AN INTEGRAL FORM OF QUANTUM TOROIDAL $\mathfrak{gl}_1$

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We consider the (direct sum over all $n \in \mathbb{N}$ of the) K-theory of the semi-nilpotent commuting variety of $\mathfrak{gl}_n$, and describe its convolution algebra structure in two ways: the first as an explicit shuffle algebra (i.e., a particular $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-submodule of the equivariant K-theory of a point) and the second as the $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-algebra generated by certain elements $\{\bar{H}_{n,d}\}_{(n,d)\in \mathbb{N}\times\mathbb{Z}}$. As the shuffle algebra over $\mathbb{Q}(q_1, q_2)$ has long been known to be isomorphic to half of an algebra known as quantum toroidal $\mathfrak{gl}_1$, we thus obtain a description of an important integral form of the quantum toroidal algebra.

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1. INTRODUCTION

1.1. Commuting stacks

Moduli spaces of quiver representations and moduli spaces of sheaves are both important settings for geometric representation theory. Moreover, they are very closely connected, in that one can see the same phenomena occur for both classes of moduli spaces. Arguably, nowhere is this more apparent than in the case of the Jordan quiver (namely, the quiver with one vertex and one loop), which corresponds to sheaves on $\mathbb{A}^2$. To be more specific, consider the commuting stack

$$\text{Comm}_n = \Big\{(X,Y) \in \text{Mat}_{n\times n}^{\times 2} \text{ s.t. } [X,Y]=0\Big\}/GL_n$$

where the action of GL_n is by simultaneous conjugation of the matrices X, Y. From the point of view of quivers, Comm_n is the cotangent bundle of the stack $\text{Mat}_{n\times n}/GL_n$ of n-dimensional representations of the Jordan quiver. From the point of view of sheaves, a point of Comm_n describes a length n sheaf on $\mathbb{A}^2$,

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as the commuting endomorphisms X and Y encode an action of $\mathcal{O}_{\mathbb{A}^2}$. Let us consider

$$K = \bigoplus_{n=0}^{\infty} K_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Comm}_n) \tag{1}$$

the (0-th) equivariant algebraic K-theory groups of all commuting stacks considered together. The torus $\mathbb{C}^* \times \mathbb{C}^*$ acts by rescaling the matrices X and Y independently, and thus K is a $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-module, where q_1, q_2 denote the standard characters of $\mathbb{C}^* \times \mathbb{C}^*$. As explained in [13], there is a convolution algebra structure on K which is additive in n (we do not need to review the construction in the present paper, but the interested reader may find an overview in [11, Section 2.3]).

1.2. *K*-theoretic Hall algebras

Upon localization with respect to the fraction field of $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$, the algebra

$$K_{\mathrm{loc}} = K \bigotimes_{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]} \mathbb{Q}(q_1, q_2) \tag{2}$$

is a well-known object in representation theory: it was shown in [13] to match the elliptic Hall algebra of [1], in [3, 12] to match half of quantum toroidal $\mathfrak{gl}_1$ (also known as the Ding–Iohara–Miki algebra, see [16] for an overview of this important algebra), and in [7] to match the shuffle algebra of [2]. However, if one has derived categories (or any other categorification) in mind, knowing K_{loc} is not good enough. Instead, one would hope to solve the following.

Problem 1.1. Describe K as a $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-algebra.

Although certain aspects of Problem 1.1 have been studied ([18, 19]), the bad news is that we do not yet know a complete solution. The good news is that in Theorem 1.2, we provide a complete solution to a closely related problem, which is relevant to the setting of categorified knot invariants and affine Hecke algebras studied in [4, 5]. To set up this closely related problem, let us note that the commuting stack has three variants of interest to us, namely

$$\mathrm{Comm}_n^{\mathrm{nilp}} \subset \mathrm{Comm}_n^{\text{semi-nilp}} \subset \mathrm{Comm}_n \tag{3}$$

where the stack on the left consists of pairs of nilpotent commuting matrices (X, Y), while the stack in the middle allows X to be arbitrary but requires Y to be nilpotent. The (direct sums over all $n \in \mathbb{N}_0$ of the) $\mathbb{C}^* \times \mathbb{C}^*$ equivariant algebraic K-theory groups of the stacks above are denoted by

$$K^{\mathrm{nilp}} \longrightarrow K^{\text{semi-nilp}} \longrightarrow K. \tag{4}$$

The maps above are simply the direct image maps induced by (3), and they are actually algebra homomorphisms with respect to the convolution product (indeed, K^{nilp} and $K^{\text{semi-nilp}}$ are algebras by the exact same construction as K of (1)). All three algebras in (4) have the same localization, i.e., are isomorphic upon tensoring with $\mathbb{Q}(q_1, q_2)$, but the middle one is described explicitly before localization.

THEOREM 1.2. *We have an isomorphism* $\iota^{\text{semi-nilp}} : K^{\text{semi-nilp}} \xrightarrow{\sim} \mathcal{S}$, *where*

$$\mathcal{S} \subset \bigoplus_{n=0}^{\infty} \mathbb{Z}\left[q_1^{\pm 1}, q_2^{\pm 1}\right]\left[z_1^{\pm 1}, \dots, z_n^{\pm 1}\right]^{\text{sym}}$$

is the $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$*-submodule determined by the conditions of Definition* 3.1, *and made into an algebra via the shuffle product* (16).

1.3. **Connection with the elliptic Hall algebra**

Our starting point in the analysis of $K^{\text{semi-nilp}}$ is the fact (proved in [14]) that it is generated as an algebra by the K-theory groups of the closed substacks

$$\big(\mathrm{Mat}_{n\times n} \times \{0\}\big)/GL_n \subset \mathrm{Comm}_n^{\text{semi-nilp}} \tag{5}$$

as n ranges over $\mathbb{N}_0$. The isomorphism $\iota^{\text{semi-nilp}}$ of Theorem 1.2 maps the K-theory group of the substack (5) to

$$\mathbb{Z}\left[q_1^{\pm 1}, q_2^{\pm 1}\right]\left[z_1^{\pm 1}, \dots, z_n^{\pm 1}\right]^{\text{sym}} \cdot F_n \subset \mathcal{S} \tag{6}$$

where

$$F_n = \prod_{1\le i,j\le n} \left(1 - \frac{q_2 z_i}{z_j}\right). \tag{7}$$

The elements F_n were first studied in [2], and we prove the surjectivity of the map $\iota^{\text{semi-nilp}}$ by showing that the elements of (6) also generate $\mathcal{S}$, as n ranges over $\mathbb{N}_0$. As the injectivity of $\iota^{\text{semi-nilp}}$ was established in [17], this proves Theorem 1.2.

Inspired by the elliptic Hall algebra of [1], it was shown in [7] that we have the following equality of $\mathbb{Q}(q_1, q_2)$-vector spaces

$$\mathcal{S}_{\text{loc}} := \mathcal{S} \bigotimes_{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]} \mathbb{Q}(q_1, q_2) = \bigoplus_{\frac{d_1}{n_1} \le \dots \le \frac{d_k}{n_k}} \mathbb{Q}(q_1, q_2) \cdot \bar{H}_{n_1,d_1} \dots \bar{H}_{n_k,d_k} \tag{8}$$

for certain elements $\{\bar{H}_{n,d}\}_{(n,d)\in\mathbb{N}\times\mathbb{Z}}$ of $\mathcal{S}_{\text{loc}}$, that we recall in Section 4. We then prove the following stronger version of the decomposition (8).

LEMMA 1.3. *We have the following equality of $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-modules*

$$\mathcal{S} = \bigoplus_{\frac{d_1}{n_1} \leq \dots \leq \frac{d_k}{n_k}} \mathbb{Z}\big[q_1^{\pm 1}, q_2^{\pm 1}\big] \cdot \bar{H}_{n_1,d_1} \dots \bar{H}_{n_k,d_k} \tag{9}$$

2. THE (SEMI-NILPOTENT) K-THEORETIC HALL ALGEBRA

2.1. Commuting stacks revisited

Let us consider the **commuting variety**

$$\overline{\mathrm{Comm}}_n \stackrel{i_n}{\hookrightarrow} \mathbb{A}^{2n^2} \tag{10}$$

[1] consisting of pairs of commuting $n \times n$ matrices X, Y. We consider the action

$$G_n := \mathbb{C}^* \times \mathbb{C}^* \times GL_n \curvearrowright \mathbb{A}^{2n^2}$$

given by

$$(t_1, t_2, g) \cdot (X, Y) = \left(\frac{1}{t_1} gXg^{-1}, \frac{1}{t_2} gYg^{-1}\right)$$

which preserves $\overline{\mathrm{Comm}}_n$. Thus, (10) induces a map on equivariant K-theory

$$K_{G_n}\big(\overline{\mathrm{Comm}}_n\big) \xrightarrow{i_{n*}} K_{G_n}\big(\mathbb{A}^{2n^2}\big). \tag{11}$$

The **commuting stack** is $\mathrm{Comm}_n = \overline{\mathrm{Comm}}_n / GL_n$, and its K-theory is given by

$$K_{\mathbb{C}^* \times \mathbb{C}^*}\big(\mathrm{Comm}_n\big) = K_{G_n}\big(\overline{\mathrm{Comm}}_n\big) \tag{12}$$

which explains our interest in the map (11).

