In memory of my mother

THE PRIME-REPRESENTING FUNCTIONS

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In this paper, we prove that there is a fixed number D such that the values $\lceil D^{d^n} \rceil$ are primes for all positive integers n and real numbers $d > 8/3$. We also prove that there exists a value e such that the values $\lceil B^{e^n} \rceil$ are primes for all positive integers n and a given any real number $B > 1$, where $\lceil . \rceil$ is the ceiling function.

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1. INTRODUCTION

Mills [2] (only with one page), showed that there is a fixed number A such that, for all positive integers n, the values $\lfloor A^{3^n} \rfloor$ are primes. Niven [3] proved that, given any real number $B > 1$, there exist a value b such that, for all positive integers n, the values $\lfloor B^{b^n} \rfloor$ are prime numbers. Tóth [4] proved that there exists a positive real constant C such that $\lceil C^{c^n} \rceil$ is a prime-representing function for $c \geq 3, c \in \mathbb{N}$ and all positive integers n. All proofs make use of the result that $p_{n+1} - p_n < K p_n^{5/8}$, where K is a fixed positive integer and p_n is the n^{th} prime. This result is due to Ingham [1].

2. THEOREMS

We show that there is a fixed number D such that the values $\lceil D^{d^n} \rceil$ are primes for all positive integers n and real numbers $d > 8/3$ (by Mills's result). Since $p_{n+1} - p_n < K p_n^{5/8}$, we derive

$$p_n > p_{n+1} - K p_n^{5/8} > p_{n+1} - K p_{n+1}^{5/8}. \tag{1}$$

LEMMA 2.1. *If d is a real number greater than $8/3$ and N is a positive integer satisfying $N > K^{\frac{8}{3d-8}} \geq 1$ (K as in [1] or (1)), then there exists a prime p such that $N^d > p > (N-1)^d + 1$.*

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Proof. First, we prove $N^d - N^{d-1} > (N-1)^d + 1$, that is, $(N-1)[N^{d-1} - (N-1)^{d-1}] > 1$. Consider $f(d) = N^{d-1} - (N-1)^{d-1}$. Since

$$f^{'}(d) = N^{d-1} lnN - (N-1)^{d-1} ln(N-1) > 0 \text{ for } d > 1,$$

f is an increasing function. We deduce that $f(d) > f(8/3) > f(2) = 1$. Hence $(N-1)[N^{d-1} - (N-1)^{d-1}] > N - 1 \geq 1$. We conclude that

$$N^d - N^{d-1} > (N-1)^d + 1. \tag{2}$$

Next, we consider $g(x) = x - Kx^{5/8}$ where $x > N^d$. Since

$$g^{'}(x) = 1 - \frac{5}{8}K\frac{1}{x^{3/8}} = \frac{8x^{3/8} - 5K}{8x^{3/8}},$$

it is easy to see that $x^{3/8} > N^{3d/8} > [K^{8/(3d-8)}]^{3d/8} = K^{\frac{3d}{3d-8}} > K$. It follows that $g^{'}(x) > 0$. Therefore, we claim that

$$x - Kx^{5/8} > N^d - KN^{5d/8}. \tag{3}$$

Finally, let p_{n+1} be the smallest prime greater than N^d. Since (1), (2) and (3), we have

$$\begin{aligned} p_{n+1} > N^d > p_n > p_{n+1} - Kp_{n+1}^{5/8} &> N^d - KN^{5d/8} > N^d - N^{\frac{3d-8}{8}} N^{5d/8} \\ &= N^d - N^{d-1} > (N-1)^d + 1. \end{aligned}$$

The lemma is proved. □

Let P_0 be a prime greater than $K^{\frac{8}{3d-8}}$. By using Lemma 2.1, there exists a prime P_1 such that we have ${P_0}^d > P_1 > (P_0 - 1)^d + 1$. On the other hand, $(P_0 - 1)^d + 1 > P_0 > K^{\frac{8}{3d-8}}$, thus, $P_1 > K^{\frac{8}{3d-8}}$ and so on, we can construct an infinite sequence of primes $P_0 < P_1 < P_2 < \cdots$, such that ${P_n}^d > P_{n+1} > (P_n - 1)^d + 1$. Let $u_n = P_n^{d^{-n}}$, $v_n = (P_n - 1)^{d^{-n}}$. Then

$$u_n > v_n; \quad u_{n+1} = P_{n+1}^{d^{-(n+1)}} < u_n = P_n^{d^{-n}};$$
$$v_{n+1} = (P_{n+1} - 1)^{d^{-(n+1)}} > v_n = (P_n - 1)^{d^{-n}}.$$

It follows that the u_n form a bounded monotone decreasing sequence. Let $D = \lim_{n\to\infty} u_n$.

THEOREM 2.2. *For all positive integers n and real numbers $d > 8/3$, the number $\lceil D^{d^n} \rceil$ is a prime.*

Proof. One can check that $u_n > D > v_n$. Therefore, $P_n > D^{d^n} > P_n - 1$, i.e., $\lceil D^{d^n} \rceil$ is a prime. □

We show that there exists a value e such that the values $\lceil B^{e^n} \rceil$ are primes for all positive integers n (by Niven's result).

