A REMARK ON CONTRACTIBLE BANACH ALGEBRAS OF OPERATORS

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For a Banach algebra A, we say that an element M in $A \otimes^{\gamma} A$ is a hyper-commutator if $(a \otimes 1)M = M(1 \otimes a)$ for every $a \in A$. A diagonal for a Banach algebra is a hyper-commutator whose image under diagonal mapping is 1. It is well known that a Banach algebra is contractible iff it has a diagonal. The main aim of this note is to show that for any Banach subalgebra $A \subseteq \mathcal{L}(X)$ of bounded linear operators on infinite-dimensional Banach space X, which contains the ideal of finite-rank operators, the image of any hyper-commutator of A under the canonical algebra-morphism $\mathcal{L}(X) \otimes^{\gamma} \mathcal{L}(X) \to \mathcal{L}(X \otimes^{\gamma} X)$, vanishes.

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1. INTRODUCTION

A Banach algebra A is called contractible (super-amenable) [8, 10] if every bounded derivation from A into any Banach A-bimodule is inner. Contractibility is a strong version of the notion of amenability. The concept of amenability (for Banach algebras) has been formulated by Johnson in his seminal paper [4] on Hochschild cohomology of Banach algebras. For various notions of amenability in Theory of Banach Algebras, see [7, 8, 10]. It is known that any finite-dimensional contractible Banach algebra is a finite direct sum of full matrix algebras [10, Theorem 4.1.4]. Until now, the only known contractible Banach algebras are of this form. Indeed, it is a longstanding question that whether every contractible Banach algebra is finite-dimensional [8, p. 224]. Also, the following special case of this question has not been answered yet [3],[8, p. 224]: Does, for any Banach space X, the contractibility of the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on X, imply that X is finite-dimensional? For information on these questions see [10, §4.1 and p. 196] and [12, 5]. We must remark that the chance that there exist infinite-dimensional contractible Banach algebras is not very small: For a long time it was a common belief that for infinite-dimensional Banach spaces X, $\mathcal{L}(X)$ cannot be amenable. But, in

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2009, Argyros and Haydon [1] found out a specific infinite-dimensional Banach space E which its dual is $\ell^1 = E^*$ and has The Scaler-Plus-Compact Property. For such a Banach space E, as it has been pointed out by Dales, $\mathcal{L}(E)$ is an amenable Banach algebra; see [9].

In this note, we introduce the notion of hyper-commutator for Banach algebras. It is well known that a Banach algebra is contractible iff it is unital and has a diagonal. By definition, a diagonal of a Banach algebra is a hyper-commutator whose image under the diagonal mapping is 1. The main aim of this note is to prove the following property of hyper-commutators: For any infinite-dimensional Banach space X, and any Banach subalgebra A of $\mathcal{L}(X)$ which contains the ideal of finite-rank operators, the image of any hyper-commutator of A, under the canonical algebra-morphism,

$$A \otimes^{\gamma} A \hookrightarrow \mathcal{L}(X) \otimes^{\gamma} \mathcal{L}(X) \to \mathcal{L}(X \otimes^{\gamma} X),$$

vanishes. For the proof, we use the famous Kadec–Snobar's estimate [2, Theorem 6.28] on operator-norms of projections.

Since our results mainly concern the contractibility of $\mathcal{L}(X)$, some known results on contractibility are organized in Section 2 for contractible central Banach algebras. (So, there is nothing special new in Section 2.) In Section 3, we prove our main result and give some new remarks on contractibility of $\mathcal{L}(X)$.

2. SOME KNOWN RESULTS ON CONTRACTIBILITY

For preliminaries on contractibility, we refer the reader to Runde's books [8, 10]. (All results in this section are well known or are simple variations of the results of [5, 8, 10, 12].) The topological dual of a Banach space X is denoted by X^*. The completed projective tensor product of Banach spaces X, Y is denoted by $X \otimes^{\gamma} Y$. The projective norm is denoted by $\|\cdot\|_{\gamma}$. The Banach space of bounded linear operators from X into Y is denoted by $\mathcal{L}(X, Y)$. For Banach algebras A, B, the Banach space $A \otimes^{\gamma} B$ is a Banach algebra with the multiplication given by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for $a, a' \in A, b, b' \in B$. The diagonal mapping $\Delta : A \otimes^{\gamma} A \to A$ for A is the unique bounded linear operator defined by $a \otimes b \mapsto ab$. A diagonal for a unital Banach algebra A is an element $M \in A \otimes^{\gamma} A$ satisfying

