INDEX THEORY OF PSEUDODIFFERENTIAL OPERATORS ON LIE STRUCTURES

KARSTEN BOHLEN

Communicated by Lucian Beznea

We review recent progress regarding the index theory of operators defined on non-compact manifolds that can be modeled by Lie groupoids. The structure of a particular type of almost regular foliation is recalled and the construction of the corresponding accompanying holonomy Lie groupoid. Using deformation groupoids, K-theoretical invariants can be defined and compared. We summarize how questions in index theory are addressed via the geometrization made possible by the use of deformation groupoids. The discussion is motivated by examples and applications to degenerate PDE's, diffusion processes, evolution equations and geometry.

AMS 2020 Subject Classification: 19K56, 46L80.

Key words: index theory, Lie structure, pseudodifferential operator.

1. INTRODUCTION

The purpose of this article is to survey the state of research and the recent solution of some of the open problems that were posed in the summary of V. Nistor [42] and that were obtained by the author of the present text with collaborators.

In the study of partial differential equations, the Fredholm index plays an important role. It codifies the dimension of the space of solutions of the equation associated to the underlying Hilbert space operator and the image of the operator in the number $\operatorname{ind}(P) = \dim\ker P - \dim\operatorname{coker} P$. Whenever both dimensions in the difference are finite, the operator is Fredholm. The most notable outcome is the Atiyah-Singer index theorem [1, 2], which expresses the index in terms of topological information reliant on the stable homotopy class of the principal symbol of the operator. This theorem is widely acclaimed due to its revelation of surprising connections among Analysis, Topology, and Geometry. Furthermore, it finds diverse applications in the realms of PDEs and theoretical physics. The validity of the result is established for compact manifolds without boundaries, where the ellipticity of the operator is synonymous

MATH. REPORTS **26(76)** (2024), *3-4*, 255–287
doi: 10.59277/mrar.2024.26.76.3.4.255

with its Fredholm property as a bounded operator acting on the appropriate Sobolev spaces. Over the subsequent decades, researchers have explored various potential avenues for generalizing the index theorem. In this note, we focus on the extensions to specific non-compact manifolds.

The geometry of non-compact manifolds with a prescribed behavior at infinity can often be effectively understood via a compactification, where the resulting object is a compact manifold endowed with a structural Lie algebra of vector fields. Such *degenerate vector fields* that give rise to Lie structures could be for example of the type $\{\partial_t, x\partial_x\}$, as occur e.g. in the evolution equation

$$\partial_t u = \Big(\mu - \frac{1}{2}\sigma^2\Big)(x\partial_x)u + \frac{1}{2}\sigma^2(x\partial_x)^2 u$$

that is a well-known example from the study of diffusion processes, cf. Section 2.1. In this example, the intuition is that x serves as a variable defining the boundary of a compact manifold with boundary M, where the vector field ∂_t is defined on the interior, M_0, of the manifold and $x\partial_x$ is tangent to the boundary ∂M. More generally, there occur manifolds with corners, i.e. many intersecting boundary strata which are endowed with a Lie algebra $\mathcal{V}$ of vector fields. To $\mathcal{V}$ we can associate a vector bundle $\mathcal{A}$, whose set of smooth sections is $\mathcal{V}$. Formally, we can associate a groupoid whose Lie algebroid is $\mathcal{A}$:

$$\mathcal{G} := M_0 \times M_0 \cup \partial M \times \partial M \times (\mathbb{R} \setminus \{0\}).$$

The object $\mathcal{G}$ can be understood as a glueing of the pair groupoid on the interior to the cylinder formed out of the boundary. Operators of the type exemplified above can be in a sense realized on this desingularized object $\mathcal{G}$. The proper context for such objects is the theory of singular foliations, i.e. manifolds endowed with a structural Lie algebra of vector fields that belong to a particular class of singular foliations.

In the abstract setup, we study foliations which stem from a Lie algebroid $\mathcal{A}$ over a compact manifold with corners M, with the property that the restriction $\mathcal{A}_{|M_0}$ to a dense open subset $M_0 \subset M$ results in the tangent bundle TM_0 and further condition on the boundary behavior of the accompanying vector fields. These Lie algebroids have universal integrating Lie groupoids, whose construction was described by C. Debord [16] and J. Pradines [46]. In addition, these foliations are the ones studied under the name *manifold with a Lie structure at infinity* or *Lie manifold* by Ammann–Lauter–Nistor [3]. In a more general context, I. Androulidakis and G. Skandalis [4] gave a construction of the holonomy groupoids. The resulting groupoid is no longer transversally smooth, but retains a smooth structure on the fibers. A singular foliation in the sense of Androulidakis–Skandalis becomes a Debord foliation after a Nash blowup [34]. As observed by A. Connes [15], the space of leaves of a foliation, which is generally topologically badly behaved, should be replaced by the corresponding integrating holonomy groupoid. Therefore, in attempting to generalize the Atiyah-Singer index theorem [1] to non-compact manifolds, groupoids play the central role. The index problem of elliptic operators, which may include pseudodifferential operators on noncompact manifolds has applications in the study of partial differential equations. For instance, in the case of stratified manifolds, the index of Lopatinskij–Shapiro type problems associated with the restrictions of the operator to lower dimensional strata gives the number of additional boundary and coboundary conditions that have to be imposed in order to obtain a well-posed problem.

The above considerations lead to three main objects of study in order to address index theory questions on foliated structures:

- The topological *pre-quantization* of singular structures in the form of Lie groupoids (or continuous family groupoids) and their corresponding deformation theory.
- The *quantization* of groupoids, often times given in the form of C^*-algebras of groupoids.
- The K-theoretic invariants furnished by deformation groupoids.

The resulting geometrization of the index theory of interesting operators occurring in the theory of partial differential equations, gives rise to powerful tools.

We study integrated Lie structures $(M, \mathcal{G})$, that is amenable Lie groupoids $\mathcal{G}$ over compact manifolds with corners such that $M_0 = M \setminus \partial M$ is saturated and $\mathcal{G}_{M_0} = M_0 \times M_0$. For instance, that covers the following situations:

- Manifolds with corners [39]. Here $\mathcal{G} = \mathcal{G}_b$ is obtained after blowing-up successively the submanifolds $H_i \times H_i$ into $M \times M$, where H_i run through the connected boundary hypersurfaces of M, and then removing the so-called lateral faces in b-geometry terminology, which equivalently amounts to consider the subspace $\mathrm{SBlup}_{r,s}(M^2, (H_i^2)_i)$ of the blow-up according to the terminology of [19].

- Manifolds with fibered corners, and thus equivalently stratified pseudomanifolds [18]. Here $\mathcal{G} = \mathcal{G}_\pi$ is obtained as before (blowing-up and removing the lateral faces), but now starting with $\mathcal{G}_b$ in which the fibered diagonals $H_i \times_\pi H_i$ are blown-up in the order prescribed by the order relation between boundary hypersurfaces, cf. Section 2.3.

• Manifolds with amenable foliated boundary. The pseudodifferential operators are studied in [48] and the corresponding groupoid $\mathcal{G}_{\mathcal{F}}$, although not directly used, is constructed. Actually, it is stated in [19] that $\mathcal{G}_{\mathcal{F}}$ is obtained by blowing-up in $M \times M$ the holonomy groupoid of the foliation on the boundary, that is $\mathcal{G}_{\mathcal{F}} = \mathrm{SBlup}_{r,s}(M \times M, \mathrm{Hol}(\mathcal{F}))$.

• Contact structures, by which we mean tuples $(M, \mathcal{G})$ where $\mathcal{G}$ is filtered, i.e. the associated Lie algebroid $\mathcal{A}(\mathcal{G})$ is equipped with a filtration by subbundles $0 = \mathcal{A}^0\mathcal{G} \leq \mathcal{A}^1\mathcal{G} \leq \cdots \leq \mathcal{A}^N\mathcal{G} = \mathcal{A}(\mathcal{G})$ such that the module of C^∞-sections $\Gamma(\mathcal{A}^\bullet\mathcal{G})$ is a filtered Lie algebra, cf. Section 3.2.

• There are many other examples related to singular spaces, see for instance [42, 14].

In the case of a contact structure, a special notion for Fredholmness and a modified pseudodifferential calculus is needed. In the other examples, there is a well-defined notion of *full ellipticity for operators, namely the principal symbol and the indicial symbols* (*restrictions* $(I_F)_{F\in\mathcal{F}_1(M)}$ *of the operator family* $(P_x)_{x\in M}$*, defined over the fibers* $(\mathcal{G}_x)_{x\in M}$*, to the closed embedded boundary faces* $F \in \mathcal{F}_1(M)$*,* $I_F\colon (P_x)_{x\in M} \mapsto (P_x)_{x\in F}$) *are both pointwise invertible, cf. Section* 4.1 in the corresponding calculus, that ensures the Fredholmness of the associated operators on M.

Although we lay a focus on the index of (fully-)elliptic operators, the theory also enables the study of non-elliptic Fredholm operators, using an adapted associated pseudodifferential calculus (on so-called contact structures which are special cases of almost injective Lie algebroids, cf. Section 3.2) and higher invariants cf. Section 6.

2. OPERATORS ON MANIFOLDS WITH CORNERS

2.1. Degenerate operators and propagators

We begin this section with several examples of degenerate operators which can be naturally defined on Lie structures. As is shown later, they occur as special cases of the (pseudo-)differential operators on Lie structures.