2.2. The shuffle algebra

If we let $\circ \in \mathbb{A}^{2n^2}$ denote the origin, then the following restriction map

$$K_{G_n}\big(\mathbb{A}^{2n^2}\big) \stackrel{|_\circ}{\cong} K_{G_n}(\mathrm{pt}) = \mathbb{Z}\big[q_1^{\pm 1}, q_2^{\pm 1}\big]\big[z_1^{\pm 1}, \dots, z_n^{\pm 1}\big]^{\mathrm{sym}} \tag{13}$$

is an isomorphism, where q_1, q_2 denote the standard characters of $\mathbb{C}^* \times \mathbb{C}^*$, $z_1, \dots, z_n$ denote the standard characters of a maximal torus of GL_n, and "sym" denotes symmetric Laurent polynomials in $z_1, \dots, z_n$. Composing (11) with (13) yields

$$K_{G_n}\big(\overline{\mathrm{Comm}}_n\big) \xrightarrow{\iota_n} \mathbb{Z}\big[q_1^{\pm 1}, q_2^{\pm 1}\big]\big[z_1^{\pm 1}, \dots, z_n^{\pm 1}\big]^{\mathrm{sym}}$$

[1]Strictly speaking, one should think of $\overline{\mathrm{Comm}}_n$ as the derived subscheme of $\mathbb{A}^{2n^2}$ cut out by the Koszul complex of the system of n^2 equations $[X, Y] = 0$, but we do not need this subtlety.

as a map of $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-modules. We abuse the notation ι_n by also using it for the composition of the map above with the equality (12)

$$K_{\mathbb{C}^* \times \mathbb{C}^*}\big(\mathrm{Comm}_n\big) \xrightarrow{\iota_n} \mathbb{Z}\big[q_1^{\pm 1}, q_2^{\pm 1}\big]\big[z_1^{\pm 1}, \dots, z_n^{\pm 1}\big]^{\mathrm{sym}} \tag{14}$$

Definition 2.1 ([2]). Consider the rational function

$$\zeta(x) = \frac{(1 - xq_1)(1 - xq_2)\left(1 - \frac{q_1 q_2}{x}\right)}{1 - x}$$

The vector space

$$\mathcal{V} = \bigoplus_{n=0}^{\infty} \mathbb{Z}\big[q_1^{\pm 1}, q_2^{\pm 1}\big]\big[z_1^{\pm 1}, \dots, z_n^{\pm 1}\big]^{\mathrm{sym}} \tag{15}$$

is made into an algebra via the following **shuffle product**

$$\begin{aligned} &R(z_1, \dots, z_n) * R'(z_1, \dots, z_{n'}) \\ &\quad = \mathrm{Sym}\left[\frac{R(z_1, \dots, z_n) R'(z_{n+1}, \dots, z_{n+n'})}{n! n'!} \prod_{1 \le i \le n < j \le n+n'} \zeta\left(\frac{z_i}{z_j}\right)\right]. \end{aligned} \tag{16}$$

(above, "Sym" refers to symmetrization with respect to $z_1, \dots, z_{n+n'}$).

The K**-theoretic Hall algebra** is defined as

$$K = \bigoplus_{n=0}^{\infty} K_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Comm}_n).$$

It is endowed with a certain convolution product ([13], [11, Section 2.3] for the construction in notation closer to ours), which has the property that the maps (14) combine to an algebra homomorphism

$$K \xrightarrow{\iota} \mathcal{V}. \tag{17}$$

Unfortunately, we do not know how to effectively describe the image of ι.

2.3. The semi-nilpotent commuting stack

In the present paper, we study a variant of the K-theoretic Hall algebra, which we are able to describe completely in terms of (the natural analogue of) the homomorphism (17). Consider the **semi-nilpotent commuting variety**

$$\overline{\mathrm{Comm}}_n^{\text{semi-nilp}} \subset \overline{\mathrm{Comm}}_n$$

parametrizing those pairs (X, Y) of commuting $n \times n$ matrices with X arbitrary and Y nilpotent. The semi-nilpotency condition initially arose in the context of K-theoretic Hall algebras in [14], but it also naturally arises in categorification via knot invariants ([4, 5]). Letting the **semi-nilpotent commuting stack** be

$$\text{Comm}_n^{\text{semi-nilp}} = \overline{\text{Comm}_n}^{\text{semi-nilp}}/GL_n$$

we may define the following analogue of the construction of the previous subsection

$$K^{\text{semi-nilp}} = \bigoplus_{n=0}^{\infty} K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Comm}_n^{\text{semi-nilp}}).$$

Then, we have the natural analogue of the map (17)

$$K^{\text{semi-nilp}} \xrightarrow{\iota^{\text{semi-nilp}}} \mathcal{V}. \tag{18}$$

One endows $K^{\text{semi-nilp}}$ with the same kind of convolution product as K, thus making (18) into an algebra homomorphism. The map (18) is well known to be injective ([17, Lemma 2.5.1]). The main purpose of the present paper is to explicitly and effectively describe its image. We actually provide two descriptions of the image: one as an explicit subalgebra $\mathcal{S} \subset \mathcal{V}$ (in Section 3) and one by producing an explicit PBW basis of $\mathcal{S}$ over the ring $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] = K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{pt})$ (in Section 4).

2.4. **Generators**

We employ the language of integer partitions from [6, Chapter 1]. For any partition $\lambda = (n_1 \geq \dots \geq n_k) \vdash n$, let

$$\text{Comm}_\lambda^{\text{semi-nilp}} \overset{i_\lambda}{\hookrightarrow} \text{Comm}_n^{\text{semi-nilp}} \tag{19}$$

be the closure of the substack consisting of pairs of commuting matrices (X, Y) which (up to conjugation) are block triangular with respect to a flag of subspaces

$$0 = V_0 \subset V_1 \subset \dots \subset V_{k-1} \subset V_k = \mathbb{C}^n$$

where $\dim V_i/V_{i-1} = n_i$; above, "block-triangular" means that

$$X(V_i) \subset V_i \qquad \text{and} \qquad Y(V_i) = V_{i-1} \tag{20}$$

for all $i \in \{1, \dots, k\}$. A well-known fact of linear algebra is that

$$\text{Comm}_n^{\text{semi-nilp}} = \bigcup_{\lambda \vdash n} \text{Comm}_\lambda^{\text{semi-nilp}}.$$

The substack corresponding to $\lambda = (n)$ is simply $\mathbb{A}^{n^2}/GL_n$, as (20) requires X to be arbitrary but Y to be 0. As such, the composition

$$K_{G_n}(\mathbb{A}^{n^2}) \xrightarrow{i_{(n)*}} K_{G_n}(\overline{\text{Comm}_n}^{\text{semi-nilp}}) \longrightarrow K_{G_n}(\mathbb{A}^{2n^2}) \overset{|_\circ}{\cong} K_{G_n}(\text{pt})$$

is simply given by mapping X to $(X,0)$ and then restricting to the origin. Because of this, the image of the composition above is the principal ideal generated by the (equivariant) Koszul complex of

$$\mathbb{A}^{n^2} \hookrightarrow \mathbb{A}^{2n^2}, \qquad X \mapsto (X,0)$$

in the ring $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}][z_1^{\pm 1}, \dots, z_n^{\pm 1}]^{\text{sym}}$. This Koszul complex is none other than

$$F_n(z_1, \dots, z_n) = \prod_{1 \le i,j \le n} \left(1 - \frac{z_i q_2}{z_j}\right).$$

Therefore, we have for all $n \in \mathbb{N}$

$$\mathbb{Z}\left[q_1^{\pm 1}, q_2^{\pm 1}\right]\left[z_1^{\pm 1}, \dots, z_n^{\pm 1}\right]^{\text{sym}} \cdot F_n \subset \text{Im } \iota^{\text{semi-nilp}}. \tag{21}$$

PROPOSITION 2.2. *As a* $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$*-algebra,* $\text{Im } \iota^{\text{semi-nilp}}$ *is generated by the elements in the left-hand side of* (21)*, as* n *goes over* $\mathbb{N}$*.*

The result above was proved at the level of Chow groups in [14, Proposition 5.12]; the adaptation of the proof of *loc. cit.* to K-theory is straightforward, so we leave it as an exercise to the reader.