$$\Delta(M) = 1, \qquad (c \otimes 1)M = M(1 \otimes c), \quad (c \in A).$$

It is well known that a Banach algebra is contractible iff it is unital and has a diagonal: Suppose that A is contractible. Let E denote the Banach A-bimodule with the underlying Banach space A and, left and right module-operations $ax := ax$ and $xa := 0$ for $a \in A, x \in E$. Then $\mathrm{id} : A \to E$ is a derivation and hence, inner. Thus, A has a right unit. Similarly, it is proved that A has a left unit and hence, A is unital. Now, consider the derivation $D : A \to \ker(\Delta)$ defined by $a \mapsto (1 \otimes a) - (a \otimes 1)$. (Note that $A \otimes^\gamma A$ is canonically a Banach A-bimodule with module-operations given by $c(a \otimes b) := (c \otimes 1)(a \otimes b)$ and $(a \otimes b)c := (a \otimes b)(1 \otimes c)$, and Δ is a bimodule-morphism.) D must be inner and thus, there is $N \in \ker(\Delta)$ with the property $aN - Na = D(a)$; hence, $M := N + (1 \otimes 1)$ is a diagonal for A. Conversely, suppose that M,

$$M = \sum_{n=1}^{\infty} a_n \otimes b_n, \qquad \sum_{n=1}^{\infty} \|a_n\| \|b_n\| < \infty, \qquad (a_n, b_n \in A) \tag{1}$$

is a diagonal for A. If $D : A \to X$ is a bounded derivation, then it can be checked that for the element $z := \sum_{n=1}^{\infty} a_n D(b_n)$ of X we have $D(a) = az - za$. Thus, D is inner.

LEMMA 2.1. *Let A be a contractible Banach algebra and let E, F be unital Banach left A-modules. Then any diagonal for A gives rise to a bounded projection $\Phi = \Phi_{E,F}$ from $\mathcal{L}(E, F)$ onto ${}_A\mathcal{L}(E, F)$.*

Here, ${}_A\mathcal{L}(E, F)$ denotes the closed linear subspace of $\mathcal{L}(E, F)$ of all bounded linear left A-module-morphisms from E into F.

Proof. Let M be a diagonal for A. For any $T \in \mathcal{L}(E, F)$, consider the bounded 3-linear mapping

$$\theta_T : A \times A \times E \to F, \quad (a, b, x) \mapsto aT(bx).$$

This gives rise to the bounded linear operator

$$\Theta_T : A \otimes^\gamma A \otimes^\gamma E \to F, \quad (a \otimes b \otimes x) \mapsto \theta_T(a, b, x).$$

We let $\Phi(T) : E \to F$ be the bounded linear operator defined by

$$\big[\Phi(T)\big](x) := \Theta_T(M \otimes x).$$

More explicitly, if M is of the form (1) then

$$\big[\Phi(T)\big](x) := \sum_{n=1}^{\infty} a_n T(b_n x) \quad (x \in E). \tag{2}$$

For $c \in A$ and $x \in E$ we have,

$$\begin{aligned}
\big[\Phi(T)\big](cx) &= \Theta_T(M \otimes cx) \\
&= \Theta_T\big((Mc) \otimes x\big) \\
&= \Theta_T\big((cM) \otimes x\big) \\
&= c\big(\Theta_T(M \otimes x)\big) \\
&= c\big[\Phi(T)\big](x).
\end{aligned}$$

Thus, $\Phi(T)$ belongs to ${}_A\mathcal{L}(E,F)$. It is easily verified that $T \mapsto \Phi(T)$ is a bounded linear operator from $\mathcal{L}(E,F)$ into ${}_A\mathcal{L}(E,F)$. If $T \in {}_A\mathcal{L}(E,F)$, then (2) shows that $\Phi(T)(x) = \sum_{n=1}^{\infty} a_n b_n T(x) = T(x)$. The proof is complete. $\square$

The center of a Banach algebra A is denoted by $\mathcal{Z}(A)$. Note that $\mathcal{Z}(A)$ is a closed subalgebra of A.