Example 2.1. 1) Let (X_t) denote the Itô diffusion process fulfilling the stochastic differential equation:

$$dX_t = \mu(X,t)\,dt + \sigma(X,t)\,dW_t.$$

Given a twice differentiable function $f(t,x)$, application of a Taylor expansion and of Itô's Lemma furnishes the Kolmogorov backward equation:

$$-\partial_t f(x,t) = \mu(x,t)\partial_x f(x,t) + \frac{1}{2}\sigma^2(x,t)\partial_x^2 f(x,t).$$

The corresponding propagator is the operator $\partial_t + \mathcal{L}^+$, where

$$\mathcal{L}^+ := \mu(x,t)\partial_x + \frac{1}{2}\sigma^2(x,t)\partial_x^2$$

denotes the Kolmogorov backward operator. In important special cases, the propagator is an elliptic parabolic operator of degenerate type. For example, this is the case if the process is assumed to be stationary, Markovian, ergodic and the corresponding diffusion matrix is nonnegative definite, i.e., X is a hypoelliptic diffusion process. We have the underlying initial value problem:

$$\partial_t f = \mathcal{L}^+ f, \ f(0,x) = g(x).$$

Formally, a solution is denoted by $u(t) = \exp(-t\mathcal{L}^+)$. Several numerical, approximate schemes, also based on the Laplace operator in Riemannian geometry (cf. e.g. [20, 23, 36, 40, 45, 50]), have been proposed to study the solution operator in various special cases of interest. Many important cases of elliptic generators $\mathcal{L}^+$ correspond to differential operators associated to specific Lie structures. We return to the complicated analysis of the associated heat operator $\exp(-tG^2)$ of a pseudodifferential operator P on an arbitrary Lie structure for an unbounded representative G in Section 5. Of interest is also the associated wave operator, which describes the diffuse behavior (cf. quantum dynamics) of the underlying system. We make some remarks regarding future research directions at the end of the paper.

2) Consider the special case of the previous example given by the Black-Scholes-Merton model with constant drift μ and constant volatility σ, corresponding to a fixed underlying asset. Let f be twice differentiable such that

$$df = \left(\mu x \partial_x f + \partial_t f + \frac{1}{2}\sigma^2 x^2 f\right) dt.$$

Note that $(x\partial_x)^2 - x\partial_x = x^2\partial_x^2$, hence

$$\begin{aligned} df &= (\mu x \partial_x f + \partial_t f + \frac{1}{2}\sigma^2 (x\partial_x)^2 f - \frac{1}{2}\sigma^2 (x\partial_x) f)\, dt \\ &= \left(\left(\mu - \frac{1}{2}\sigma^2\right)(x\partial_x) f + \frac{1}{2}\sigma^2 (x\partial_x)^2 f + \partial_t f\right) dt \\ &=: ((\mathcal{L}^+_{\mu,\sigma} + \partial_t) f)\, dt. \end{aligned}$$

Here

$$\mathcal{L}^+_{\mu,\sigma} := \left(\mu - \frac{1}{2}\sigma^2\right)(x\partial_x) + \frac{1}{2}\sigma^2 (x\partial_x)^2$$

is the corresponding elliptic generator. The vector fields $\{\partial_t, x\partial_x\}$ generate a smooth foliation; by the Serre-Swan theorem these generators give rise to a vector bundle $T_b M$ over a compact manifold with boundary M.

3) Various forms of the Laplacian operator make their appearance as differential operators adapted to different Lie structures.

- We can consider the Laplacian in cylindrical coordinates (ϱ, θ, z) on $\mathbb{R}^3$ written as

$$\Delta u = \varrho^{-2}\big((\varrho\partial_\varrho)^2 u + \partial_\theta^2 u + (\varrho\partial_z)^2\big).$$

The local generators are $\{\varrho\partial_\varrho, \partial_\theta, \varrho\partial_z\}$ which yield the Lie structure of edge type. The behavior of partial differential equations near the edges of certain polyhedral domains can be described in this way. We can consider boundary value problems on such domains with the help of a so-called wedge structure. This is modeled on particular manifolds with corners in three dimensions.

- The Laplace operator in polar coordinates (ϱ, θ) on $\mathbb{R}^2$ is written

$$\Delta u = \varrho^{-2}(\varrho^2\partial_\varrho^2 u + \varrho\partial_\varrho u + \partial_\theta^2 u).$$

Therefore, we have the local generators $\{\varrho^{-1}\partial_\varrho, \partial_\theta\}$ which correspond to the Lie structure of b-type near $\varrho = \infty$. This example is relevant when considering boundary value problems on domains with conical ends [42, 26].

- Similarly, we can write the Laplacian in spherical coordinates (ϱ, x'), $\varrho > 0,\ x' \in S^2$

$$-\Big(\Delta + \frac{Z}{\varrho}\Big)u = -\varrho^{-2}(\varrho^2\partial_\varrho^2 u + 2\varrho\partial_\varrho u + \Delta_{x'} u + Z\varrho u).$$

We obtain the Schrödinger operator with Newton potential in three dimensions, see also [42].

2.2. **Manifolds with corners**

A manifold with corners is a manifold modeled on the space $[0,\infty)^n = \overline{\mathbb{R}}_+^n$. To develop the theory of such manifolds one starts by identifying the points $x \in M$ of different codimension. A point x has a neighborhood of the form $(0,\infty)^k \times \mathbb{R}^{n-k}$, where by k we designate the codimension of the point. Connected components of codimension k points are referred to as codimension k faces. The closed faces of codimension k are the closures of open faces of codimension k. A convenient supplementary condition is to require faces to be in addition embedded in M. By that we mean that the closed, embedded hyperface F has a corresponding smooth function $\rho_F\colon M \to \overline{\mathbb{R}}_+$ that vanishes on F and $d\rho$ is non-vanishing when restricted to F. The nice feature of such faces is that they have again the structure of a manifold with corners. We denote by $\mathcal{F}_k(M)$ the closed codimension k-faces. The C^∞-structure on M

can be determined by a smooth neighborhood (a smooth manifold without boundary) $\widetilde{M}$ into which M is embedded and which by pullback furnishes the C^∞-structure on M. We recall the notion of a *tame submersion* for a C^∞ map $f : M \to N$ between manifolds with corners, that at any point $x \in M$ we have:

$$df_x(T_xM) = T_{f(x)}N \quad \text{and} \quad (df_x)^{-1}\big(T^+_{f(x)}N\big) = T^+_xM. \tag{1}$$

Here, TM denotes the ordinary tangent vector bundle and T^+M its subset of inward pointing vectors. Under such assumptions, f preserves the codimension of points and its fibers have no boundary.

2.3. Integrated Lie structures

By a Lie structure, we mean a structural Lie algebra of vector fields that forms a projective, finitely generated C^∞-module, or equivalently a Lie algebroid with additional properties. Such structures are special kinds of singular foliations that are in particular almost injective. The Lie structures were introduced by B. Ammann, R. Lauter and V. Nistor as an axiomatization of compactifications of certain non-compact manifolds with a controlled structure at infinity. The motivation stems from singular analysis and the Melrose quantization problem, a program that was detailed in the talk [38]. In general, to any singular foliation one can construct a holonomy groupoid, i.e., an integrating groupoid which is minimal or universal in a precise sense. However, the groupoid may fail to be transversally smooth and Hausdorff, but still possesses smooth range/source fibers (without corners), cf. I. Androulidakis and G. Skandalis [4]. Almost injective Lie algebroids on the other hand have a Hausdorff and C^∞ integrating holonomy groupoid, cf. the construction detailed by C. Debord [16]. By a Lie groupoid over a compact manifold with embedded corners $\mathcal{G}_0 = M$ we mean the structural maps

$$(r, s)\colon \mathcal{G} \rightrightarrows M, u\colon \mathcal{G}_0 \hookrightarrow \mathcal{G}, i\colon \mathcal{G} \xrightarrow{\sim} \mathcal{G},$$
$$m\colon \mathcal{G} *_s^r \mathcal{G} = \big\{(\gamma, \eta) \in \mathcal{G} \times \mathcal{G} : s(\gamma) = r(\eta)\big\} \to \mathcal{G}$$

fulfilling the axioms

i) $(s \circ u)_{|\mathcal{G}_0} = (r \circ u)_{|\mathcal{G}_0} = \mathrm{id}_{\mathcal{G}_0}$,

ii) For each $\gamma \in \mathcal{G}$

$$m\big((u \circ r)(\gamma), \gamma\big) = \gamma, \; m\big(\gamma, (u \circ s)(\gamma)\big) = \gamma,$$

iii) For $(\gamma, \eta) \in \mathcal{G} *_s^r \mathcal{G}$ we have

$$r\big(m(\gamma, \eta)\big) = r\big(\gamma\big), \; s\big(m(\gamma, \eta)\big) = s(\eta),$$

iv) For $(\gamma_1, \gamma_2), \; (\gamma_2, \gamma_3) \in \mathcal{G} *_s^r \mathcal{G}$, we have

$$m\big(m(\gamma_1, \gamma_2), \gamma_3\big) = m\big(\gamma_1, m(\gamma_2, \gamma_3)\big),$$

v) $r \circ i = s,\ s \circ i = r$,

vi) For all $\gamma \in \mathcal{G}$

$$m\big(i(\gamma), \gamma\big) = \mathrm{id}_{s(\gamma)},\ m\big(\gamma, i(\gamma)\big) = \mathrm{id}_{r(\gamma)},$$

such that the arrows $\mathcal{G} = \mathcal{G}_1$ form a C^∞ manifold with corners, all structural maps are C^∞ maps of manifolds with corners and the source map is a tame surjective submersion. By the axiom $r = s \circ i$, also the range map is a tame surjective submersion. Thereby, the source and range fibers $s^{-1}(x),\ r^{-1}(x)$ for each $x \in \mathcal{G}_0 = M$ are smooth manifolds without corners. It is convenient to denote multiplication by $m(\gamma, \eta) = \gamma \cdot \eta$, whenever the arrows are composable and by $i(\gamma) = \gamma^{-1}$ the inverse operation. The associated Lie algebroid is defined as the pullback by the unit inclusion $\mathcal{A}(\mathcal{G}) := u^* T^s\mathcal{G}$, where by $T^s\mathcal{G} = \ker ds$ we denote the s-vertical tangent bundle. The anchor is defined as $\varrho = dr \circ u^* \colon \mathcal{A}(\mathcal{G}) \to TM$. On the other hand, given an abstract Lie algebroid $(\pi \colon \mathcal{A} \to M, \varrho \colon A \to TM)$, we call $\mathcal{A}$ integrable if there is a Lie groupoid $\mathcal{G}$ such that $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$.