3. THE SHUFFLE ALGEBRA

3.1. An integral version of wheel conditions

The main purpose of the present section is to identify the image of the map (18). Proposition 2.2 implies that

$$\text{Im } \iota^{\text{semi-nilp}} = \left\langle \mathbb{Z}\left[q_1^{\pm 1}, q_2^{\pm 1}\right]\left[z_1^{\pm 1}, \dots, z_n^{\pm 1}\right]^{\text{sym}} \cdot F_n \right\rangle_{n \in \mathbb{N}} \subset \mathcal{V}. \tag{22}$$

Definition 3.1. Consider the $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-submodule $\mathcal{S} \subset \mathcal{V}$ consisting of symmetric Laurent polynomials $R(z_1, \dots, z_n)$ such that for any partition $(n_1 \ge \dots \ge n_k) \vdash n$, the quantity

$$R\left(x_1, x_1 q_2, \dots, x_1 q_2^{n_1 - 1}, \dots, x_k, x_k q_2, \dots, x_k q_2^{n_k - 1}\right) \tag{23}$$

is divisible by

$$\prod_{i=1}^{k} \left[(1 - q_2)^{n_i} \prod_{s=1}^{n_i - 1} \zeta(q_2^s)^{n_i - s}\right] \tag{24}$$

$$\prod_{1 \le i < j \le k} \left[\prod_{a=1}^{n_i - 1} \prod_{b=0}^{n_j - 1} \left(x_i q_1 - x_j q_2^{b-a}\right)\right] \left[\prod_{a=1}^{n_i - 1} \prod_{b=0}^{n_j - 1} \left(x_j q_1 - x_i q_2^{a-b-1}\right)\right]$$

in the ring $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}][x_1^{\pm 1}, \dots, x_k^{\pm 1}]$. We call $\mathcal{S}$ the (integral) **shuffle algebra**.

Remark 3.2. Upon tensoring with $\mathbb{Q}(q_1, q_2)$, all scalars $1 - q_2$ and $\zeta(q_2^s)$ become invertible, and the fact that the specialization (23) is divisible by (24) reduces to

$$R\big(x, xq_2, xq_1q_2, z_4, \dots, z_n\big) = R\big(x, xq_2, xq_1^{-1}, z_4, \dots, z_n\big) = 0 \tag{25}$$

(which is none other than the particular case of equation (24) for the partition $(2, 1, \dots, 1) \vdash n$). Conditions (25) are precisely the well-known wheel conditions ([2]) for the shuffle algebra associated to quantum toroidal $\mathfrak{gl}_1$ over the field $\mathbb{Q}(q_1, q_2)$.

Remark 3.3. In the context of integral forms of quantum affine groups, divisibility conditions on integral shuffle algebras were first studied in [15, Definition 3.37].

3.2. **The inclusion $\subseteq$ of Theorem 1.2**

The following two propositions immediately establish the fact that

$$\text{Im } \iota^{\text{semi-nilp}} \subseteq \mathcal{S}. \tag{26}$$

Proposition 3.4. *For any $n \in \mathbb{N}$, we have $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}][z_1^{\pm 1}, \dots, z_n^{\pm 1}]^{\text{sym}} \cdot F_n \subset \mathcal{S}$.*

Proof. Since F_n vanishes whenever we set $z_i = q_2 z_j$ (for any $i \neq j$), then the conditions in Definition 3.1 for any multiple of F_n are trivially satisfied. □

Proposition 3.5. *The submodule $\mathcal{S} \subset \mathcal{V}$ is a subalgebra with respect to equation* (16).

Proof. Let us write $\mathcal{S}_n \subset \mathcal{S}$ for the graded part consisting of Laurent polynomials in n variables (i.e., the n-th direct summand of (15)). We need to prove that

$$R \in \mathcal{S}_{n'} \quad \text{and} \quad R' \in \mathcal{S}_{n-n'} \qquad \Rightarrow \qquad R * R' \in \mathcal{S}_n.$$

Consider the specialization of the set of variables $\{z_1, \dots, z_n\}$ at

$$\big\{x_1, x_1q_2, \dots, x_1q_2^{n_1-1}, \dots, x_k, x_kq_2, \dots, x_kq_2^{n_k-1}\big\} \tag{27}$$

for some $n_1 + \dots + n_k = n$. By (16), to plug this specialization into $R * R'$ means to sum over all ways to permute the variables (27) and to plug them into

$$\frac{R(z_1, \dots, z_{n'})R'(z_{n'+1}, \dots, z_n)}{n'!(n-n')!} \prod_{1 \leq i \leq n' < j \leq n} \zeta\Big(\frac{z_i}{z_j}\Big). \tag{28}$$

However, because $\zeta(q_2^{-1}) = 0$, such a permutation can produce a non-zero contribution only if the variables

$$x_i, x_i q_2, \dots, x_i q_2^{m_i-1} \quad \text{are plugged into the variables of } R'$$
$$x_i q_2^{m_i}, x_i q_2^{m_i+1}, \dots, x_i q_2^{n_i-1} \quad \text{are plugged into the variables of } R$$

for some $m_i \in \{0, \dots, n_i\}$, for all $i \in \{1, \dots, k\}$. The contribution of such a permutation to the specialization of (28) is then

$$R\big(\dots, x_i q_2^{m_i}, x_i q_2^{m_i+1}, \dots, x_i q_2^{n_i-1}, \dots\big) R'\big(\dots, x_i, x_i q_2, \dots, x_i q_2^{m_i-1}, \dots\big).$$

$$\prod_{i=1}^{k} \prod_{a=m_i}^{n_i-1} \prod_{b=0}^{m_i-1} \zeta(q_2^{a-b}) \Big[\prod_{1 \le i \neq j \le k} \prod_{a=m_i}^{n_i-1} \prod_{b=0}^{m_j-1} \zeta\Big(\frac{x_i q_2^a}{x_j q_2^b}\Big) \Big]. \tag{29}$$

It remains to show that, for any $m_i \in \{0, \dots, n_i\}$, the expression (29) is divisible by (24). Because $R \in \mathcal{S}_{n'}$ and $R' \in \mathcal{S}_{n-n'}$, the first line of (29) is divisible by

$$(1-q_2)^n \prod_{i=1}^{k} \Big[\prod_{s=1}^{m_i-1} \zeta(q_2^s)^{m_i-s} \prod_{s=1}^{n_i-m_i-1} \zeta(q_2^s)^{n_i-m_i-s} \Big].$$

Together with the various $\zeta(q_2^{a-b})$ on the second line of (29), this precisely establishes divisibility by the expression on the first line of (24). Then it remains to prove that (29) is divisible by the expression on the second line of (24). To this end, note that the formula in square brackets in (29) is divisible by

$$\prod_{1 \le i \neq j \le k} \Big[\prod_{a=m_i}^{n_i-1} \prod_{b=0}^{m_j-1} \big(x_i q_1 - x_j q_2^{b-a}\big) \prod_{a=0}^{m_i-1} \prod_{b=m_j}^{n_j-1} \big(x_i q_1 - x_j q_2^{b-a-1}\big) \Big]. \tag{30}$$

Meanwhile, the second line of (24) can be rewritten in a more symmetric way as

$$\prod_{1 \le i \neq j \le k} \Big[\prod_{a=\min(n_i-n_j,0)+1}^{n_i-1} \prod_{b=0}^{\min(n_i,n_j)-1} \big(x_i q_1 - x_j q_2^{b-a}\big) \Big]. \tag{31}$$

As a consequence, Definition 3.1 implies that the first line of (29) is divisible by

$$\prod_{1 \le i \neq j \le k} \Big[\prod_{a=\min(m_i-m_j,0)+1}^{m_i-1} \prod_{b=0}^{\min(m_i,m_j)-1} \big(x_i q_1 - x_j q_2^{b-a}\big) \prod_{a=\min(m_i-m_j,n_i-n_j)+1}^{n_i-m_j-1} \prod_{b=0}^{\min(n_i-m_i,n_j-m_j)-1} \big(x_i q_1 - x_j q_2^{b-a}\big) \Big]. \tag{32}$$