PROPOSITION 2.2. *Let A be a contractible Banach algebra. Then any diagonal for A gives rise to a canonical bounded linear operator $\Psi : A \to \mathcal{Z}(A)$ with $\Psi(1) = 1$.*

Proof. Consider A as a Banach left A-module in the canonical fashion. For any $c \in A$, let $\ell_c : A \to A$ denote the left multiplication operator by c. By the notations of Lemma 2.1,

$$\Phi_{A,A}(\ell_c) : A \to A, \qquad x \mapsto \sum_{n=1}^{\infty} a_n c b_n x$$

is a left module-morphism and hence, there is a $\tilde{c} \in A$ such that $\Phi_{A,A}(\ell_c) = r_{\tilde{c}}$ where $r_{\tilde{c}} : A \to A$ denotes the right multiplication operator by $\tilde{c}$. It is clear that $\tilde{c} = \sum_{n=1}^{\infty} a_n c b_n$ and $\tilde{c} \in \mathcal{Z}(A)$. We let Ψ to be defined by $c \mapsto \tilde{c}$. $\square$

Note that a unital Banach algebra is called central if its center is the one-dimensional subalgebra generated by 1. It is an elementary fact that for any Banach space X, $\mathcal{L}(X)$ is central. The following result is a variation of [5, Proposition 5.1].

PROPOSITION 2.3. *Let A be a contractible central Banach algebra. Then any diagonal for A gives rise to a canonical bounded linear functional $\psi \in A^*$ with $\psi(1) = 1$.*

Proof. We have $\mathcal{Z}(A) = \mathbb{C}1$. With the notations of Proposition 2.2, ψ is defined by

$$\Psi(c) = \sum_{n=1}^{\infty} a_n c b_n = \psi(c)1, \qquad (c \in A). \qquad \square$$

By an "ideal" in a Banach algebra, we mean a closed two-sided ideal. It is an elementary fact that any proper ideal in a unital Banach algebra is contained in at least one maximal ideal. Thus, any nonzero unital Banach algebra has at least one maximal ideal.

THEOREM 2.4. *Let A be a contractible central Banach algebra. Then A has a unique maximal ideal $\mathcal{M}_A$.*

Proof. Let

$$\mathcal{M}_A := \text{ closed linear span of } \{c \in A : c \text{ belongs to a proper ideal of } A\}.$$

It is clear that $\mathcal{M}_A$ is an ideal of A which contains every proper ideal of A. With ψ as in Proposition 2.3, for any $c \in A$ which is contained in a proper ideal J of A, we must have $\psi(c) = 0$, because otherwise we must have $1 = \psi(c)^{-1} \sum_{n=1}^{\infty} a_n c b_n \in J$, a contradiction. Thus, $\mathcal{M}_A \subseteq \ker(\psi)$ and hence, $\mathcal{M}_A$ is a proper ideal of A. □

A closed linear subspace F of a Banach space E is called topologically complemented if there is a closed linear subspace F' of E such that $E = F \oplus F'$. In this case, F' is called a topological complement for F. F is topologically complemented in E iff there is a bounded linear projection from E onto F.

LEMMA 2.5. *Let A be a contractible Banach algebra and let E be a unital Banach left A-module. Suppose that E is compactly generated, i.e., there exists a norm-compact subset K of E such that every $x \in E$ is of the form $x = ay$ for some $a \in A, y \in K$. Suppose that E has approximation property. Then E is finite-dimensional.*

Proof. Let M be a diagonal for A of the form (1). We can suppose that $\|b_n\| \to 0$ and $\sup_{n\geq 1} \|a_n\| < \infty$. Continuity of the module-operation implies that the set $\cup_{n\geq 1} b_n K \subset E$ is contained in a compact subset of E. Let $\Phi_{E,E}$ be defined as in the proof of Lemma 2.1. The approximation property for E means that there exists a net $(S_\lambda)_\lambda$ of finite-rank operators in $\mathcal{L}(E)$ such that $S_\lambda \to \mathrm{id}_E$ uniformly on compact subsets of E. The above assumptions imply that $\Phi_{E,E}(S_\lambda)$ is a net of compact operators on E such that it converges uniformly to id_E on K. Now, since $\Phi_{E,E}(S_\lambda)$'s are module-morphisms and K generates E, we have $\Phi_{E,E}(S_\lambda) \to \mathrm{id}_E$ in operator-norm. Thus, id_E is a compact operator. Hence, E is finite-dimensional. □