Definition 2.2. We recall the definition of a Lie structure.

1. Let M be a compact manifold with embedded corners. A *Lie structure* is a Lie algebroid $(\pi \colon \mathcal{A} \to M, \varrho \colon \mathcal{A} \to TM)$ such that $\mathcal{A}_{|M_0} = \pi^{-1}(M_0)$ the restriction to the interior $M_0 := M \setminus \partial M$ is isomorphic, via the anchor map ϱ, to the tangent bundle on the interior TM_0 (almost injectivity) and in addition the structural Lie algebra of vector fields $\mathcal{V} := \Gamma(\mathcal{A})$ consists of vector fields tangent to all faces in M.

2. A Lie structure $\mathcal{A}$ is called *integrated* if there is an s-connected Lie groupoid $\mathcal{G} \rightrightarrows M$ such that the associated Lie algebroid is canonically isomorphic to $\mathcal{A}$, i.e., $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$.

Remark 2.3. Given a C^∞-manifold M, the Serre-Swan theorem furnishes a category equivalence between C^∞-modules that are finitely generated projective over M and the category of finite rank C^∞ vector bundles over M. For a structural Lie algebra of vector fields that is a projective $C^\infty(M)$-module, this furnishes a vector bundle $\mathcal{A} \to M$ that carries the natural Lie algebroid structure. Note that, we have not made use of the compactness of M.

Lie structures can be classified by the generators of the structural Lie algebra of C^∞ vector fields $\mathcal{V} := \Gamma(\mathcal{A})$, cf. Table 1.

Table 1 – Singular manifolds

Local generators	Lie structure
∂_{x_i}	smooth, compact
$x\partial_x,\ x\partial_{y_i}$	asymptotically euclidean
$x\partial_x,\ \partial_{y_i}$	b-type
$x^n\partial_x,\ \partial_{y_i}$	general cusps ($n \geq 2$)
$x^2\partial_x,\ x\partial_{y_i}$	scattering
$x^l\partial_x,\ x^l\partial_{y_i},\ \partial_{z_j}$	edge (l-fold)
$x^2\partial_x,\ x\partial_{y_i},\ \partial_{z_j}$	fibered cusp
etc.	

Example 2.4 (Scattering Lie structure). Let M be a compact manifold with boundary endowed with the Lie structure of scattering vector fields, i.e. the module of vector fields $\mathcal{V}_{sc} = p\mathcal{V}_b$ where p is the boundary defining function. In local coordinates where $x_1 = p$ the generators of these vector fields can be chosen as $\{x_1^2\partial_{x_1}, x_1\partial_{x_j}\}$, $j > 1$. An integrating groupoid in this case is written as a set

$$\mathcal{G}_{sc} = T_{\partial M}M \cup (M_0 \times M_0) \rightrightarrows M.$$

Here the tangent bundle restricted to ∂M is a Lie groupoid which is glued to the pair groupoid on the interior. If M is a compact manifold with corners the scattering groupoid takes the form

$$\mathcal{G}_{sc} = \left(\bigcup_{F \in \mathcal{F}_1(M)} T_F M \right) \cup (M_0 \times M_0) \rightrightarrows M.$$

Example 2.5 (Generalized cusp Lie structure). On the compact manifold with corners M we consider the Lie structure $\mathcal{V}_{c_n}$ of generalized cusps for $n \geq 2$. The local generators of vector fields of $\mathcal{V}_{c_n}$ are given by $\{x_1^n\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$. Then an integrating groupoid is defined as

$$\Gamma_n(M) = \left\{ (x, y, \lambda) \in M \times M \times (\mathbb{R}_+)^I : \lambda_i p_i(x)^n p_i(y)^n = p_i(x)^n - p_i(y)^n, \forall\, i \in I \right\}.$$

The structural maps are defined in the same way as in the b-groupoid case. We set $\mathcal{G}_n(M) := \mathcal{C}_s\Gamma_n(M)$, the s-connected component of the groupoid $\Gamma_n(M)$.

Example 2.6 (Fibered cusp Lie structure). Another interesting case is that of manifolds with iterated fibered boundary, in particular the following example is based on [35] and the definitions are from [18]. For the construction of the integrating Lie groupoid of a different type of fibered cusp Lie structure we refer to [22]. We assume again that M is a compact manifold with corners, but this time the boundary strata are assumed to be fibered in the following sense. Let $\{F_i\}_{i\in I}$ the boundary hyperfaces of M and denote by $\pi = (\pi_1, \dots, \pi_N)$ an iterated boundary fibration structure: There is a partial order defined on $\{F_i\}_{i\in I}$, $\pi\colon F_i \to B_i$ are fibrations where B_i is the base, a compact manifold with corners. The Lie structure is defined via

$$\mathcal{V}_\pi := \left\{V \in \mathcal{V}_b : V_{|F_i} \text{ tangent to fibers of } \pi_i\colon F_i \to B_i,\ V p_i \in p_i^2 C^\infty(M)\right\}$$

where $\{p_i\}_{i\in I}$ denotes the boundary defining functions as usual. Then $\mathcal{V}_\pi$ is a finitely generated $C^\infty(M)$-module and a Lie sub-algebra of $\Gamma^\infty(TM)$. The corresponding groupoid is amenable [18, Lemma 4.6]; as a set it is defined as

$$\mathcal{G}_\pi(M) := (M_0 \times M_0) \cup \left(\bigcup_{i=1}^N (F_i \times_{\pi_i} T^\pi B_i \times_{\pi_i} F_i) \times \mathbb{R}\right),$$

where $T^\pi B_i$ denotes the algebroid of B_i.

Example 2.7. We model space time as a $3+1$ dimensional manifold M_0, i.e., a manifold endowed with a metric g_0. The metric in most cases is the Schwartzschild or Kerr metric and describes a manifold which is *asymptotically euclidean.* This is a particular example of a Lie structure. In [25], the authors consider this manifold to prove the uniqueness of smooth stationary black holes in general relativity under certain assumptions.

2.4. **Pseudodifferential operators**

Abstractly, a Lie structure $\mathcal{V}$ gives rise to an enveloping algebra, which provides us with a class of differential operators. In fact, the class of differential operators of order m is recursively defined as

$$\begin{aligned}\mathrm{Diff}_{\mathcal{V}}^m(M) &:= \mathcal{V}\mathrm{Diff}_{\mathcal{V}}^{m-1} + \mathrm{Diff}_{\mathcal{V}}^{m-1}(M)\\ &= \left\{f V_1 \cdots V_m : f \in C^\infty(M),\ V_j \in \mathcal{V},\ 1 \le j \le m\right\}.\end{aligned}$$

We set $\mathrm{Diff}_{\mathcal{V}}^0(M) := C^\infty(M)$. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, define by $\mathrm{Diff}^m(\mathcal{G})$ the right-equivariant, linear operators $C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G})$ which are generated by vector fields in $\Gamma(T^s\mathcal{G})$ as a $C_c^\infty(\mathcal{G})$-module. The bigger class of pseudodifferential operators is essentially given by a family of pseudodifferential operators on the fibers of the groupoid.

Since the Lie groupoids $\mathcal{G} \rightrightarrows \mathcal{G}_0$ under consideration have by definition the source and range given by tame submersions, the fibers $\mathcal{G}_x$ are smooth manifolds without corners. Imposing a uniform support property implies that the operator is properly supported when considered as an operator acting on $C_c^\infty(\mathcal{G})$. We obtain furthermore, by imposition of a right equivariance property of the family $P = (P_x)_{x\in\mathcal{G}_0}$, a reduced kernel k_P, which is a compactly supported distribution on $\mathcal{G}$ conormal to $\mathcal{G}_0$. If P is such an operator, note that by right equivariance

$$k_{r(\gamma)}(\gamma_1,\gamma_2) = k_{s(\gamma)}(\gamma_1\gamma^{-1},\gamma_2\gamma^{-1}) \text{ for } \gamma_1,\gamma_2 \in \mathcal{G}_{r(\gamma)},\ \gamma\in\mathcal{G}$$

it follows that $k_{s(\gamma)}(\gamma,\eta) = k_{r(\gamma)}(\mathrm{id}_{r(\gamma)},\eta\gamma^{-1}) =: k_P(\eta\gamma^{-1})$. In particular, the reduced kernel $k_P(\eta\gamma^{-1}) = k_{s(\gamma)}(\eta,\gamma)$ depends only on $\eta\gamma^{-1}\in\mathcal{G}$ for each $(\eta,\gamma^{-1})\in\mathcal{G}*_s^r\mathcal{G}$ as can be shown by the right-invariance of a given Haar system. Thus the $\mathcal{G}$-operator $P\colon C_c^\infty(\mathcal{G})\to C_c^\infty(\mathcal{G})$ is defined as

$$Pu(\gamma) = \int_{\mathcal{G}_{s(\gamma)}} k_P(\gamma\eta^{-1})u(\eta)\,d\mu_{s(\gamma)}(\eta)$$

for $u\in C_c^\infty(\mathcal{G})$ and $(\mu_x)_{x\in\mathcal{G}_0}$ is a smooth right invariant system of Haar measures which is uniquely determined up to Morita equivalence. The class of pseudodifferential operators $\Psi^m(\mathcal{G})$ on $\mathcal{G}$ is defined in the following fashion, where the kernel k_P turns out to be a distribution on $\mathcal{G}$ that is conormal to M (which is embedded in $\mathcal{G}$, via the unit inclusion).

Definition 2.8. A pseudodifferential operator on $\mathcal{G}$ is a C^∞-family of pseudodifferential operators on the s-fibers of $\mathcal{G}$, equivariant with respect to the action of $\mathcal{G}$.