We claim that the product of (30) and (32) is divisible by (31), for any choice of numbers $m_i \in \{0,\dots,n_i\}$, exactly what we needed to prove in order to conclude Proposition 3.5. This follows from the fact that the Laurent polynomial

$$\sum_{a=m_i}^{n_i-1}\sum_{b=0}^{m_j-1} z^{b-a} + \sum_{a=0}^{m_i-1}\sum_{b=m_j}^{n_j-1} z^{b-a-1} + \sum_{a=\min(m_i-m_j,0)+1}^{m_i-1}\ \sum_{b=0}^{\min(m_i,m_j)-1} z^{b-a}$$
$$+ \sum_{a=\min(m_i-m_j,n_i-n_j)+1}^{n_i-m_j-1}\ \sum_{b=0}^{\min(n_i-m_i,n_j-m_j)-1} z^{b-a}$$
$$- \sum_{a=\min(n_i-n_j,0)+1}^{n_i-1}\ \sum_{b=0}^{\min(n_i,n_j)-1} z^{b-a}.$$

has non-negative coefficients, as it is equal to $\sum_{a=\min(n_j-n_i,m_j-m_i)}^{\max(0,m_j-m_i)-1} z^a$. $\square$

3.3. The inclusion $\supseteq$ of Theorem 1.2

We now prove the opposite inclusion to (26), thus concluding the proof of Theorem 1.2.

PROPOSITION 3.6. *We have* Im $\iota^{\text{semi-nilp}} \supseteq \mathcal{S}$.

Proof. We refine the argument of [7, Proposition 2.4], itself based on [2]. For any partition $\lambda = (n_1 \geq \dots \geq n_k) \vdash n$, consider the linear map

$$\mathcal{S}_n \xrightarrow{\varphi_\lambda} \mathbb{Z}\big[q_1^{\pm 1}, q_2^{\pm 1}\big]\big[x_1^{\pm 1},\dots,x_k^{\pm 1}\big]$$

$$R(z_1,\dots,z_n \mapsto R\big(x_1 q_2^{n_1-1},\dots,x_1 q_2, x_1,\dots,x_k q_2^{n_k-1},\dots,x_k q_2, x_k\big). \tag{33}$$

We consider the total lexicographic order on partitions of size n, where

$$(m_1 \geq \dots \geq m_l) > (n_1 \geq \dots \geq n_k)$$

means that there exists i such that $m_1 = n_1,\dots,m_i = n_i, m_{i+1} > n_{i+1}$. The sets

$$\mathcal{S}_\lambda = \bigcap_{\mu > \lambda} \mathrm{Ker}\ \varphi_\mu$$

yield an increasing filtration of $\mathcal{S}_n = \mathcal{S}_{(n)}$.

Claim 3.7. For any $R \in \mathcal{S}_\lambda$, there exists $R' \in (\mathrm{Im}\ \iota^{\text{semi-nilp}}) \cap \mathcal{S}_\lambda$ such that

$$\varphi_\lambda(R) = \varphi_\lambda(R'). \tag{34}$$

Iterating Claim 3.7 for all partitions λ in decreasing lexicographic order allows us to take any $R \in \mathcal{S}_n$, and by subtracting various elements in Im $\iota^{\text{semi-nilp}}$, ensure that it lies in the kernel of φ_λ for smaller and smaller λ. As soon as we pass $\lambda = (1, \dots, 1)$, then we have made R equal to 0 by subtracting various elements in Im $\iota^{\text{semi-nilp}}$, and the proof of Proposition 3.6 would be complete.

Let us now prove Claim 3.7. If we write $\lambda = (n_1 \geq \dots \geq n_k)$, then the transposed partition $\lambda' = (t_1 \geq \dots \geq t_p)$ is defined by the equation

$$n_i = \Big|\big\{u \in \{1, \dots, p\} \mid t_u \geq i\big\}\Big| \tag{35}$$

for all i. Let us write $s_i = t_1 + \dots + t_i$ for all i, and define

$$R'(z_1, \dots, z_n) = \mathrm{Sym}\big[r(z_1, \dots, z_n)\big] \tag{36}$$

where for any $\rho \in \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}][z_1^{\pm 1}, \dots, z_n^{\pm 1}]^{\text{psym}}$, we set

$$r = \rho(z_1, \dots, z_n) \prod_{i=1}^{p} F_{t_i}(z_{s_{i-1}+1}, \dots, z_{s_i}) \prod_{1 \leq i < j \leq p} \prod_{a=s_{i-1}+1}^{s_i} \prod_{b=s_{j-1}+1}^{s_j} \zeta\Big(\frac{z_a}{z_b}\Big)$$

(the superscript "psym" means that we require ρ to be symmetric in the set $z_{s_{i-1}+1}, \dots, z_{s_i}$ for all $i \in \{1, \dots, p\}$ separately). We claim that

$$R' \in (\mathrm{Im}\ \iota^{\text{semi-nilp}}) \cap \mathcal{S}_\lambda. \tag{37}$$

Note that $R' \in \mathrm{Im}\ \iota^{\text{semi-nilp}}$, as it is a linear combination of shuffle products of Laurent polynomials divisible by $F_{t_1}, \dots, F_{t_p}$. Next, we claim that $R' \in \mathcal{S}_\lambda$; to see this, we must show that R' is annihilated by φ_μ for any $\mu > \lambda$. Computing $\varphi_\mu(R')$ for some $\mu = (m_1 \geq \dots \geq m_l)$ entails specializing the variables of R' to

$$\big\{x_i, x_i q_2, \dots, x_i q_2^{m_i - 1}\big\}_{\{1, \dots, l\}}. \tag{38}$$

Equivalently, this amounts to inserting the variables (38) among the arguments of r in an arbitrary order. Let us call such an insertion "good" if for each $i \in \{1, \dots, l\}$, the variables $x_i q_2^{m_i - 1}, \dots, x_i q_2, x_i$ are plugged in successive sets among

$$\big\{z_1, \dots, z_{s_1}\big\}, \big\{z_{s_1+1}, \dots, z_{s_2}\big\}, \dots, \big\{z_{s_{p-1}+1}, \dots, z_n\big\}. \tag{39}$$

Because $\zeta(q_2^{-1}) = 0$ and $F_t(\dots, x, xq_2, \dots) = 0$ for all t, only good insertions have the property that r specializes to a non-zero value. However, $\mu > \lambda$ means that

$$\begin{aligned}
m_1 = n_1 &= \Big|\big\{u \in \{1, \dots, p\} \mid t_u \geq 1\big\}\Big| \\
&\dots \\
m_i = n_i &= \Big|\big\{u \in \{1, \dots, p\} \mid t_u \geq i\big\}\Big| \\
m_{i+1} > n_{i+1} &= \Big|\big\{u \in \{1, \dots, p\} \mid t_u \geq i+1\big\}\Big|
\end{aligned}$$

for some i, and thus good insertions cannot exist. This establishes (37).

It remains to show that we can choose the Laurent polynomial ρ in the definition of r so that (34) holds. Recall that $\varphi_\lambda(R')$ is calculated by inserting the variables $x_i q_2^{n_i-1}, \ldots, x_i q_2, x_i$ in the arguments of r. Repeating the argument in the preceding paragraph shows that the only good insertions contributing to $\varphi_\lambda(R')$ are

$$\left\{z_{s_{i-1}+1}, z_{s_{i-1}+2}, \ldots, z_{s_i}\right\} = \left\{x_1 q_2^{n_1-i}, x_2 q_2^{n_2-i}, \ldots, x_{t_i} q_2^{n_{t_i}-i}\right\}$$

for all $i \in \{1, \ldots, p\}$. Thus, we conclude that

$$\varphi_\lambda(R') = \varphi_\lambda(\rho) \prod_{i=1}^{p} \prod_{1\le a,b\le t_i} \left(1 - \frac{x_a q_2^{n_a+1}}{x_b q_2^{n_b}}\right) \prod_{1\le i<j\le p} \prod_{a=1}^{t_i} \prod_{b=1}^{t_j} \zeta\left(\frac{x_a q_2^{n_a-i}}{x_b q_2^{n_b-j}}\right).$$