Note that any Banach space X considered as a unital Banach left $\mathcal{L}(X)$-module in the canonical fashion, is generated by any of its nonzero vectors. Also, for any unital Banach algebra A and any closed left ideal J of A, the quotient Banach left A-module A/J is generated by the class of 1 in A/J.

LEMMA 2.6. *Let A be a contractible Banach algebra and let E be a unital Banach left A-module. Suppose that $F \subset E$ is a closed submodule which is (as a Banach space) topologically complemented in E. Then F has a topological complement in E which is also a closed submodule.*

Proof. Let M be a diagonal of the form (1) for A, and let $\Phi_{E,F}$ be the corresponding operator as in Lemma 2.1. There exists a bounded linear projection p from E onto F. By Lemma 2.1, $\Phi_{E,F}(p)$ is a module-morphism from E into F, and hence, $\ker \Phi_{E,F}(p)$ is a closed submodule of E. Since F is a submodule, for every $y \in F$ and $b \in A$, we have $p(by) = by \in F$. Thus

$$[\Phi_{E,F}(p)](y) = \sum_{n=1}^{\infty} a_n p(b_n y) = \sum_{n=1}^{\infty} a_n b_n y = y, \quad (y \in F).$$

This shows that $\Phi_{E,F}(p)$ is a projection from E onto F. Thus, $\ker \Phi_{E,F}(p)$ is the desired complement for F. $\square$

If A, A' are contractible Banach algebras with diagonals M, M' of the forms as in (1), then $\sum_{n,m=1}^{\infty} a_n \otimes a'_m \otimes b_n \otimes b'_m$ is a diagonal for $A \otimes^\gamma A'$. Also, $\sum_{n=1}^{\infty} b_n \otimes a_n$ is a diagonal for A^{op}, the opposite algebra of A. Thus, if A is contractible, then $A \otimes^\gamma A^{\mathrm{op}}$ is contractible.

The analogue of Lemma 2.6 is satisfied for bimodules:

LEMMA 2.7. *Let A be a contractible Banach algebra and E a unital Banach A-bimodule. If F is a closed sub-bimodule of E which is topologically complemented, then it has a complement in E which is also a sub-bimodule.*

Proof. Any unital Banach A-bimodule E may be considered as unital Banach left $A \otimes^\gamma A^{\mathrm{op}}$-module with module operation given by $(a \otimes b)x := axb$ $(a \in A, b \in A^{\mathrm{op}}, x \in E)$. In this fashion, any A-bimodule-morphism is a left $A \otimes^\gamma A^{\mathrm{op}}$-module-morphism. The converses of this facts are also satisfied. Now, the desired result follows from Lemma 2.6. $\square$

In the following result, we consider some properties of any nonzero proper ideal J of a contractible central Banach algebra A. Note that the existence of J implies that A is infinite-dimensional. Indeed, if A is finite-dimensional then it follows from [10, Theorem 4.1.2] that A is isomorphic to a full matrix algebra and hence, A has not nontrivial ideals.

THEOREM 2.8. *Let A be a contractible central Banach algebra. Suppose that J is a proper and nonzero ideal of A. The following statements hold:*

(i) *J is not topologically complemented in A.*

(ii) *A/J has not approximation property.*

(iii) *If J is compactly generated as left (respectively, right) A-module, then J has not approximation property.*

Proof. (i): If J is topologically complemented in A, then by Lemma 2.7 there is a closed, proper, and nonzero ideal J' such that $A = J \oplus J'$. Thus, we have $J, J' \subset \mathcal{M}_A$, a contradiction. (Note that (i) may be concluded from centrality of A. Indeed, if $A = J \oplus J'$, then there exist orthogonal nonzero central idempotents $e \in J, e' \in J'$ with $e + e' = 1$.) (ii): If A/J has approximation property, then by Lemma 2.5, A/J is finite-dimensional and hence, J is topologically complemented in A, a contradiction with (i). (iii) follows from Lemma 2.5, similarly. $\square$

The following corollary follows from the above results.