We refer to [43] for the technical details regarding the uniform support condition contained in the notion of being a C^∞-family which entails that k_P is uniformly supported. We denote by $\Psi_{\mathcal{V}}^*(M;E_0,E_1)$ the algebra of pseudodifferential operators of Lie type on $M_0 = M\setminus\partial M$ acting between the sections of vector bundles $E_j\to M$ [3]. It coincides with the image of $\Psi_{\mathcal{G}}^*(E_0,E_1)$ by the vector representation $r_\#$ when $\mathcal{G}$ is s-connected [3]. By a slight abuse of notation, we continue to set $\Psi_{\mathcal{V}}^* = r_\#\Psi_{\mathcal{G}}^*$ in the general case. Equivalently, $\Psi_{\mathcal{V}}^*$ is isomorphic to the space obtained by restricting elements of $\Psi_{\mathcal{G}}^*$ to the fiber of $\mathcal{G}$ over any arbitrary interior point x. The isomorphism comes then from the diffeomorphism $r:\mathcal{G}_x\to M_0$. One observes that the pseudodifferential operators of order <0 are contained in $C^*(\mathcal{G})$. The 0-order operators $\Psi^0(\mathcal{G})$ are a subalgebra of the multiplier algebra $M(C^*(\mathcal{G}))$ which makes it possible to consider the norm closure $\overline{\Psi^0}(\mathcal{G})$. Since the groupoids under consideration are assumed to be amenable, we do not need to make any distinction between the reduced and the full C^*-algebras of a Lie groupoid. More details and proofs of these assertions are contained in the survey [19].

3. DESINGULARIZATION AND RENORMALIZATION

3.1. Renormalization

In order to obtain finite values, upon integration of forms that are generated by a given Lie structure, the need arises to suitably renormalize the singular denominators. The renormalization procedure depends on the order of the singularities considered (referred to as the degeneracy index below).

Example 3.1. Consider the case of the b vector fields $\mathcal{V} = \mathcal{V}_b$ on a manifold M_0 with cylindrical end $(-\infty, 0]_s \times Y$. The Kondratiev transform $x = e^s$ furnishes a compactification $\widehat{M_0} = M$ to a manifold with boundary, where $s \to -\infty$ corresponds to $x \to 0$. Close to the boundary we have the density $ds = \frac{dx}{x}$ with $\partial_s = x\partial_x$. The singular structure is encoded in a Riemannian metric g (a compatible metric on the b-tangent bundle $T_bM \to M$) which is product type close to the boundary

$$g = ds^2 + h = \left(\frac{dx}{x}\right)^2 + h.$$

Notice that $\frac{dx}{x}$ is not integrable over $[0,1]_x$. We use therefore the renormalization by observing that for $\Re z > 0$ the function x^z is integrable with regard to $\frac{dx}{x}$ over $[0,1]_x$. By setting

$$G(f)(z) = \int_M x^z f \, dg, \ \ f \in C^\infty(M), \ \ \Re(z) > 0$$

we define the b-trace as the regularized value of $G(f)(z)$ in $z = 0$ by which we mean that $G(f)$ extends to a meromorphic function of $z \in \mathbb{C}$, and we consider the constant Laurent coefficient at 0.

These issues motivate the following general definition of renormalizability. To keep the presentation short, we exclude here the case of fibered corners. However, in [12] renormalizability is defined more generally for a class of Lie structures of fibration type, see also [42, 37].

Definition 3.2. Let $(M, \mathcal{V})$ be a compact manifold with corners, endowed with Lie structure $\mathcal{V}$ and boundary defining functions $(\rho_F)_{F \in \mathcal{F}_1(M)}$. We set $\rho := \prod_{F \in \mathcal{F}_1(M)} \rho_F$. Then, the Lie structure $\mathcal{V}$ is called *renormalizable* if for $\omega \in {}^{\mathcal{V}}\Omega^p(M)$ there is a $k \geq 1$ such that $G_\rho(\omega)(z) := \int_M \rho^z \omega$ is holomorphic on $\{z \in \mathbb{C}^{\mathcal{F}_1(M)} : \Re(z_F) > k - 1\}$ and admits a meromorphic extension to $\mathbb{C}^{\mathcal{F}_1(M)}$. We call the minimal $k_{\deg}(\mathcal{V}) := k$ for which this property holds, the *degeneracy index* of $\mathcal{V}$. We call the boundary defining functions a *choice of renormalization*. Define

$$\text{(2)} \qquad {}^{\mathcal{V}}\!\!\!\!\!\!\!\!\;\;\fint_M \omega := \operatorname{Reg}_{z=0} \int_M \rho^z \omega$$

as the regularized value at $z = 0$ of $G_\rho(\omega)(z)$.

Remark 3.3. i) Up to a remainder term, involving the integral and derivatives of ω restricted to the boundary components, the value depends on the choice of boundary defining functions [29, Section 4].

ii) We make in what follows the assumption that $(M, \mathcal{G}, \mathcal{A})$ is an integrated Lie manifold with connected boundary ∂M and trivial fibration $p \colon \partial M \to \{pt\}$. Assume that the underlying Lie structure $\mathcal{V}_k := \Gamma(\mathcal{A}(\mathcal{G}_k))$ has generators of the form $\{x^k \partial_x, \partial_y\}$ where $k \in \mathbb{N}$. We refer to this as the c_k*-structure*. Denote by $\mathcal{V}_b$ the Lie structure consisting of smooth vector fields that are tangent to the boundary ∂M. Hence the c_1-structure corresponds to the b-structure. The degeneracy indices are $k_{\mathrm{deg}}(\mathcal{V}_k) = k$. According to [28, Section 15.3], the norm closure of the calculus for the c_k structure with $k \geq 2$ is the same as the norm closure of the b-structure. In particular, the Lie groupoid C^*-algebras are equal. In the more general case of a fibered cusp Lie structure, given a geometric Dirac operator that is fully elliptic, the index problem can be reduced to a fully elliptic geometric Dirac operator on a b-manifold, by deforming the exact metrics, cf. [32]. In view of these facts, we restrict the following discussion of the renormalized integrals to the case of the integrated Lie structure $(\mathcal{G}, M, \mathcal{A})$ with $\mathcal{V} = \mathcal{V}_b = \Gamma(\mathcal{A})$. For this case, we can compare the previously defined renormalization with the Hadamard partie finie procedure. To M attach the semi-infinite cylinder $\partial M \times (-\infty, 0]$ yielding a manifold with cylindrical ends M_{cyl}. We take the metric $g = g_{\partial M} + dt^2$ and map the cylindrical end via the transform $r = e^t$ to a tubular neighborhood of the boundary, obtaining the cylindrical end metric $g = g_{\partial M} + (r^{-1} dr)^2$ near the boundary. We consider the set ${}^{\mathrm{b}}\mathrm{C}^\infty(M_0)$ defined to consist of $f \in C^\infty(M_0)$ such that there are $f_0^-, f_1^-, \dots \in C^\infty(\partial M)$ with

$$\begin{aligned} \widetilde{f}(x, -) &\sim_{x \to -\infty} f_0^- + f_1^- e^x + f_2^- e^{2x} + \cdots, \\ \widetilde{f}(x, -) &= f(e^x, -), \end{aligned}$$

cf. [32, Section 1.6] for the details. Accordingly, we define the p-forms ${}^{\mathrm{b}}\Omega^p(M_0)$, i.e. forms that in local coordinates are written as

$$\omega = f(x, y) dx \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_{p-1}} + g(x, y) dy_{j_1} \wedge \cdots \wedge d_{y_{j_p}}$$

where $1 \leq i_1 < \cdots < i_{p-1} \leq n$ and $1 \leq j_1 < \cdots < j_p \leq n$ and f, g are contained in ${}^{\mathrm{b}}\mathrm{C}^\infty(M_0)$. Consider the functions smooth up to the boundary, $C^\infty(M)$. The restriction to the interior induces an isomorphism of algebras $C^\infty(M) \xrightarrow{\sim} {}^{\mathrm{b}}\mathrm{C}^\infty(M_0)$ and the inclusion of the boundary $\partial M \to M_{\mathrm{cyl}}$ induces a pullback $i^* \colon {}^{\mathrm{b}}\Omega^p(M_0) \to \Omega^p(\partial M)$. Given a p-form ω on the manifold with cylindrical end, the renormalized integral is defined via the Hadamard finite part renormalization

$$ {}^{\mathrm{b}}\!\!\int_{M_{\mathrm{cyl}}} \omega = \text{ finite part of } \int_{x \geq -R} \omega \text{ as } R \to \infty. $$

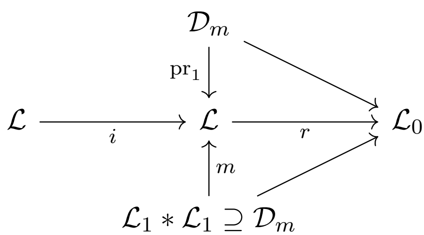
3.2. **Holonomy groupoid**

Let us briefly sketch the construction of the holonomy groupoid as described by C. Debord [16], given an almost injective Lie algebroid $\mathcal{A}$.

The Lie groupoid is reconstructed via a sheaf of germs of equivalence classes of partial Morita equivalences that fulfills an additional technical condition. Given an almost injective Lie algebroid $(\pi \colon \mathcal{A} \to M, \varrho)$ over a compact manifold with corners with dense interior $M_0 = M \setminus \partial M$. As demonstrated in the work C. Debord [16] and J. Pradines [46], a sheaf of germs can be constructed and used to glue the holonomy groupoid. We recall this construction in the present section.

We consider spans $A \leftarrow Z \to B$ that are formed out of the tame surjective submersions, in the category of C^∞ manifolds with corners. The notion of Lie groupoids that are final objects in the category of such spans goes back to B. Bigonnet under the name of quasi-graphoid. J. Renault [48] refers to them as essentially principal groupoids and J. Pradines calls them monograph groupoids.

