_1$$

which, repeated $k-1$ more times, leads to

$$\left\| \left(Q_{\frac{t}{k}}^k \sigma \right) (x) \right\|_{E_x} \leq \int_M \mathrm{d}y_1 \, h\Big(\frac{t}{k}, x, y_1\Big) \ldots \int_M \mathrm{d}y_k \, h\Big(\frac{t}{k}, y_{k-1}, y_k\Big) \|\sigma(y_k)\|_{E_{y_k}}$$
$$= (\mathrm{e}^{-\frac{t}{k} H_{\mathrm{d}}} \ldots \mathrm{e}^{-\frac{t}{k} H_{\mathrm{d}}} \|\sigma\|)(x) = (\mathrm{e}^{-tH_{\mathrm{d}}} \|\sigma\|)(x),$$

where $\|\sigma\|$ is the function $y \mapsto \|\sigma(y)\|_{E_y}$. (One has $\|\sigma\| \in L^2(M)$ tautologically because $\sigma \in \Gamma^2(E)$.)

We have already shown that $\mathrm{e}^{-tH_\nabla} \sigma = \lim_{k\to\infty} Q_{\frac{t}{k}}^k \sigma$ in $\Gamma^2(E)$, therefore there exists a subsequence $(k_i)_{i\geq 0}$ such that $(\mathrm{e}^{-tH_\nabla} \sigma)(x) = \lim_{i\to\infty} (Q_{\frac{t}{k_i}}^{k_i} \sigma)(x)$ for almost all $x \in M$, whence

$$\|(\mathrm{e}^{-tH_\nabla} \sigma)(x)\|_{E_x} = \lim_{i\to\infty} \left\| \left(Q_{\frac{t}{k_i}}^{k_i} \sigma \right)(x) \right\|_{E_x} \leq (\mathrm{e}^{-tH_{\mathrm{d}}} \|\sigma\|)(x)$$

for almost all $x \in M$. $\square$

The diamagnetic inequality just proved is not new, but the proof presented above is. Alternative proofs based on stochastic techniques may be found in [4, Section 9] and in [6, Proposition 2.2], and a more abstract treatment, in the line of thought described in [9], may be found in [8]. Another proof which uses only functional-analytic techniques (but completely different from the one shown in this article) may be found in [7, Section VII.3].

Acknowledgments. I would like to thank Prof. Radu Purice of the "Simion Stoilow" Institute of Mathematics of the Romanian Academy for the invaluable help provided in the elaboration of this work in the course of my doctoral studies, and Prof. Lucian Beznea of the same institute for the discussions which led to the writing down of the present text.

REFERENCES

[1] I. Chavel, *Eigenvalues in Riemannian Geometry.* Pure and Applied Mathematics **115**, Academic Press, Orlando, 1984.

[2] I. Chavel, *Riemannian Geometry. A Modern Introduction,* Second Edition. Cambridge Stud. Adv. Math. **98**, Cambridge Univ. Press, Cambridge, 2006.

[3] E.B. Davies, *One-Parameter Semigroups.* London Math. Soc., Monographs, **15**, Academic Press, London, 1980.

[4] B. Driver and A. Thalmaier, *Heat equation derivative formulas for vector bundles.* J. Funct. Anal. **183**, *1*, 42–108.

[5] A. Grigor'yan, *Heat Kernel and Analysis on Manifolds.* AMS/IP Studies in Advanced Mathematics **47**, American Mathematical Society (AMS), Providence, 2009.

[6] B. Güneysu, *Kato's inequality and form boundedness of Kato potentials on arbitrary Riemannian manifolds.* Proc. Amer. Math. Soc. **142**, *4*, 1289–1300.

[7] B. Güneysu, *Covariant Schrödinger Semigroups on Riemannian Manifolds.* Oper. Theory Adv. Appl. **264**, Birkhäuser, Cham, 2017.

[8] I. Shigekawa, *L^p contraction semigroups for vector valued functions*. J. Funct. Anal. **147** (1997), 69–108.

[9] B. Simon, *Kato's inequality and the comparison of semigroups*. J. Funct. Anal. **32** (1979), 97–101.

Received 9 September 2022

"Simion Stoilow" Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, Bucharest, RO-014700, Romania
Alexandru.Mustatea@imar.ro