Although ρ is not itself an element of $\mathcal{S}_n$, the notation $\varphi_\lambda(\rho)$ is defined just like (33). We may now move the products in a, b from the inside to the outside of the above formula, and obtain (after clearing various cancelations involving ζ factors)

$$\varphi_\lambda(R') = \varphi_\lambda(\rho) \cdot \Pi_1 \Pi_2 \Pi_3 \tag{40}$$

where

$$\Pi_1 = \prod_{a=1}^{k} \left[(1-q_2)^{n_a} \prod_{u=1}^{n_a-1} \zeta(q_2^u)^{n_a-u}\right]$$

$$\Pi_2 = \prod_{1\le a\ne b\le k} \left[\prod_{\substack{0\le u<n_a \\ 0\le v<n_b}}^{u-v>n_a-n_b} \left(1 - \frac{x_a q_1}{x_b q_2^{v-u}}\right) \prod_{\substack{1\le u\le n_a \\ 0\le v<n_b}}^{u-v\le n_a-n_b} \left(1 - \frac{x_a q_1}{x_b q_2^{v-u}}\right)\right]$$

$$\Pi_3 = \prod_{1\le a\ne b\le k} \prod_{u=\max(n_a-n_b,0)+1}^{n_a} \left(1 - \frac{x_a q_2^u}{x_b}\right).$$

Clearly, Π_1 is precisely the first line of (24), while it is elementary to see that Π_2 matches the second line of (24) up to an overall monomial. The fact that $R \in \mathcal{S}_n$ implies that $\varphi_\lambda(R)$ is divisible by (24), and thus is divisible by $\Pi_1\Pi_2$. However, the fact that $R \in \mathcal{S}_\lambda$ implies certain additional divisibilities: whenever

$$x_i q_2^{-1} \quad \text{or} \quad x_i q_2^{n_i} \qquad \text{is set equal to} \qquad x_j, x_j q_2, \ldots, x_j q_2^{n_j-1}$$

for some $i < j$, the quantity $\varphi_\lambda(R)$ must vanish (indeed, this is because if we enlarge n_i and diminish n_j by some positive amount, the resulting partition μ is larger than λ). This precisely entails the fact that $\varphi_\lambda(R)$ is divisible by Π_3, so we conclude that there exists a Laurent polynomial $A(x_1, \ldots, x_k)$ such that

$$\varphi_\lambda(R) = A(x_1, \ldots, x_k) \cdot \Pi_1 \Pi_2 \Pi_3. \tag{41}$$

Moreover, $A(x_1, \dots, x_k)$ is symmetric in x_a and x_b if $n_a = n_b$, because R is symmetric in all of its variables. Thus, we must choose ρ such that

$$\varphi_\lambda(\rho) = A(x_1, \dots, x_k) \tag{42}$$

and then (40) and (41) would imply (34). We may assume that A is a polynomial in $x_1, \dots, x_k$, by multiplying (42) with a sufficiently high monomial. Thus, if the partition λ consists of d_1 times 1, d_2 times 2 etc, we may assume that

$$A(x_1, \dots, x_k) = m_{\nu_1}(x_k, \dots, x_{k-d_1+1}) m_{\nu_2}(x_{k-d_1}, \dots, x_{k-d_1-d_2+1}) \dots$$

where $m_\nu(z_1, z_2, \dots) = \sum_{\sigma(i)<\sigma(j) \text{ if } \nu_i = \nu_j}^{\sigma \in S(\infty)} z_{\sigma(1)}^{\nu_1} z_{\sigma(2)}^{\nu_2} \dots$ denotes the monomial symmetric function associated to the partition $\nu = (\nu_1 \geq \nu_2 \geq \dots)$. If we define

$$\rho'(z_1, \dots, z_n) = m_{\nu_1}(z_1, \dots, z_{t_1}) m_{\nu_2}(z_{t_1+1}, \dots, z_{t_1+t_2}) \dots$$

then it is straightforward to see that

$$\varphi_\lambda(\rho') = q_2^{\text{some integer}} \cdot A(x_1, \dots, x_k) + B(x_1, \dots, x_k)$$

where B is a polynomial, symmetric in x_a and x_b if $n_a = n_b$, for which the sequence

$$\left(\underset{x_k, \dots, x_{k-d_1}}{\text{hom deg}} B, \underset{x_{k-d_1+1}, \dots, x_{k-d_1-d_2}}{\text{hom deg}} B, \dots \right)$$

is lexicographically smaller than the analogous sequence for A (while the total homogeneous degree of B is the same as that of A). Therefore, we may repeat the argument above for B instead of A; after finitely many iterations of this procedure, we would obtain a polynomial ρ for which (42) holds precisely. □

Proof of Theorem 1.2. It follows from (26) and Proposition 3.6. □

4. THE PBW BASIS

4.1. A basis indexed by convex paths

For any $(n, d) \in \mathbb{N} \times \mathbb{Z}$, consider the Laurent polynomial

$$P_{n,d} = \text{Sym} \left[\frac{\prod_{i=1}^n z_i^{\lfloor \frac{id}{n} \rfloor - \lfloor \frac{(i-1)d}{n} \rfloor} \sum_{s=0}^{t-1} \frac{z_{a(t-1)+1} \dots z_{a(t-s)+1}}{q_2^s z_{a(t-1)} \dots z_{a(t-s)}}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}}{z_i q_2}\right)} \prod_{1 \leq i < j \leq n} \zeta\left(\frac{z_i}{z_j}\right) \right] \tag{43}$$

where we write $t = \gcd(n, d)$ and $a = \frac{n}{t}$. With the notation above, let

$$\gamma_{n,d} = \frac{q_2^t - 1}{(q_1^t - 1)(q_3^t - 1)} \cdot (q_1 - 1)^n (q_3 - 1)^n \tag{44}$$

with $q_3 = \frac{1}{q_1 q_2}$. Let us define the following rescaled versions of (43)

$$\bar{P}_{n,d} = \gamma_{n,d} \cdot P_{n,d}. \tag{45}$$

A sequence $v = \{(n_1, d_1), \dots, (n_k, d_k)\} \subset \mathbb{N} \times \mathbb{Z}$ is called a **convex path** if

$$\frac{d_1}{n_1} \leq \dots \leq \frac{d_k}{n_k}.$$

We always consider convex paths up to the equivalence generated by permuting lattice points of the same slope. This is motivated by the fact that $P_{n,d}$ and $P_{n',d'}$ commute if (n,d) and (n',d') have the same slope ([7]), and thus the expressions

$$P_v = P_{n_1,d_1} * \dots * P_{n_k,d_k} \tag{46}$$

$$\bar{P}_v = \bar{P}_{n_1,d_1} * \dots * \bar{P}_{n_k,d_k} \tag{47}$$

only depend on the equivalence class of a convex path. It was shown in [7] that

$$\mathcal{S}_{\text{loc}} := \mathcal{S} \bigotimes_{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]} \mathbb{Q}(q_1, q_2) = \bigoplus_{v \text{ convex path}} \mathbb{Q}(q_1, q_2) \cdot P_v \tag{48}$$

following the analogous result of [1] for the elliptic Hall algebra [2]. Because relation (48) is taken over $\mathbb{Q}(q_1, q_2)$, it also holds with the P's replaced by $\bar{P}$'s.

Remark 4.1. The following formulas are proved in [10, (2.34) and (2.35)]

$$P_{n,d} = \gamma'_{n,d} \cdot \text{Sym}\left[\frac{\prod_{i=1}^n z_i^{\lfloor \frac{id}{n} \rfloor - \lfloor \frac{(i-1)d}{n} \rfloor} \sum_{s=0}^{t-1} \frac{z_{a(t-1)+1} \cdots z_{a(t-s)+1}}{q_1^s z_{a(t-1)} \cdots z_{a(t-s)}}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}}{z_i q_1}\right)} \prod_{1 \leq i < j \leq n} \zeta\left(\frac{z_i}{z_j}\right)\right]$$

$$= \gamma''_{n,d} \cdot \text{Sym}\left[\frac{\prod_{i=1}^n z_i^{\lfloor \frac{id}{n} \rfloor - \lfloor \frac{(i-1)d}{n} \rfloor} \sum_{s=0}^{t-1} \frac{z_{a(t-1)+1} \cdots z_{a(t-s)+1}}{q_3^s z_{a(t-1)} \cdots z_{a(t-s)}}}{\prod_{i=1}^{n-1} \left(1 - \frac{z_{i+1}}{z_i q_3}\right)} \prod_{1 \leq i < j \leq n} \zeta\left(\frac{z_i}{z_j}\right)\right]$$

where we recall that $q_3 = \frac{1}{q_1 q_2}$, $t = \gcd(n,d)$, $a = \frac{n}{t}$, and define

$$\gamma'_{n,d} = \frac{q_1^t - 1}{(q_1 - 1)^n} \cdot \frac{(q_2 - 1)^n}{q_2^t - 1} \qquad \text{and} \qquad \gamma''_{n,d} = \frac{q_3^t - 1}{(q_3 - 1)^n} \cdot \frac{(q_2 - 1)^n}{q_2^t - 1}.$$

4.2. **From power sum functions to complete symmetric functions**

For any coprime $(n,d) \in \mathbb{N} \times \mathbb{Z}$, the following power series identities

$$1 + \sum_{t=1}^{\infty} \frac{H_{nt,dt}}{x^t} = \exp\left(\sum_{t=1}^{\infty} \frac{P_{nt,dt}}{t x^t}\right) \tag{49}$$

[2]Indeed, [1] interpreted the fact that convex paths index a linear basis of (half of) the elliptic Hall algebra as an analogue of the classic fact that unordered collections of positive roots index a linear basis of (half of) quantum groups of finite type.