Definition 3.4. A *partial Lie groupoid* is given by the data $(\mathcal{L}_1, \mathcal{L}_0, s)$ where $\mathcal{L}_1, \mathcal{L}_0$ are C^∞-manifolds (with corners) and $s \colon \mathcal{L}_1 \to \mathcal{L}_0$ is a (tame) surjective submersion, as well as $\mathcal{D}_m \subset \mathcal{L}_1 * \mathcal{L}_1$, which is an open subset. The involution $i \colon \mathcal{L}_1 \to \mathcal{L}_1$, as well as the remaining structure maps $m \colon \mathcal{D}_m \to \mathcal{L}_1$, $r \colon \mathcal{L}_1 \to \mathcal{L}_0$, $s \colon \mathcal{L}_1 \to \mathcal{L}_0$, $u \colon \mathcal{L}_0 \to \mathcal{L}_1$ fulfill the usual axioms of a groupoid, whenever the corresponding operations are defined.



Definition 3.5. A *generalized atlas* $\mathfrak{A} := \{\mathcal{L}_i \rightrightarrows U_i\}_{i \in I}$ consists of partial Lie groupoids $\mathcal{L}_i \rightrightarrows U_i$ and a covering $\{U_i\}_{i \in I}$ by open subsets, such that:

1. For each $i \in I$ the partial Lie groupoids $\mathcal{L}_i$ is a partial spanoid, i.e. a final object in spans.

2. For each $i, j \in I$ there are partial sub groupoids $H_i^j \subset \mathcal{L}_i$, $H_j^i \subset \mathcal{L}_j$ and partial span isomorphisms $\varphi_{ij} \colon H_i^j \xrightarrow{\sim} H_j^i$.

Remark 3.6. The partial spanoid property implies that the φ_{ij} are unique, i.e., there is a maximal open subset $\mathcal{D}(\varphi_{ij}) \subset \mathcal{L}_i$ such that $U_i \cap U_j \subset \mathcal{D}(\varphi_{ij})$, referred to as the *domain* of φ_{ij}. We assume throughout that all domains of φ_{ij} are maximal.

We proceed to the construction of a pseudogroup, associated to a given generalized atlas. To this end we form the following category $\widetilde{\Phi}(X) \rightrightarrows \mathcal{O}(X)$:

$$\widetilde{\Phi}(X) := \begin{cases} \widetilde{\Phi}(X)_0 = \mathcal{O}(X), \\ \widetilde{\Phi}(X)_1 = \text{span isomorphisms } \mathcal{L}_0 \xrightarrow{\sim} \mathcal{L}_1,\ U \subset \mathcal{L}_0, V \subset \mathcal{L}_1. \end{cases}$$

The category $(\widetilde{\Phi}(X), \circ)$ is endowed with the fiber-product, furnishing the composition of the category. In the next step, we augment this category to a groupoid $(\Phi(X), \circledcirc)$:

$$\Phi(X) := \begin{cases} \Phi(X)_0 = \mathcal{O}(X), \\ \Phi(X)_1 = \text{arrows } f \colon \mathcal{L}_0 \dashrightarrow \mathcal{L}_1 \text{ partial Morita isomorphisms}, \\ \qquad\qquad U \subset \mathcal{L}_0, V \subset \mathcal{L}_1. \end{cases}$$

The groupoid is endowed with the generalized tensor product $\circledcirc$, which is the composition of arrows.

Definition 3.7. Given a generalized atlas $\mathfrak{A}$, we define the associated pseudogroup $\Phi_{\mathfrak{A}}$ to be the wide subgroup of $\Phi(X)$ consisting of partial Morita isomorphisms between elements of $\mathfrak{A}$ such that the identity is contained in $\Phi_{\mathfrak{A}}$ and such that $\Phi_{\mathfrak{A}}$ is stable with regard to inversion, partial composition and restriction.

The pseudogroup $\Phi_{\mathfrak{A}}$ gives rise to a sheaf of germs $\mathcal{G}_{\Phi_{\mathfrak{A}}} := \Phi_{\mathfrak{A}}/\sim$, consisting of germs of partial Morita isomorphisms. The global object of this sheaf realizes a Lie groupoid. To this end, the equivalence relation on $\Phi_{\mathfrak{A}}$ is defined as: Given two partial Morita isomorphisms f and g, then $f \sim_{\gamma_f} g$ if there are open subsets $V_f, W_f \subset Z_f$ and an isomorphism of spans $\varphi \colon V_f \xrightarrow{\sim} W_f$ with $\varphi(\gamma_f) = \gamma_f$. The operation of generalized tensor product induces the necessary algebraic structure on the partial Morita isomorphisms which induces a groupoid structure on the global object. The partial spanoid property is then decisive to yield a compatible C^∞-structure that, by tameness, induces a smooth structure (without corners) on the fibers of the resulting holonomy groupoid.

Definition 3.8. Given a Lie algebroid $(\pi\colon \mathcal{A} \to M, \varrho)$, we define the *holonomy groupoid* $\mathcal{G} \rightrightarrows M$ of $\mathcal{A}$ by the following properties:

1. The Lie groupoid $\mathcal{G}$ *integrates* $\mathcal{A}$, i.e. $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}$.

2. It is *minimal*, i.e. for any Lie groupoid $\mathcal{H} \rightrightarrows M$ which integrates $\mathcal{A}$ there is a surjective morphism of Lie groupoids $\mathcal{H} \to \mathcal{G}$.

THEOREM 3.9 (Debord). *Let* $(M, \mathcal{A})$ *be an almost injective Lie algebroid. Then there is a spanoid Lie groupoid* $\mathcal{G} \rightrightarrows M$ *which integrates* $\mathcal{A}$.

Example 3.10. i) We refer to the examples of Section 2.3 for concrete Lie groupoids whose existence may be inferred by the construction recalled above.

ii) Let $(M, \mathcal{A})$ such that the Lie algebroid $\mathcal{A}$ is equipped with a filtration by subbundles $0 = \mathcal{A}^0 \leq \mathcal{A}^1 \leq \cdots \leq \mathcal{A}^N = \mathcal{A}$ such that the module of C^∞-sections $\Gamma(\mathcal{A}^\bullet \mathcal{G})$ is a filtered Lie algebra. We introduce the Lie algebroid $\mathfrak{A}_{H\leq}$ as a set by $\mathcal{A} \times \mathbb{R}^* \cup \mathcal{A}_H \times \{0\}$, where we denote the graded Lie algebroid bundle $\mathcal{A}_H := \oplus_{i=1}^N \mathcal{A}^i/\mathcal{A}^{i-1}$. The C^∞ sections of vector fields are defined by

$$\Gamma(\mathfrak{A}_{H\leq}) := \big\{X \in \Gamma(\mathcal{A} \times \mathbb{R}) : \partial_t^i X_{|t=0} \in \Gamma(\mathcal{A}^i),\ i \geq 0\big\}.$$

Then $\mathfrak{A}_{H\leq}$ is an almost injective Lie algebroid and we can find an integrating Lie groupoid $\mathcal{G}_{H\leq} \rightrightarrows M \times \mathbb{R}$ by application of Theorem 3.9. There are various different explicit constructions of the groupoid that can be found in the literature. As an example, we refer to [21] for an elegant coordinate-free approach.

4. REDUCTION THEORY

4.1. Deformation groupoids and geometric reduction

Let $F \subset M$ be a closed subspace which is saturated, by which we mean that $s^{-1}(F) = r^{-1}(F)$, and set $O := M \setminus F$. Then $\mathcal{G}_F$ is a continuous family groupoid [44, 27]. By restricting over F, we get the F-indicial symbol map:

$$I_F \colon \Psi_{\mathcal{G}}^*(E_0, E_1) \to \Psi_{\mathcal{G}_F}^*\big(E_0|_F, E_1|_F\big).$$

Gathering both symbol maps, we get the *F-joint symbol map*:

(3)

$$\sigma_{F,m} = (\sigma_{\mathrm{pr}}, I_F) \colon \Psi_{\mathcal{G}}^m(E_0, E_1) \longrightarrow C\big(S(\mathcal{A}^*), \overline{\pi}^* \mathrm{Hom}(E_0, E_1)\big) \times \overline{\Psi_{\mathcal{G}_F}^m}\big(E_0|_F, E_1|_F\big).$$

The range $\Sigma_F^m(E_0,E_1)$ of $\sigma_{F,m}$ is called the *F-joint symbols space* and its closure is denoted by $\overline{\Sigma_F^m}(E_0,E_1)$. We write Σ_F for $\overline{\Sigma_F^0}$.

This gives the short exact sequence of C^*-algebras [27]:

$$C^*\big(\mathcal{G}_O, \mathrm{End}(E)\big) \rightarrowtail \Psi_{\mathcal{G}}(E) \xrightarrow{\sigma_F} \Sigma_F(E), \tag{4}$$

where $C^*(\mathcal{G}_O, \mathrm{End}(E))$ is the closure of $C_c^\infty(G_O, r^*\mathrm{End}(E))$ into $\Psi_{\mathcal{G}}(E)$. The adiabatic groupoid $\mathcal{G}^{\mathrm{ad}} \rightrightarrows \mathcal{G}_{0,\mathrm{ad}} := \mathcal{G}_0 \times [0,1]$ is the natural Lie groupoid integrating the Lie algebroid (cf. Figure 1) $(\mathcal{A}^{\mathrm{ad}}, \varrho^{\mathrm{ad}})$ given by:

$$\mathcal{A}^{\mathrm{ad}} = \mathcal{A} \times [0,1] \text{ and } \varrho^{\mathrm{ad}}\colon \mathcal{A}^{\mathrm{ad}} \to T\mathcal{G}_0 \times T[0,1],$$
$$\mathcal{A}^{\mathrm{ad}} \ni (x,v,t) \mapsto (x,tv,t,0) \in T\mathcal{G}_0 \times T[0,1] = TG_{0,\mathrm{ad}}.$$

More precisely,

$$\mathcal{G}^{\mathrm{ad}} = \mathcal{A} \times \{0\} \cup \mathcal{G} \times (0,1] \text{ and } \mathcal{A}(\mathcal{G}^{\mathrm{ad}}) \cong \mathcal{A}^{\mathrm{ad}},$$

see also [15, 24, 43]. Out of the adiabatic groupoid we construct the *F-Fredholm groupoid*:

$$\mathcal{G}_F^{\mathcal{F}} := \mathcal{G}^{\mathrm{ad}} \setminus \big(\mathcal{G}_F \times \{1\}\big) \rightrightarrows \mathcal{G}_{0,F}^{\mathcal{F}} = \big(\mathcal{G}_0 \times [0,1]\big) \setminus \big(F \times \{1\}\big) \tag{5}$$

This is again a Lie groupoid (as an open subset of $\mathcal{G}^{\mathrm{ad}}$). The *noncommutative tangent bundle* is defined by:

$$\mathcal{T}_F\mathcal{G}_0 := \mathcal{G}_F^{\mathcal{F}} \setminus \big(\mathcal{G}_O \times (0,1]\big) \rightrightarrows \mathcal{G}_{0,F} = \mathcal{G}_{0,F}^{\mathcal{F}} \setminus \big(O \times (0,1]\big). \tag{6}$$

It is a $C^{\infty,0}$ groupoid [44]. To gain more insight, consider the data encoded by the K-theory of the C^*-algebra of the noncommutative tangent bundle $K_0(C^*(\mathcal{T}))$ for the case of $F = \partial M$ connected boundary, cf. Figure 2. On the one hand, for $t = 0$ it encodes the homotopy class of the principal symbol of the given operator P. For $0 < t < 1$, it encodes the homotopy data coming from the indicial (boundary) symbol of the operator. The K-theory of the C^*-algebra of the Fredholm groupoid (Figure 3) additionally encodes the values of the geometric index map at $t = 1$, i.e. the K-theory of the compact operators that are given by $C^*(M_0 \times M_0) \cong \mathcal{K}_{M_0}$.