COROLLARY 2.9. *Let X be an infinite-dimensional Banach space. If $\mathcal{L}(X)$ is contractible then, (i) X has not approximation property; (ii) $\mathcal{L}(X)$ has a unique maximal ideal $\mathcal{M}$; (iii) $\mathcal{M}$ is not topologically complemented; and (iv) $\mathcal{L}(X)/\mathcal{M}$ has not approximation property.*

A contractible Banach algebra A is called symmetrically contractible if A has a symmetric diagonal; that is, a diagonal M satisfying $\mathcal{F}_A(M) = M$ where $\mathcal{F}_A : A \otimes^\gamma A \to A \otimes^\gamma A$ denotes flip, i.e., the unique bounded linear mapping defined by $(a \otimes b) \mapsto (b \otimes a)$. The matrix algebra $\mathbf{M}_n$ is symmetrically contractible. Indeed, it is well known that $\mathbf{M}_n$ has the unique diagonal $n^{-1} \sum_{i,j=1}^n \delta_{ij} \otimes \delta_{ji}$ where δ_{ij}'s denote the standard basis of $\mathbf{M}_n$. We know from [10, Theorem 4.1.2] that any finite-dimensional contractible Banach algebra is a finite direct sum of full matrix algebras; hence, any such a Banach algebra is also symmetrically contractible. Note that if for $i = 1, \ldots, k$, A_i is a contractible Banach algebra with diagonal M_i, then

$$(M_1, \ldots, M_k) \in \oplus_{i=1}^k (A_i \otimes^\gamma A_i) \subset (\oplus_{i=1}^k A_i) \otimes^\gamma (\oplus_{i=1}^k A_i),$$

is a diagonal for the Banach algebra $\oplus_{i=1}^k A_i$. The following result is a variation of [5, Proposition 5.3].

THEOREM 2.10. *Let A be a symmetrically contractible Banach algebra. Then any symmetric diagonal of A gives rise to a bounded normalized $\mathcal{Z}(A)$-valued trace for A. If A is central, then A has a normalized trace $\psi \in A^*$.*

Proof. Let M be a symmetric diagonal for A of the form (1). We saw in Proposition 2.2 that the assignment $c \mapsto \sum_{n=1}^\infty a_n c b_n$ defines a bounded linear mapping $\Psi : A \to \mathcal{Z}(A)$ with $\Psi(1) = 1$. For every $c, c' \in A$, we have $\sum_{n=1}^\infty b_n \otimes a_n cc' = \sum_{n=1}^\infty cb_n \otimes a_n c'$ and hence $\sum_{n=1}^\infty a_n cc' \otimes b_n = \sum_{n=1}^\infty a_n c' \otimes cb_n$. Thus, we have

$$\Psi(cc') = \Delta\left(\sum_{n=1}^\infty a_n cc' \otimes b_n\right) = \Delta\left(\sum_{n=1}^\infty a_n c' \otimes cb_n\right) = \Psi(c'c).$$

For the second assertion in the statement, if A is a central Banach algebra, then we apply the above reasoning with Ψ replaced by ψ as given by Proposition 2.3. $\square$

For the matrix algebra $\mathbf{M}_n$, the unique diagonal of $\mathbf{M}_n$ gives rise to the ordinary trace.

3. A NULL-PROPERTY OF DIAGONALS

Let X be a Banach space. Consider the unique bounded linear operator

$$\Upsilon : \mathcal{L}(X) \otimes^\gamma \mathcal{L}(X) \to \mathcal{L}(X \otimes^\gamma X),$$

defined by

$$[\Upsilon(T \otimes S)](x \otimes y) = T(x) \otimes S(y), \qquad (T, S \in \mathcal{L}(X), x, y \in X).$$

Then Υ is an algebra-morphism between Banach algebras. We denote the image under Υ of any element $N \in \mathcal{L}(X) \otimes^\gamma \mathcal{L}(X)$, by $N^{\rm op}$. It follows from properties of projective tensor product, that $\|\Upsilon\| = 1$ and hence $\|N^{\rm op}\| \leq \|N\|_\gamma$. Note that, in general, Υ is not one-to-one. (This can be concluded from the fact that the canonical mapping from $X^* \otimes^\gamma X^*$ onto the space of nuclear bilinear functionals on $X \times X$ is not necessarily one-to-one [11, §2.6].)