(50)
$$1+\sum_{t=1}^{\infty}\frac{\bar{H}_{nt,dt}}{x^t}=\exp\left(\sum_{t=1}^{\infty}\frac{\bar{P}_{nt,dt}}{tx^t}\right)$$

define elements $\{H_{n,d},\bar{H}_{n,d}\}_{(n,d)\in\mathbb{N}\times\mathbb{Z}}\in\mathcal{S}_{\text{loc}}$. In [9, Formula (2.9)], we showed that

(51)
$$H_{n,d}=\text{Sym}\left[\frac{\prod_{i=1}^n z_i^{\lfloor\frac{id}{n}\rfloor-\lfloor\frac{(i-1)d}{n}\rfloor}}{\prod_{i=1}^{n-1}\left(1-\frac{z_{i+1}}{z_iq_2}\right)}\prod_{1\le i<j\le n}\zeta\left(\frac{z_i}{z_j}\right)\right]$$

$\forall(n,d)\in\mathbb{N}\times\mathbb{Z}$. Similarly, the following formula can be found in [8, Exercise 3.18]

(52)
$$\bar{H}_{n,d}=(q_1-1)^n(q_2-1)^n$$
$$\cdot\,\text{Sym}\left[\frac{\prod_{i=1}^n z_i^{\lfloor\frac{id}{n}\rfloor-\lfloor\frac{(i-1)d}{n}\rfloor}\prod_{s=1}^{t-1}\left(q_1^s-\frac{z_{as+1}}{z_{as}q_3}\right)}{(q_1-1)\dots(q_1^t-1)\prod_{i=1}^{n-1}\left(1-\frac{z_{i+1}}{z_iq_3}\right)}\prod_{1\le i<j\le n}\zeta\left(\frac{z_i}{z_j}\right)\right].$$

By switching the roles of q_1 and q_3, one also obtains the following analogous formula

(53)
$$\bar{H}_{n,d}=(q_2-1)^n(q_3-1)^n$$
$$\cdot\,\text{Sym}\left[\frac{\prod_{i=1}^n z_i^{\lfloor\frac{id}{n}\rfloor-\lfloor\frac{(i-1)d}{n}\rfloor}\prod_{s=1}^{t-1}\left(q_3^s-\frac{z_{as+1}}{z_{as}q_1}\right)}{(q_3-1)\dots(q_3^t-1)\prod_{i=1}^{n-1}\left(1-\frac{z_{i+1}}{z_iq_1}\right)}\prod_{1\le i<j\le n}\zeta\left(\frac{z_i}{z_j}\right)\right].$$

PROPOSITION 4.2. *We have $\bar{H}_{n,d}\in\mathcal{S}$ for all $(n,d)\in\mathbb{N}\times\mathbb{Z}$.*

Proof. Consider any partition $(n_1,\dots,n_k)\vdash n$. In the Laurent polynomial

$$\bar{H}_{n,d}(z_1,\dots,z_n)\cdot(1+q_3)(1+q_3+q_3^2)\dots(1+q_3+\dots+q_3^{t-1})$$

(where $t=\gcd(n,d)$), let us specialize the variables $z_1,\dots,z_n$ to

(54)
$$x_1,x_1q_2,\dots,x_1q_2^{n_1-1},\dots,x_k,x_kq_2,\dots,x_kq_2^{n_k-1}.$$

Using formula (53), this amounts to permuting the variables (54) arbitrarily, and then inserting them instead of $z_1,\dots,z_n$ into a certain expression of the form

(55)
$$\frac{(1-q_2)^n\cdot\text{Laurent polynomial}}{\prod_{i=1}^{n-1}\left(1-\frac{z_{i+1}}{z_iq_1}\right)}\prod_{1\le i<j\le n}\zeta\left(\frac{z_i}{z_j}\right).$$

Because $\zeta(q_2^{-1})=0$, the only insertions which produce a non-zero contribution are those for which $x_iq_2^{n_i-1},\dots,x_i$ are plugged into $z_{a_1},\dots,z_{a_{n_i}}$ for certain indices $a_1<\dots<a_{n_i}$, for each $i\in\{1,\dots,k\}$. As such, it is clear that

the resulting specialization is divisible by the expression on the first line of (24). Moreover, for any $i \neq j$ such that $n_i \geq n_j$, let us zoom in on a fixed $b \in \{0, \dots, n_j - 1\}$ and assume that the variables (54) are permuted in the order

$$x_i q_2^{n_i-1}, \dots, x_i q_2^u, x_j q_2^b, x_i q_2^{u-1}, \dots, x_i$$

for some u. Then the product of ζ functions in (55) is a multiple of

$$(x_i q_1 - x_j q_2^{b-n_i+1}) \dots (x_i q_1 - x_j q_2^{b-u-1}) \underline{(x_i q_1 - x_j q_2^{b-u})}$$
$$(x_i q_1 - x_j q_2^{b-u})(x_i q_1 - x_j q_2^{b-u+1}) \dots (x_i q_1 - x_j q_2^{b-1}).$$

As for the denominator in (55), it can at most cancel the underlined term above. The resulting expression is a multiple of $\prod_{a=1}^{n_i-1}(x_i q_1 - x_j q_2^{b-a})$; taking the product over $b \in \{0, \dots, n_j - 1\}$'s shows that the overall specialization is a multiple of the first product on the second line of (24). One shows that the specialization is a multiple of the second product on the second line of (24) analogously. Thus

$$\frac{\bar{H}_{n,d}(x_1, \dots, x_1 q_2^{n_1-1}, \dots, x_k, \dots, x_k q_2^{n_k-1})}{\text{expression (24)}} \in$$
$$\mathbb{Z}\big[q_1^{\pm 1}, q_2^{\pm 1}\big]_{(1+q_3+\dots+q_3^{s-1})_{s\in\mathbb{N}}}\big[x_1^{\pm 1}, \dots, x_k^{\pm 1}\big].$$

Repeating the argument with the roles of q_1 and q_3 switched (i.e., using (52) instead of (53)) shows that the ratio above has coefficients in the localization

$$\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]_{(1+q_1+\dots+q_1^{s-1})_{s\in\mathbb{N}}}.$$

We conclude that the coefficients are actually in $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$, as we needed to show. □

4.3. A basis of the semi-nilpotent K-theoretic Hall algebra

By analogy with (46) and (47), let us write for any convex path v

$$H_v = H_{n_1,d_1} * \dots * H_{n_k,d_k} \tag{56}$$

$$\bar{H}_v = \bar{H}_{n_1,d_1} * \dots * \bar{H}_{n_k,d_k}. \tag{57}$$

Clearly, the elements $P_{n,d}$ may be replaced by either $H_{n,d}$ and $\bar{H}_{n,d}$ in (48), to produce a valid basis of $\mathcal{S}_{\text{loc}}$ as a $\mathbb{Q}(q_1, q_2)$-vector space. However, our main interest is in the following $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$-submodule of $\mathcal{S}_{\text{loc}}$

$$\mathcal{A} = \bigoplus_{v \text{ convex path}} \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] \cdot \bar{H}_v.$$

We are now ready to prove Lemma 1.3, which provides an integral version of (48).