The exact sequence:

$$C^*\big(\mathcal{G}_O \times (0,1]\big) \rightarrowtail C^*(\mathcal{G}_{\mathcal{F}}) \xrightarrow{e_0} C^*(\mathcal{T}_F M) \tag{7}$$

possessing a contractible kernel and a nuclear quotient, we get an isomorphism $e_0 : K_0(C^*(\mathcal{G}_F^{\mathcal{F}})) \to K_0(C^*(\mathcal{T}_F M))$, and moreover, $e_0 \in KK(C^*(\mathcal{G}_F^{\mathcal{F}}), C^*(\mathcal{T}_F M))$ provides a KK-equivalence (that is, is invertible). Considering the restriction $e_1 : C^*(\mathcal{G}_F^{\mathcal{F}}) \to C^*(\mathcal{G}_O)$, we get another index map:

$$\mathrm{ind}_F^{\mathcal{F}} := (e_1)_* \circ (e_0)_*^{-1}\colon K_0\big(C^*(\mathcal{T}_F M)\big) \to K_0\big(C^*(G_O)\big).$$

We have the following Poincaré duality type isomorphism, cf. [11, Theorem 2.4].

THEOREM 4.1. *There is a group isomorphism:*

$$\mathrm{pd}_F\colon \mathrm{Ell}_F(\mathcal{G}) \to K_0\big(C^*(\mathcal{T}_F M)\big) \tag{8}$$

such that $e_0(\mathrm{pd}_F[P]_F) = [P]_{\mathrm{pr,ev}} \in K_c^0(\mathcal{A}^*)$ *and* $\mathrm{ind}_F^{\mathcal{F}}(\mathrm{pd}_F[P]_F) = \mathrm{ind}_F([P]_F)$. *Here* e_0 *is the restriction map* $C^*(\mathcal{T}_F M) \to C^*(\mathcal{A})$.

The analogy to the commutative case is that Σ_F is the *noncommutative cosphere bundle*, relative to F, and $\mathcal{T}_F M$ is the *noncommutative tangent bundle*, relative to F, associated with $\mathcal{G} \rightrightarrows M$, see also [27, Section 5]. Denote by $\partial\colon K_1(\Sigma_F) \to K_0(C^*(G_O))$ the connecting "index" map in the K-theory six-term exact sequence associated to the short exact sequence of the full symbol map (4). By [17], we have that

$$\mathrm{ind}_{\mathcal{F}}^F\big(\mathrm{pd}_F[P]_F\big) = \partial\big[\sigma_F(P)\big]_1$$

where by $[\sigma_F(P)]_1$, we denote the K_1-class of $P \in \Psi_{\mathcal{G}}(E)$ under the full symbol map σ_F. By Theorem 4.1 we obtain that $\mathrm{ind}_F([P]_F) = \partial[\sigma_F(P)]_1$. In particular for $F = \partial M$ and $\mathcal{G}$ an integrated Lie manifold, the index map ind_F recovers the Fredholm index. We consider here and below the data: $\varphi\colon \Sigma \to M$, where by Σ we denote a compact manifold with corners and by φ a tame surjective submersion. By virtue of another straightforward deformation construction (cf. Figures 4, 5), we obtain the following pushforward homomorphisms:

$$\varphi_!^{\mathcal{F}} : K_*\big(C^*\big(({}^{\varphi}\mathcal{G})^{\mathcal{F}}\big)\big) \longrightarrow K_*\big(C^*\big(\mathcal{G}^{\mathcal{F}}\big)\big), \tag{9}$$

$$\varphi_!^{\mathrm{nc}} : K_*\big(C^*(\mathcal{T}\Sigma)\big) \longrightarrow K_*\big(C^*(\mathcal{T}M)\big), \tag{10}$$

$$\varphi_!^{0} : K_*\big(C^*({}^{\varphi}\mathcal{A})\big) \longrightarrow K_*\big(C^*(\mathcal{A})\big). \tag{11}$$

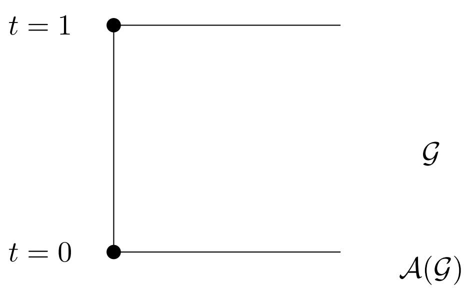


Figure 1 – Adiabatic groupoid.

Noting that the operation on a Lie groupoids given by pullback (by a tame surjective submersion) does not commute with the operation of taking

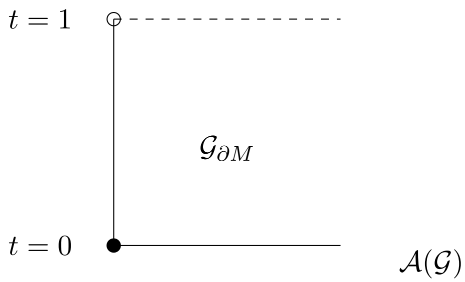


Figure 2 – Noncommutative tangent bundle.

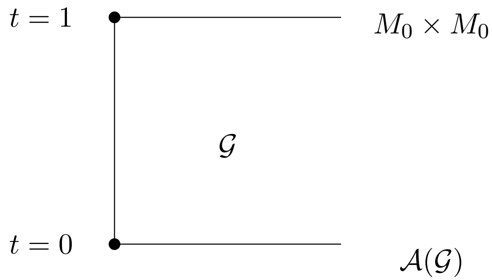


Figure 3 – Fredholm groupoid.

the adiabatic deformation, we make use of another deformation groupoid:

$$\mathcal{L}_\varphi = ({}^\varphi\mathcal{G})^{\mathrm{ad}} \times \{u = 0\} \cup {}^{\varphi_1}(\mathcal{G}^{\mathrm{ad}}) \times (0, 1]_u \rightrightarrows \Sigma \times [0, 1]_t \times [0, 1]_u$$

to facilitate the pushforward $\varphi_!^0$. Denote by $\varphi_1 := \varphi \times \mathrm{Id}_{[0,1]_t}$.

The groupoid $\mathcal{L}_\varphi^{\mathrm{nc}}$ facilitating the pushforward $\varphi_!^{\mathrm{nc}}$ is the restriction of $\mathcal{L}_\varphi$ to $tz = 0$, where z denotes the lift of a boundary defining function $\varrho_{\partial M} \colon M \to \overline{\mathbb{R}}_+$ by φ, i.e. $z = \varrho_{\partial M} \circ \varphi$.

4.2. **Clutching data**

Let $P \in \Psi_{\mathcal{V}}^m(M; E_0, E_1)$ be a pseudodifferential operator that is fully elliptic. We define next the *clutching data* associated to P as the quadruple $(\Sigma_{\mathcal{A}}, \varphi, E_\sigma, \mathcal{C})$.

The *clutching space* is defined as the sphere bundle $\Sigma_{\mathcal{A}} := S(\mathcal{A} \oplus 1_{\mathbb{R}})$ which equals $\mathcal{B}(\mathcal{A})_+ \cup_{S(\mathcal{A})} \mathcal{B}(\mathcal{A})_-$, where $\mathcal{B}(\mathcal{A})$ denotes the ball bundle in $\mathcal{A}$ and $\mathcal{B}(\mathcal{A})_\pm$ denote the upper and lower hemispheres respectively that are glued along the hypersurface $S(\mathcal{A})$ to obtain the clutching space. Denote by $\varphi \colon \Sigma_{\mathcal{A}} \to M$ the

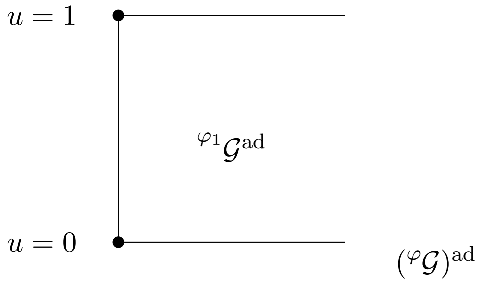


Figure 4 – Groupoid facilitating pushforward $\varphi_!^0$.