Proposition 3.1. *Let $\Lambda \in \mathcal{L}(X \otimes^\gamma X)$ be such that for every rank-one operator $T \in \mathcal{L}(X)$,*

$$(T \otimes 1)^{\rm op} \Lambda = \Lambda (1 \otimes T)^{\rm op}.$$

Then there is a unique operator Γ in $\mathcal{L}(X)$ such that $\Lambda = (1 \otimes \Gamma)^{\rm op} \mathcal{F}_X$.

Proof. Let y be a nonzero vector in X, and let $f \in X^*$ be such that $f(y) = 1$. Let $T \in \mathcal{L}(X)$ to be defined by $x \mapsto f(x)y$. For $x \in X$, we have

$$(T \otimes 1)^{\rm op} \Lambda(x \otimes y) = \Lambda(x \otimes y). \tag{3}$$

X has the decomposition $< y > \oplus \ker(f)$ where $< y >$ denotes the subspace generated by y. There exist $z \in \ker(f) \otimes^\gamma X$ and $w \in X$ such that

$$\Lambda(x \otimes y) = y \otimes w + z.$$

It follows from (3) that $\Lambda(x \otimes y) = y \otimes w$. Since the mapping $x \mapsto \Lambda(x \otimes y)$ is linear and bounded, there is $\Gamma_y \in \mathcal{L}(X)$ such that $\Lambda(x \otimes y) = y \otimes \Gamma_y(x)$. Now, suppose that y, y' in X are linearly independent. We have

$$\Lambda\big(x \otimes (y + y')\big) = y \otimes \Gamma_y(x) + y' \otimes \Gamma_{y'}(x),$$
$$\Lambda\big(x \otimes (y + y')\big) = (y + y') \otimes \Gamma_{y+y'}(x).$$

Thus, $\Gamma_y = \Gamma_{y'}$. Also, it can be checked that for every nonzero scalar λ, we have $\Gamma_{\lambda y} = \Gamma_y$. Thus, there exists $\Gamma \in \mathcal{L}(X)$ such that $\Lambda(x \otimes y) = y \otimes \Gamma(x)$ for every $x, y \in X$. The proof is complete. $\square$

Corollary 3.2. *Let M be an element of $\mathcal{L}(X)\otimes^{\gamma}\mathcal{L}(X)$ that satisfies*

$$(4)\qquad (T\otimes 1)M = M(1\otimes T), \qquad (T\in\mathcal{L}(X) \text{ of rank one}).$$

Then there exists $\Gamma\in\mathcal{L}(X)$ such that $M^{\mathrm{op}} = (1\otimes\Gamma)^{\mathrm{op}}\mathcal{F}_X$. Moreover, if M is symmetric (i.e., $\mathcal{F}_{\mathcal{L}(X)}(M)=M$) then there exists a scaler λ such that

$$M^{\mathrm{op}} = \lambda\mathcal{F}_X.$$

Proof. The first part follows directly from Proposition 3.1. Suppose that M is symmetric. It follows from the identity $[\mathcal{F}_{\mathcal{L}(X)}(M)]^{\mathrm{op}} = \mathcal{F}_X M^{\mathrm{op}}\mathcal{F}_X$, that

$$\mathcal{F}_X(1\otimes\Gamma)^{\mathrm{op}} = (1\otimes\Gamma)^{\mathrm{op}}\mathcal{F}_X.$$

Thus, for every $x,y\in X$, we have $\Gamma(y)\otimes x = y\otimes\Gamma(x)$. This means that Γ is a scalar multiple of identity. The proof is complete. □

Let Y,Y',Z be finite-dimensional Banach spaces. Similar to the mapping Υ above, we denote by $\Upsilon: N\mapsto N^{\mathrm{op}}$ the unique bounded linear mapping

$$\mathcal{L}(Y,Z)\otimes^{\gamma}\mathcal{L}(Z,Y')\to\mathcal{L}(Y\otimes^{\gamma}Z, Z\otimes^{\gamma}Y'),$$

given by

$$(T\otimes S)^{\mathrm{op}}(y\otimes z) = \big(T(y)\otimes S(z)\big).$$

We know that this is a linear isomorphism.