LEMMA 2.3. *If a is a real number greater than 8 and M is a positive integer satisfying $M > K$ (K as in [1] or (1)), then there exists a prime p such that $M^a > p > (M-1)^a + 1$.*

Proof. First, we prove that $M^a - M^{a-1} > (M-1)^a + 1$; this means that $(M-1)[M^{a-1} - (M-1)^{a-1}] > 1$. Consider $f(a) = M^{a-1} - (M-1)^{a-1}$. Since $f^{'}(a) = M^{a-1}lnM - (M-1)^{a-1}ln(M-1) > 0$ for $a > 1$. We deduce that $f(a) > f(8) > f(2) = 1$. Hence $(M-1)[M^{a-1} - (M-1)^{a-1}] > M - 1 \geq 1$. We conclude that

$$M^a - M^{a-1} > (M-1)^a + 1. \tag{4}$$

Next, we consider $g(x) = x - Kx^{5/8}$ where $x > M^a$. Since

$$g^{'}(x) = 1 - \frac{5}{8}K\frac{1}{x^{3/8}} = \frac{8x^{3/8} - 5K}{8x^{3/8}},$$

it is easy to see that $x^{3/8} > M^{3a/8} > K^{3a/8} > K$. It follows that $g^{'}(x) > 0$. Therefore, we claim that

$$x - Kx^{5/8} > M^a - KM^{5a/8}. \tag{5}$$

Finally, let p_{n+1} be the smallest prime greater than M^a. Since (1), (4) and (5), we have

$$\begin{aligned} p_{n+1} > M^a > p_n > p_{n+1} - Kp_{n+1}^{5/8} > M^a - KM^{5a/8} \\ > M^a - MM^{5a/8} > M^a - M^{a-1} > (M-1)^a + 1. \end{aligned}$$

The lemma is proved. □

Given a real number $B > 1$, we write $\log x$ for the logarithm of x to base B, that is, $\log x = \log_B x$ for every real number $x > 0$. Let P_1 be a prime greater than $\max\{B^8 + 1, K\}$, hence $\log(P_1 - 1) > 8$. By using Lemma 2.3, there exists a prime P_2 such that $P_1^{\log P_1} > P_2 > (P_1 - 1)^{\log P_1} + 1$. Moreover,

$$P_2 > (P_1 - 1)^{\log P_1} + 1 > (P_1 - 1)^{\log(P_1 - 1)} + 1 > P_1 > K. \tag{6}$$

Since $P_1^{\log P_1} > P_2$, we deduce that $\{\log P_1\}\{\log P_1\} > \log P_2$, thus

$$\log P_1 > \{\log P_2\}^{1/2}. \tag{7}$$

Since $P_2 > (P_1 - 1)^{\log(P_1 - 1)} + 1$, that is, $P_2 - 1 > (P_1 - 1)^{\log(P_1 - 1)}$. Therefore, $\log(P_2 - 1) > \{\log(P_1 - 1)\}\{\log(P_1 - 1\}$. It follows that

$$\{\log(P_2 - 1)\}^{1/2} > \log(P_1 - 1) > 8. \tag{8}$$

From (7) and (8), we get

$$\log P_1 > \{\log P_2\}^{1/2} > \{\log(P_2 - 1)\}^{1/2} > \log(P_1 - 1) > 8. \tag{9}$$

By (6) and (9), we have $P_2 > P_1 > \max\{B^8+1, K\}$ and $\{\log P_2\}^{1/2} > 8$, respectively. Similarly, by the lemma there exists a prime P_3 satisfying $P_2^{\{\log P_2\}^{1/2}} > P_3 > (P_2-1)^{\{\log P_2\}^{1/2}}+1 > (P_2-1)^{\{\log(P_2-1)\}^{1/2}}+1 > P_2 > K$. We also have

$$\{\log P_2\}^{1/2} > \{\log P_3\}^{1/3} > \{\log(P_3-1)\}^{1/3} > \{\log(P_2-1)\}^{1/2} > 8.$$

And so on, we can construct an infinite sequence of primes $P_1 < P_2 < P_3 < \cdots$, such that

(10)

$$P_n^{\{\log P_n\}^{1/n}} > P_{n+1} > (P_n-1)^{\{\log P_n\}^{1/n}}+1 > (P_n-1)^{\{\log(P_n-1)\}^{1/n}}+1 > P_n.$$

Let $u_n = \{\log P_n\}^{1/n}$, $v_n = \{\log(P_n-1)\}^{1/n}$. From (10), one can easily check that $u_n > u_{n+1} > v_{n+1} > v_n > 8$. It follows that the u_n form a bounded monotone decreasing sequence. Let $e = \lim_{n\to\infty} u_n$.

THEOREM 2.4. *Given any real number $B > 1$, there exists a value e such that, for all positive integers n, the values $\lceil B^{e^n}\rceil$ are prime numbers.*

Proof. It is easy to see that $u_n > e > v_n$. Therefore, $P_n > B^{e^n} > P_n - 1$, i.e., $\lceil B^{e^n}\rceil$ is a prime. □

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