Proof of Lemma 1.3. By Propositions 3.5 and 4.2, we have

$$\mathcal{S} \supseteq \mathcal{A}.$$

It remains to prove the opposite inclusion, namely

$$\mathcal{S} \subseteq \mathcal{A}. \tag{58}$$

To this end, recall the symmetric pairing defined in [7, Formula (4.7)]

$$\mathcal{S}_{\text{loc}} \otimes \mathcal{S}_{\text{loc}} \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{Q}(q_1, q_2) \tag{59}$$

by the formula [3]

$$\langle R, R' \rangle = \frac{1}{(q_2 - 1)^n} \int_{|z_1| \gg \dots \gg |z_n|} \frac{r(z_1, \dots, z_n) R'(z_1^{-1}, \dots, z_n^{-1})}{\prod_{1 \le i < j \le n} \zeta\left(\frac{z_j}{z_i}\right)} \prod_{i=1}^{n} Dz_i \tag{60}$$

(where $Dz = \frac{dz}{2\pi i z}$), for any $R' \in \mathcal{S}_{\text{loc}}$ and

$$R = \text{Sym}\left[r(z_1, \dots, z_n) \prod_{1 \le i < j \le n} \zeta\left(\frac{z_i}{z_j}\right) \right] \tag{61}$$

where r is an arbitrary Laurent polynomial with coefficients in $\mathbb{Q}(q_1, q_2)$. Since any element $R \in \mathcal{S}_{\text{loc}}$ can be written in the form (61) for some Laurent polynomial r with coefficients in $\mathbb{Q}(q_1, q_2)$ (as proved in [7, Theorem 2.5]), formula (60) determines the pairing (59) completely. It was shown in [7, Proposition 5.4] that $\{P_v\}_{v \text{ convex}}$ is an orthogonal basis with respect to the pairing (59), satisfying

$$\langle P_v, \bar{P}_v \rangle = \prod_{\mu \in \mathbb{Q}} z_{\lambda_v^\mu}. \tag{62}$$

Let us explain the notation in the right-hand side of (62): for any convex path $v = \{(n_1, d_1), \dots, (n_k, d_k)\}$ and any

$$\mu = \frac{d}{n} \in \mathbb{Q}$$

(assume $\gcd(n, d) = 1$), those elements of v of slope μ is of the form

$$(nt_1, dt_1), \dots, (nt_k, dt_k)$$

for some partition $\lambda_v^\mu = (t_1 \ge \dots \ge t_k)$. As μ goes over the infinitely many rational numbers, all but finitely many of these partitions are empty. Finally, for any partition $\lambda = (t_1 \ge \dots \ge t_k)$, we set

$$z_\lambda = t_1 \dots t_k \prod_{u \in \mathbb{N}} (\text{number of } u\text{'s in } \lambda)!$$

[3]Note that our normalization of (60) differs from that of *loc. cit.* by $(q_1 - 1)^n (q_3 - 1)^n$. Moreover, the order of variables in our contour integral is opposite to that of *loc. cit.* (i.e., $|z_1| \gg \dots \gg |z_n|$ instead of $|z_1| \ll \dots \ll |z_n|$); this is simply a matter of convenience for us, as formula (60) holds with either order (compare with [11, Formulas (3.2) and (3.31)]).

and this completes the explanation of the right-hand side of (62).

Claim 4.3. An element $R \in \mathcal{S}_{loc}$ is a linear combination of $\bar{H}_v$'s with coefficients in $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$ if and only if

$$\langle R, H_v \rangle \in \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$$

for all convex paths v.

Formula (62) reduces Claim 4.3 to the following well-known fact about symmetric functions: the Hall inner product of a symmetric function f with all products of complete symmetric functions h_n are integral if and only if f is an integral linear combination of products of complete symmetric functions (indeed, products of complete symmetric functions yield the dual basis to monomial symmetric functions). Thus, Claim 4.3 reduces (58) to showing that

$$\langle R, H_v \rangle \in \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] \tag{63}$$

for any $R \in \mathcal{S}$ and any convex path v. The remainder of the proof deals with establishing (63). To this end, formula (51) implies that H_v is a particular element of the shuffle algebra of the form

$$R'(z_1, \ldots, z_n) = \text{Sym}\left[\frac{p(z_1, \ldots, z_n)}{\prod_{i=1}^{n-1}\left(1 - \frac{z_{i+1}}{z_i q_2}\right)} \prod_{1 \le i < j \le n} \zeta\left(\frac{z_i}{z_j}\right)\right] \tag{64}$$

where $p(z_1, \ldots, z_n)$ is an arbitrary Laurent polynomial with coefficients in $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$.

Claim 4.4. For any $R \in \mathcal{S}_{loc}$ and any R' as in (64), we have

$$\langle R, R' \rangle = \sum_{n_1 + \cdots + n_k = n} \int_{|x_1| \ll \cdots \ll |x_k|}$$

$$\left[\underset{\{z_{n_1+\cdots+n_{i-1}+1}, \ldots, z_{n_1+\cdots+n_i}\}_{1 \le i \le k} = \{x_i q_2^{n_i - 1}, \ldots, x_i\}_{1 \le i \le k}}{Res} \right. \tag{65}$$

$$\left. \frac{1}{(q_2 - 1)^n} \cdot \frac{R(z_1, \ldots, z_n) \cdot p(z_1^{-1}, \ldots, z_n^{-1})}{\prod_{i=1}^{n-1}\left(1 - \frac{z_i}{z_{i+1} q_2}\right) \prod_{1 \le i < j \le n} \zeta\left(\frac{z_i}{z_j}\right)} \right] \prod_{i=1}^{k} Dx_i$$

where

$$\underset{\{z_1, \ldots, z_n\} = \{x q_2^{n-1}, \ldots, x\}}{Res}$$

denotes the iterated residue first at $z_{n-1} = z_n q_2$, then at $z_{n-2} = z_n q_2^2$, ..., finally at $z_1 = z_n q_2^{n-1}$, followed by relabeling the variable z_n by x.

Let us first indicate how Claim 4.4 implies (63). When $R \in \mathcal{S}$, Definition 3.1 tells us that the expression in square brackets of (65) is a Laurent polynomial with coefficients in $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$, divided by various linear terms of the form

$$x_i q_1^{\cdots} - x_j q_2^{\cdots}$$

for $i \neq j$. As we take the integral of such an expression in the next limit $|x_1| \ll \cdots \ll |x_k|$, the result is still an element of $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$, thus establishing formula (63).

Proof of Claim 4.4. It suffices to prove (65) for R of the form (61), as such elements span $\mathcal{S}_{\text{loc}}$. Then the right-hand side of (60) is equal to $(q_2-1)^{-n}$ times

$$\int_{|z_1|\gg\cdots\gg|z_n|} \frac{r(z_1,\dots,z_n)}{\prod_{1\le i<j\le n}\zeta\left(\frac{z_j}{z_i}\right)} \sum_{\sigma\in S(n)} \frac{p(z_{\sigma(1)}^{-1},\dots,z_{\sigma(n)}^{-1})\prod_{1\le i<j\le n}\zeta\left(\frac{z_{\sigma(j)}}{z_{\sigma(i)}}\right)}{\prod_{i=1}^{n-1}\left(1-\frac{z_{\sigma(i)}}{z_{\sigma(i+1)}q_2}\right)} \prod_{i=1}^n Dz_i$$

$$= \int_{|z_1|\gg\cdots\gg|z_n|} \sum_{\sigma\in S(n)} \frac{r(z_1,\dots,z_n)\cdot p(z_{\sigma(1)}^{-1},\dots,z_{\sigma(n)}^{-1})}{\prod_{i=1}^{n-1}\left(1-\frac{z_{\sigma(i)}}{z_{\sigma(i+1)}q_2}\right)} \prod_{1\le i<j\le n}^{\sigma(i)>\sigma(j)} \frac{\zeta\left(\frac{z_{\sigma(j)}}{z_{\sigma(i)}}\right)}{\zeta\left(\frac{z_{\sigma(i)}}{z_{\sigma(j)}}\right)} \prod_{i=1}^n Dz_i$$

$$= \sum_{\sigma\in S(n)} \int_{|w_{\sigma^{-1}(1)}|\gg\cdots\gg|w_{\sigma^{-1}(n)}|} \frac{r(w_{\sigma^{-1}(1)},\dots,w_{\sigma^{-1}(n)})p(w_1^{-1},\dots,w_n^{-1})}{\prod_{i=1}^{n-1}\left(1-\frac{w_i}{w_{i+1}q_2}\right)}$$

$$\prod_{1\le i<j\le n}^{\sigma(i)>\sigma(j)} \frac{\zeta\left(\frac{w_j}{w_i}\right)}{\zeta\left(\frac{w_i}{w_j}\right)} \prod_{i=1}^n Dw_i$$

where in the last equality we changed the variables to $w_i = z_{\sigma(i)}$. As we move the contours from $|w_{\sigma^{-1}(1)}| \gg \cdots \gg |w_{\sigma^{-1}(n)}|$ toward $|w_1| \ll \cdots \ll |w_n|$ in the integral above, we note that the only poles we might pick up are those of the form