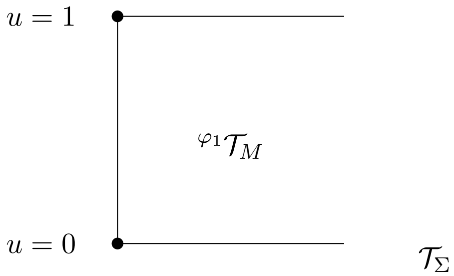


Figure 5 – Groupoid facilitating pushforward $\varphi_!^{\mathrm{nc}}$.

projection which yields a tame surjective submersion. The third component is the *clutched bundle* $E_\sigma \to \Sigma_{\mathcal{A}}$ over $\Sigma_{\mathcal{A}}$. By ellipticity of P, we have an isomorphism

$$\sigma_{\mathrm{pr}}(P) \colon \pi^* E_0 \xrightarrow{\sim} \pi^* E_1.$$

Let $\widehat{\mathcal{A}} = \mathcal{A} \cup S(\mathcal{A})$ be the radial compactification of $\mathcal{A}$ and $\widehat{\pi} \colon \widehat{\mathcal{A}} \to M$ the corresponding projection map. The clutched bundle $E_\sigma \to \Sigma_{\mathcal{A}}$ is defined by the glueing of pullbacks of E_0 and E_1, along the boundary stratum $S(\mathcal{A})$, using $\sigma_{\mathrm{pr}}(P)$:

$$E_\sigma = \widehat{\pi}^* E_0 \cup_{S(\mathcal{A})} \widehat{\pi}^* E_1.$$

In addition, we define the *taming Callias operator* $\mathcal{C}$ over the spin$^{\mathbb{C}}$-manifold $\Sigma_{\mathcal{A}}$. The data $(\Sigma_{\mathcal{A}}, {}^{\varphi}\mathcal{G}, {}^{\varphi}\mathcal{A})$, where ${}^{\varphi}\mathcal{G}$ denotes the pullback Lie groupoid over $\Sigma_{\mathcal{A}}$ integrating the pullback Lie algebroid ${}^{\varphi}\mathcal{A}$, furnishes an integrated Lie structure [11, Theorem 3.6] that is endowed with the spin structure $S \to \Sigma_{\mathcal{A}}$. The bundle ${}^{\varphi}\mathcal{A}$ is spin$^{\mathbb{C}}$, since we can write ${}^{\varphi}\mathcal{A} = \varphi^*(\mathcal{A}) \oplus \ker d\varphi$ and since $\Sigma_{\mathcal{A}} = S(\mathcal{A} \oplus 1_{\mathbb{R}})$ is a stably almost complex manifold, which can be seen by using the $\mathcal{A}$-metric with normal coordinates to define an almost complex structure on $\mathcal{A}$. The vector bundle

$$W := E_\sigma \otimes S \oplus \varphi^* E_1 \otimes (-S)$$

is a Clifford module over the Clifford bundle $\mathrm{Cl}({}^\varphi\mathcal{A})$. Denote by ${}^\varphi\nabla^W$ an ${}^\varphi\mathcal{A}$-connection, i.e. a Levi-Civita connection on the pull back Lie algebroid ${}^\varphi\mathcal{A}$ that is compatible with the Clifford action:

$$(12)\quad {}^\varphi\nabla_X^W\big(c(Y)f\big) = c({}^\varphi\nabla_X Y)f + c(Y)({}^\varphi\nabla_X^W f),\ X, Y \in \Gamma({}^\varphi\mathcal{A}),\ f \in \Gamma(W).$$

A geometric Dirac operator over the Lie structure $(\Sigma_\mathcal{A}, {}^\varphi\mathcal{A})$ is then defined via $D = c \circ (\mathrm{id} \otimes \sharp) \circ {}^\varphi\nabla^W$, where $\sharp$ is the isomorphism ${}^\varphi\mathcal{A} \cong {}^\varphi\mathcal{A}^*$ induced by the fixed compatible metric g, ${}^\varphi\nabla^W$ denotes the ${}^\varphi\mathcal{A}$-connection and c the Clifford multiplication:

$$\Gamma(W) \xrightarrow{{}^\varphi\nabla^W} \Gamma(W \otimes {}^\varphi\mathcal{A}^*) \xrightarrow{\mathrm{id}\otimes\sharp} \Gamma(W \otimes {}^\varphi\mathcal{A}) \xrightarrow{c} \Gamma(W).$$

Since c is a ${}^\varphi\mathcal{V}$-operator of order 0 and ${}^\varphi\nabla^W$ is a ${}^\varphi\mathcal{V}$-operator of order 1, we see that D is in $\mathrm{Diff}^1_{{}^\varphi\mathcal{V}}(\Sigma_\mathcal{A}; W)$. Additionally, $\sigma_1(D)\xi = ic(\xi) \in \mathrm{End}(W)$, hence invertible for $\xi \neq 0$, and D is elliptic.

Definition 4.2. We call $\mathcal{C} = (D \oplus D') + R$ a *taming Callias type operator* of D, if (D', W') is a Dirac bundle of the same parity as (D, W) and the residual operator $R \in \Psi^{-\infty}_{{}^\varphi\mathcal{V}}(\Sigma_\mathcal{A}; W \oplus W')$ is such that

$$\mathcal{C} := (D \oplus D') + R \in \Psi^1_{{}^\varphi\mathcal{V}}(\Sigma_\mathcal{A}; W \oplus W')$$

is fully elliptic and the principal symbol classes in relative K-theory agree, i.e. $[D]_{\mathrm{pr}} = [\mathcal{C}]_{\mathrm{pr}}$.

The clutching data gives rise to a geometric K-homology theory, i.e. we can describe a group ${}^\mathcal{V}\mathrm{K}^{geo}(M)$ generated by data (Σ, φ, E, B), for a given (odd) dimensional compact manifold with corners M; a tame surjective submersion $\varphi : \Sigma \longrightarrow M$; an even (odd) Dirac bundle (E, D) on $(\Sigma, {}^\varphi\mathcal{G})$; a self-adjoint even (odd) Dirac $\partial\Sigma$-taming B of D. We call such a tuple a geometric cycle and introduce an equivalence relation on the set of geometric cycles (of fixed parity) that is generated by the operations: isomorphisms of geometric cycles, direct sums, cobordisms and vector bundle modifications. We refer to [11, Section 4] for the technical details.

4.3. **Reduction to Callias operators**

There are three main ingredients to the main result which we are going to record separately. The first is the commutativity of the geometric Fredholm index map with pushforwards ([11, Theorem 3.7]).

THEOREM 4.3. *Let $(M, \mathcal{G})$ be an integrated Lie manifold and $\varphi : \Sigma \to M$ a tame surjective submersion. Then the map $\varphi_!^{\mathrm{nc}}$ commutes with Fredholm index:*

$$\mathrm{ind}_{\partial M}^{\mathcal{F}} \circ \varphi_!^{\mathrm{nc}} = \mathrm{ind}_{\partial\Sigma}^{\mathcal{F}} \tag{13}$$

Here, the target group of the index maps $\mathrm{ind}_{\bullet}^{\mathcal{F}}$ is replaced by $\mathbb{Z}$ after applying the obvious Morita equivalences.

In other words, if $B \in \Psi^*_{\varphi\mathcal{G}}(\widetilde{E}_+, \widetilde{E}_-)$ and $P \in \Psi^*_{\mathcal{G}}(E_+, E_-)$ are fully elliptic operators and satisfy $\varphi_!^{\mathrm{nc}}[B]_{\partial\Sigma,\mathrm{ev}} = [P]_{\partial\Sigma,\mathrm{ev}}$ then B and P have the same Fredholm index.

The second ingredient is the so-called functoriality of the pushforward maps that we introduced in the previous subsection. One may also refer to this as a compatibility condition, with respect to the appropriate Thom isomorphisms, cf. [11, Theorem 3.8].

THEOREM 4.4. *Let $(M, \mathcal{G})$ be an integrated Lie manifold, $\pi : V \to M$ be a real vector bundle and denote by $\varphi : \Sigma = S(V \oplus \mathbb{R}) \to M$ the associated clutching and $i : V \hookrightarrow \Sigma$ the embedding as an open hemisphere. Then*

1. *$\pi_!^{nc}$ and $\pi_!^0$ are isomorphisms, the latter being the inverse of the Thom isomorphism of the complex bundle ${}^{\pi}\mathcal{A} \to \mathcal{A}$.*
2. *We have the identities:*

$$\varphi_!^{\bullet} \circ \left(i_* \circ (\pi_!^{\bullet})^{-1}\right) = \mathrm{Id}, \qquad \text{with } \bullet = nc \text{ or } 0. \tag{14}$$

Finally, a so-called *Diracification Theorem* ([11, Theorem 3.3]):

THEOREM 4.5. *Let (S, D) be a Dirac bundle. Then for any F-fully elliptic operator $P \in \Psi_{\mathcal{G}}$ such that $[P]_{\mathrm{pr},*} = [D]_{\mathrm{pr},*} \in K^*(\mathcal{A})$, there exists a F-taming B of (S, D) such that*

$$[B]_{F,*} = [P]_{F,*} \in K_*\big(C^*(\mathcal{T}_F M)\big) \simeq K(\mu_F). \tag{15}$$

Consideration of the commutative diagram

$$\begin{array}{ccccc}
K_c^0(\mathcal{A}^*) & \xrightarrow{\text{Thom}} & K_c^0({}^{\pi}\mathcal{A}^*) & \xrightarrow{-\otimes[i]} & K_c^0({}^{\varphi}\mathcal{A}^*) \\
\uparrow{\scriptstyle \mathrm{ev}_{t=0}} & & \uparrow{\scriptstyle \mathrm{ev}_{t=0}} & & \uparrow{\scriptstyle \mathrm{ev}_{t=0}} \\
K_0(C^*(\mathcal{T}M)) & \xrightarrow{(\pi_!^{nc})^{-1}} & K_0(C^*(\mathcal{T}\mathcal{A})) & \xrightarrow{-\otimes[i]} & K_0(C^*(\mathcal{T}\Sigma_{\mathcal{A}}))
\end{array} \tag{16}$$

together with the previous facts, then furnishes the main (reduction) result ([11, Theorem 5.2, Theorem 5.4].