Lemma 3.3. *By the above assumptions, suppose that* $\dim(Y)=\dim(Y')$. *Suppose that $T: Y\to Y'$ is a linear isomorphism. For every finite-dimensional Banach space Z, let the linear mapping $\tilde{T}_Z$ be given by*

$$\tilde{T}_Z : Y\otimes^{\gamma}Z\to Z\otimes^{\gamma}Y', \qquad (y\otimes z)\mapsto\big(z\otimes T(y)\big).$$

There is a numerical positive constant c such that c is independent from Z (independent from norm and dimension of Z) and such that:

$$\|\Upsilon^{-1}(\tilde{T}_Z)\|_{\gamma}\geq c^{-1}\dim(Z).$$

Proof. Suppose that $y_1,\dots,y_k$ and $z_1,\dots,z_m$ are vector basis respectively, for Y and Z, and let $y_i' = T(y_i)$. Let the linear operators

$$S_{ij}: Y\to Z, \quad S_{ji}': Z\to Y', \qquad (1\leq i\leq k, 1\leq j\leq m)$$

be given by

$$S_{ij}(y_i) = z_j, S_{ij}(y_q)=0, \quad (q\neq i), \qquad S_{ji}'(z_j) = y_i', S_{ji}'(z_q) = 0, \quad (q\neq j).$$

Let $N := S_{ij}\otimes S_{ji}'$. Then $N^{\mathrm{op}} = \tilde{T}_Z$ and hence $\Upsilon^{-1}(\tilde{T}_Z) = N$.

Let ν denote the linear functional on $\mathcal{L}(Y,Y')$ that associates to any operator $Y\to Y'$, the normalized trace of its matrix in the bases $y_1,\dots,y_k$

and $y'_1, \ldots, y'_k$ of Y and Y'. Suppose that c denotes the functional-norm of ν. It is clear that $c \neq 0$. Consider the bilinear functional

$$\mu : \mathcal{L}(Y, Z) \times \mathcal{L}(Z, Y') \to \mathbb{C}, \qquad (P, Q) \mapsto \nu(QP).$$

Then, we have $\|\mu\| \leq c$ and hence $\|c^{-1}\mu\| \leq 1$. Now, it follows from the properties of projective tensor-norm that

$$\|N\|_\gamma \geq |c^{-1}\mu(N)| = c^{-1}m. \quad \square$$

Proposition 3.4. *Let X be an infinite-dimensional Banach space. Let $M \in \mathcal{L}(X) \otimes^\gamma \mathcal{L}(X)$ be an element that satisfies* (4). *Then $M \in \ker(\Upsilon)$. In other notation, $M^{\mathrm{op}} = 0$.*

Proof. Suppose that $\Gamma \in \mathcal{L}(X)$ is as in Corollary 3.2. Suppose that $M^{\mathrm{op}} \neq 0$ and hence $\Gamma \neq 0$. Let y, y' be two nonzero vectors in X such that $\Gamma(y) = y'$. Suppose that Y, Y' denote the one-dimensional subspaces of X generated respectively, by y, y', and suppose that $T : Y \to Y'$ is defined by $T(y) = y'$. Let Z be an arbitrary finite-dimensional subspace of X. Suppose that $E_Y : Y \to X$ and $E_Z : Z \to X$ denote the embedding-maps and $P_{Y'} : X \to Y'$ is an arbitrary continuous projection from X onto Y'. By Kadec–Snobar's Theorem [2, Theorem 6.28], we know that there exists a continuous projection $P_Z : X \to Z$, from X onto Z, such that $\|P_Z\| < 1 + \sqrt{\dim(Z)}$. Let

$$N := (P_Z \otimes P_{Y'})M(E_Y \otimes E_Z) \in \mathcal{L}(Y, Z) \otimes^\gamma \mathcal{L}(Z, Y').$$