$$\left\{w_i = w_{i+1}q_2\right\}_{i\in\{1,\dots,n-1\}}.$$

Thus, we conclude that the integral above is equal to

$$\sum_{n_1+\cdots+n_k=n} \int_{|x_1|\ll\cdots\ll|x_k|} \left[\underset{\{w_{n_1+\cdots+n_{i-1}+1},\dots,w_{n_1+\cdots+n_i}\}_{1\le i\le k}=\{x_i q_2^{n_i-1},\dots,x_i\}_{1\le i\le k}}{\text{Res}} \right.$$

$$\left. \sum_{\sigma\in S(n)} \frac{r\left(w_{\sigma^{-1}(1)},\dots,w_{\sigma^{-1}(n)}\right)\cdot p\left(w_1^{-1},\dots,w_n^{-1}\right)}{\prod_{i=1}^{n-1}\left(1-\frac{w_i}{w_{i+1}q_2}\right)} \prod_{1\le i<j\le n}^{\sigma(i)>\sigma(j)} \frac{\zeta\left(\frac{w_j}{w_i}\right)}{\zeta\left(\frac{w_i}{w_j}\right)} \right] \prod_{i=1}^k Dx_i.$$

The second line of the expression above is

$$\frac{p(w_1^{-1},\dots,w_n^{-1})}{\prod_{i=1}^{n-1}\left(1-\frac{w_i}{w_{i+1}q_2}\right)\prod_{1\le i<j\le n}\zeta\left(\frac{w_i}{w_j}\right)} \sum_{\sigma\in S(n)} r\left(w_{\sigma^{-1}(1)},\dots,w_{\sigma^{-1}(n)}\right) \prod_{\sigma(i)>\sigma(j)} \zeta\left(\frac{w_j}{w_i}\right)$$

which directly implies (65) for R of the form (61), as we needed to show. □

□

4.4. **Slope subalgebras**

In [7], we introduced **slope** subalgebras

$$\mathcal{B}^\mu \subset \mathcal{S}_{\text{loc}} \tag{66}$$

for any $\mu \in \mathbb{Q}$, which are isomorphic to the algebra

$$\Lambda = \mathbb{Q}(q_1, q_2)[p_1, p_2, \dots]$$

of symmetric polynomials in infinitely many variables (above, p_t is interpreted as the t-th power sum function). Explicitly, we have an algebra isomorphism

$$\tau^{\frac{d}{n}} : \Lambda \xrightarrow{\sim} \mathcal{B}^{\frac{d}{n}} \tag{67}$$

for any coprime $(n, d) \in \mathbb{N} \times \mathbb{Z}$, determined by the assignment

$$\tau^{\frac{d}{n}}(p_t) = \bar{P}_{nt,dt}.$$

If we let h_t denote the t-th complete symmetric function, the power series identity

$$1 + \sum_{t=1}^{\infty} \frac{h_t}{x^t} = \exp\left(\sum_{t=1}^{\infty} \frac{p_t}{tx^t}\right)$$

and formula (50) imply that $\tau^{\frac{d}{n}}(h_t) = \bar{H}_{nt,dt}$ for all coprime (n, d) and all $t \in \mathbb{N}$.

Remark 4.5. For any $\mu \in \mathbb{Q}$, the isomorphism (67) allows one to transport the usual Hall coproduct on Λ to a coproduct Δ_μ on $\mathcal{B}^\mu$; the latter coproduct was given a shuffle algebra interpretation in [7]. In particular, this allows us to prove that

$$\bar{H}_{n,0} = q_1^{\frac{n(n-1)}{2}} F_n \tag{68}$$

as both LHS and RHS are uniquely determined by the fact that they are group-like for Δ_0 in $\mathcal{B}^0$, and are annihilated by the linear map φ (with $q_1 \leftrightarrow q_2$) of *loc. cit.*

4.5. **Ribbon skew Schur functions**

The ring Λ is rich in automorphisms, as one can rescale the generators p_t independently and arbitrarily. We refer to the particularly important rescaling

$$p'_t = p_t(q_1^t - 1)$$

as a **plethysm**. We therefore obtain elements $h'_t \in \Lambda$ via the usual formula

$$1 + \sum_{t=1}^{\infty} \frac{h'_t}{x^t} = \exp\left(\sum_{t=1}^{\infty} \frac{p'_t}{tx^t}\right).$$

It was shown in [9, Section 2.3] that

$$\tau^{\frac{d}{n}}(h'_t) = \bar{H}'_{nt,dt}$$

for all coprime $(n, d) \in \mathbb{N} \times \mathbb{Z}$ and all $t \in \mathbb{N}$, where

$$\bar{H}'_{n,d} = (q_1 - 1)^n (q_2 - 1)^n \cdot \text{Sym}\left[\frac{\prod_{i=1}^n z_i^{\lfloor \frac{id}{n} \rfloor - \lfloor \frac{(i-1)d}{n} \rfloor}}{\prod_{i=1}^{n-1}\left(1 - \frac{z_{i+1}}{z_i q_3}\right)} \prod_{1 \le i < j \le n} \zeta\left(\frac{z_i}{z_j}\right)\right] \tag{69}$$

[4] for any $(n, d) \in \mathbb{N} \times \mathbb{Z}$. Moreover, one can associate ribbon skew Schur functions

$$s'_\varepsilon \in \Lambda$$

to any sequence ε consisting of zeroes and ones [5], completely determined by

$$s'_\varepsilon s'_{\varepsilon'} = s'_{\varepsilon 0 \varepsilon'} + s'_{\varepsilon 1 \varepsilon'} \tag{70}$$

and the normalization $s'_{(0^{t-1})} = h'_t$ for all t. It was shown in [7, Section 6.6] that

$$\tau^{\frac{d}{n}}(s'_\varepsilon) = S'_{n,d,\varepsilon}$$

where

$$S'_{n,d,(\varepsilon_1 \dots \varepsilon_{t-1})} = (q_1 - 1)^n (q_2 - 1)^n \cdot \text{Sym}\left[\frac{\prod_{i=1}^n z_i^{\lfloor \frac{id}{n} \rfloor - \lfloor \frac{(i-1)d}{n} \rfloor} \prod_{s=1}^{t-1}\left(-\frac{z_{as+1}}{z_{as} q_3}\right)^{\varepsilon_s}}{\prod_{i=1}^{n-1}\left(1 - \frac{z_{i+1}}{z_i q_3}\right)} \prod_{1 \le i < j \le n} \zeta\left(\frac{z_i}{z_j}\right)\right] \tag{71}$$

for any $(n, d) \in \mathbb{N} \times \mathbb{Z}$ with $\gcd(n, d) = t$ and $a = \frac{n}{t}$.

Remark 4.6. Comparing (71) with (52) yields the identity

$$\bar{H}_{n,d} = \sum_{\varepsilon_1, \dots, \varepsilon_{t-1} \in \{0,1\}} \frac{q_1^{\sum_{s=1}^{t-1} s(1-\varepsilon_s)}}{(q_1 - 1) \dots (q_1^t - 1)} \cdot S'_{n,d,(\varepsilon_1 \dots \varepsilon_{t-1})}$$

which is simply $\tau^{\frac{d}{n}}$ applied to the symmetric function identity

$$h_t = \sum_{\varepsilon_1, \dots, \varepsilon_{t-1} \in \{0,1\}} \frac{q_1^{\sum_{s=1}^{t-1} s(1-\varepsilon_s)}}{(q_1 - 1) \dots (q_1^t - 1)} \cdot s'_{(\varepsilon_1 \dots \varepsilon_{t-1})}.$$

[4] Note that our $\bar{H}'_{n,d}$ is $(1 - q_1) H_{n,d}$ of [4].

[5] Explicitly, for any $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{t-1})$, we define s'_ε as the skew Schur function associated to the size t skew Young diagram whose first box is arbitrary, and whose $i+1$-th box is either to the right or below the i-th box, depending on whether ε_i is 0 or 1 (see [7, Section 6.12] for details).

The upshot of the discussion above is that the elements $\{\bar{H}_{nt,dt}, \bar{H}'_{nt,dt}\}_{t\in\mathbb{N}}$ play the roles of ones and the same symmetric functions for any coprime $(n, d) \in \mathbb{N} \times \mathbb{Z}$, under the isomorphisms (67). Thus, understanding these elements for one slope (say 0) would yield an understanding for all slopes.

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