We have

$$\|N\|_\gamma \leq \|P_Z\|\|P_{Y'}\|\|M\|_\gamma, \qquad N^{\mathrm{op}} = \tilde{T}_Z,$$

where $\tilde{T}_Z$ is as in Lemma 3.3. Now, by Lemma 3.3, we have

$$\frac{\dim(Z)}{c\|P_{Y'}\|(1 + \sqrt{\dim(Z)})} < \|M\|_\gamma.$$

This implies that $\|M\|_\gamma = \infty$, a contradiction. Thus, we have $M^{\mathrm{op}} = 0$. $\square$

Definition 3.5. Let A be a Banach algebra and let $M \in A \otimes^\gamma A$. We say that M is a hyper-commutator for A if

$$aM = Ma \qquad (a \in A).$$

By definition, diagonals are hyper-commutators. Following the discussion of Section 1, the next question is very natural.

Question 3.6. Which Banach algebras have nonzero hyper-commutators?

Example 3.7. Let A and C be Banach algebras. Suppose that A has a nonzero hyper-commutator M. Then the Banach algebra $A \oplus C$ has a nonzero hyper-commutator. Indeed, if M is of the form (1) then

$$\sum_{n=1}^{\infty}(a_n, 0) \otimes (b_n, 0) \in (A \oplus C) \otimes^{\gamma} (A \oplus C)$$

is a hyper-commutator for $A \oplus C$. Thus, in particular, there exist infinite-dimensional Banach algebras with nonzero hyper-commutators.

The next theorem, which is the main result of this note, establishes a null-property of hyper-commutators.

THEOREM 3.8. *Let X be an infinite-dimensional Banach space. Next, let $A \subseteq \mathcal{L}(X)$ be a Banach subalgebra such that it contains the ideal of finite-rank operators. Then the image of any hyper-commutator of A under the canonical algebra-morphism*

$$A \otimes^{\gamma} A \hookrightarrow \mathcal{L}(X) \otimes^{\gamma} \mathcal{L}(X) \to \mathcal{L}(X \otimes^{\gamma} X),$$

vanishes.

Proof. It follows directly from Proposition 3.4. □

Note that for any Banach algebra A as in Theorem 3.8, we have

$$\mathcal{Z}(A) = 0 \quad \text{or} \quad \mathcal{Z}(A) = 1\mathbb{C}.$$

Remark 3.9. Suppose that $\mathcal{L}(X)$ is contractible. By Theorem 3.8, to prove that X is finite-dimensional, it is enough to prove that at least one of the diagonals of $\mathcal{L}(X)$ is invertible as a member of the Banach algebra $\mathcal{L}(X) \otimes^{\gamma} \mathcal{L}(X)$. Note that for the unique diagonal M of $\mathbf{M}_n$ we have, in the Banach algebra $\mathbf{M}_n \otimes^{\gamma} \mathbf{M}_n$, $n^2 M^2 = 1 \otimes 1$.

Remark 3.10. Suppose that X is an infinite dimensional Banach space for which the canonical mapping Υ is one-to-one. Then, by Theorem 3.8, any Banach subalgebra of $\mathcal{L}(X)$ containing the ideal of finite-rank operators, is not contractible.

Remark 3.11. Let X be an infinite dimensional Banach space. If subalgebra $A = \mathcal{L}(X)$ has at least two maximal ideals, then by Corollary 2.9, we know that A is not contractible. (See [6] for some examples of such Banach spaces.) Suppose that A has only one maximal ideal J. To prove that A is not contractible it is enough to show that the closer $\tilde{J}$ of the ideal $(J \otimes A) + (A \otimes J) \subset A \otimes^{\gamma} A$ is a maximal ideal of $A \otimes^{\gamma} A$: Indeed, if A is contractible then $A \otimes^{\gamma} A$ is contractible and since $A \otimes^{\gamma} A$ is central (this fact can be checked by considering projections onto finite-dimensional subspaces of X similar to the first part of the proof of Lemma 3.3) then it must have a unique maximal ideal. Thus, we must have $\ker(\Upsilon) \subseteq \tilde{J}$ and hence, for any diagonal M of A, M belongs to $\tilde{J}$. Therefore, we have $1 = \Delta(M) \in \Delta(\tilde{J}) \subset J$ that contradicts the properness of J.

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