Theorem 4.6. *Let $(M, \mathcal{G})$ be an integrated Lie structure and consider $P \in \Psi^m_{\mathcal{G}}(E_0, E_1)$ a fully elliptic operator. Then there exists an even Dirac bundle (E, D) on $(\Sigma_{\mathcal{A}}, {}^{\varphi}\mathcal{G})$ with boundary taming B such that*

$$\varphi^{nc}_!\big([B]_{\partial\Sigma_{\mathcal{A}},\mathrm{ev}}\big) = [P]_{\partial M,\mathrm{ev}} \in K_0\big(C^*(\mathcal{T}M)\big). \tag{17}$$

In particular, B_+ and P have the same Fredholm index.

Corollary 4.7. *The diagram*

$$\begin{array}{ccc} K_*(\mu) & & \\ \downarrow c & \overset{\mathrm{pd}}{\searrow} & \\ {}^{\mathcal{V}}\mathrm{K}^{geo}(M) & \xrightarrow{\ \lambda\ } & K(C^*(\mathcal{T}_M)) \end{array}$$

commutes. In particular, if $P \colon C^\infty(M, E_0) \to C^\infty(M, E_1)$ denotes a fully elliptic pseudodifferential operator on the integrated Lie structure $(M, \mathcal{G})$, then there is a taming $\mathcal{C}$ on the integrated Lie structure $(\Sigma_{\mathcal{A}}, {}^{\varphi}\mathcal{G})$ such that we have $\mathrm{ind}(\mathcal{C}) = \mathrm{ind}(P)$.

5. ATIYAH–PATODI–SINGER INDEX FORMULA

Let $P \in \Psi^m_{\mathcal{V}}(M; E_0, E_1)$ be a pseudodifferential operator that is fully elliptic. Since $\mathcal{G}$ is by assumption s-connected, the algebra of pseudodifferential operators of Lie type $\Psi^*_{\mathcal{V}}$ coincides with the image of $\Psi^*_{\mathcal{G}}(r^*E_0, r^*E_1)$ by the vector representation $r_\sharp$. Also, we fix the clutching data $(\Sigma_{\mathcal{A}}, \varphi, E_\sigma, \mathcal{C})$ as described in Subsection 4.2.

5.1. Heat kernel

Let $G \in \Psi^1(\mathcal{G}, r^*E)$ be an elliptic operator that is in addition symmetric and viewed as an unbounded operator on the Hilbert $C^*(\mathcal{G})$-module $\mathcal{E} = C^*(\mathcal{G}, r^*E)$. Consider the initial value problem [13, 12]:

$$\begin{aligned} &(\partial_t - G^2)u_t = 0, \ u_t \in C^0(\mathcal{G}, E), \\ &\lim_{t\to 0} u_t * u = u, \ u \in C^\infty_c(\mathcal{G}, E). \end{aligned} \tag{18}$$

Denote by Γ the curve parametrized by $\mathbb{R}_+ \ni t \mapsto -1 + t(1 \pm i)$. The Ansatz for the solution of the problem is the heat operator, defined via functional calculus

$$e^{-tG^2} := \frac{1}{2\pi i} \int_\Gamma e^{-t\lambda^2} (G^2 - \lambda)^{-1} \, d\lambda. \tag{19}$$

S. Vassout has introduced the residual class $\Psi_{\mathcal{G}}^{-\infty}(r^*E)$ which is a (non-unital) symmetric, continuously embedded Fréchet subalgebra of $C^\infty(\mathcal{G}, r^*E) \cap C^*(\mathcal{G}, r^*E)$, that is also closed with respect to holomorphic functional calculus (a Ψ^*-algebra). To make sense of the expression (19), note that the operator G is closable. By [51], closure is a regular selfadjoint unbounded morphism on $\mathcal{E}$. We then apply the continuous functional calculus of [49], by which the integral is absolutely convergent in $\mathrm{Mor}(\mathcal{E})$ and $G^k e^{-tG^2}, e^{-tG^2} G^k$ belong to $\mathrm{Mor}(\mathcal{E})$ as well, for any $k \in \mathbb{N}$. This shows that e^{-tG^2} is contained in the residual class $\Psi_{\mathcal{G}}^{-\infty}(r^*E)$.

We can associate an unbounded representative in K-homology to the given pseudodifferential operator P, cf. [12, Section 3]. To this end, note that by choosing $\Delta_0 \in \Psi_{\mathcal{G}}^2(E_0)$ as an elliptic nonnegative differential operator, we obtain by [51] that $(1 + \Delta_0)^{1/2} \in \Psi_{\mathcal{G}}^1(E_0)$ is an invertible operator. We can therefore set $G_+ = P(1 + \Delta_0)^{1/2}$ and $G_- = (G_+)^*$. We can summarize this in the following Lemma (cf. [12]).

Lemma 5.1. *Given the data as specified, there is an unbounded representative $G = G_P$ in K-homology of the form*

$$G = \begin{pmatrix} 0 & G_- \\ G_+ & 0 \end{pmatrix}$$

over $E := E_0 \oplus E_1$ such that:

1. *$(G_+)^* = G_-$, in particular G is self-adjoint and elliptic.*
2. *$G \in \Psi^1 = \Psi_c^1 + \Psi_{\mathcal{G}}^{-\infty}$.*
3. *$\mathrm{ind}(P) = \mathrm{ind}(G_+)$.*

In the computation of the index $\mathrm{ind}(P)$, we expect an error term to arise, that is a non-local contribution. To make an Ansatz for the non-local contribution, we use again the functional calculus and the unbounded representative.

Using G_P, as specified in Lemma 5.1 as well as the functional calculus of [33], we define the operators

$$A_0(t) := G e^{-\frac{t}{2} G^2},$$

$$A_1(t) := \int_t^\infty G e^{\left(\frac{t}{2}-s\right)G^2}\, ds$$

that by virtue of the above considerations and the functional calculus are elements of $\Psi_{\mathcal{G}}^{-\infty}(E)$ for $t > 0$. We make the Ansatz:

$$^{\nu}\eta(G_P)(t) := -\frac{1}{2}{}^{\nu}\mathrm{Tr}_s\big[A_0(t), A_1(t)\big]. \tag{20}$$

5.2. **Chern–Weil construction**

$\mathcal{A}$-connections

Let $(\mathcal{A}, \varrho)$ be an almost injective Lie algebroid over the compact manifold with corners M and let g be a positive definite, symmetric 2-tensor on $\mathcal{A}$ that extends a Riemannian metric g_0 that is defined on the (dense) interior $M_0 \subset M$. Then by [42, Proposition 3.11], there is a Levi-Civita connection on $\mathcal{A}$ that extends the Levi-Civita connection on TM_0.

PROPOSITION 5.2. *There is a Levi-Civita (LC) $\mathcal{A}$ connection, i.e. a differential operator $\nabla \colon \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A} \otimes \mathcal{A}^*)$ that extends the LC-connection on the interior such that the following conditions hold:*

1. $X \mapsto \nabla_X(fY) = f\nabla_X(Y) + X(f)Y$.
2. $\nabla_{fX}(Y) = f \cdot \nabla_X(Y)$.
3. $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

   Where $\langle \cdot, \cdot \rangle$ is the inner product on $\Gamma(\mathcal{A})$ induced by g.

Remark 5.3. Note that the proof also works for so-called open Lie manifolds, a variant of the Lie structure where M is not assumed to be compact and the definition of the Lie structure is modified to contain the smooth sections with compact support, cf. [42, Definition 3.3]. On the other hand, the condition of boundary tangency of [42] is not required for the existence of an $\mathcal{A}$-connection.

Chern–Weil theory is constructed in complete analogy to the standard case of smooth manifolds, by making use of $\mathcal{A}$-connections. We recall briefly the definition of Lie algebroid cohomology as well as the Chern character and Todd class.

Cohomology

We set

$$\Omega^k(\mathcal{A}) := \Gamma^\infty\Big(\mathcal{G}_0; \bigwedge^k \mathcal{A}^*\Big).$$

Observe that the connection ∇ is a linear operator $\nabla\colon \Omega^0(\mathcal{A}) \to \Omega^1(\mathcal{A})$. The curvature tensor $R(\nabla) \in \Omega^2(\mathrm{End}(\mathcal{A}))$ is defined via

$$R(\nabla)(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

The operators

$$d_{\mathcal{A}}\colon \Omega^k(\mathcal{A}) \to \Omega^{k+1}(\mathcal{A}),$$

are defined via the Koszul formula

$$\begin{aligned} d_{\mathcal{A}}\alpha(X_0,\dots,X_k) &:= \sum_{i=0}^k (-1)^i \nabla_{X_i}\alpha(X_0,\dots,\widehat{X}_i,\dots,X_k) \\ &+ \sum_{i<j}(-1)^{i+j-1}\alpha\big([X_i,X_j],X_1,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_k\big). \end{aligned} \tag{21}$$

The corresponding cohomology groups are denoted by $H^\bullet(\mathcal{A})$.

Remark 5.4. The theory is a simultaneous generalization of several other situations, e.g. if $\mathcal{A}$ is the tangent bundle TM, we recover de Rham cohomology of M and if $\mathcal{A}$ is a Lie algebroid over a point, we recover the Lie algebra Chevalley–Eilenberg cohomology.

Characteristic classes

The Chern–Weil construction of characteristic classes, applied to the Lie algebroid case, proceeds as follows. Let ∇ be an $\mathcal{A}$-connection. The Chern–Weil theorem states that for a given degree k polynomial $P\colon M_{\mathrm{rk}(\mathcal{A})}(\mathbb{C}) \to \mathbb{C}$ that is invariant with respect to the conjugacy action of $\mathrm{GL}_{\mathrm{rk}(\mathcal{A})}(\mathbb{C})$, we have that $P(R(\nabla)) \in \Omega^{2k}(\mathcal{A})$ is a well-defined, closed form, that furnishes a cohomology class in $H^{2k}(\mathcal{A})$ that is independent of the choice of connection. Note that via the pullback along the anchor, the Chern–Weil construction for a connection over a vector bundle corresponds to the construction for Lie algebroids. We recall the definition of the primary characteristic classes for Lie algebroids, which are the pullbacks of characteristic classes along the anchor map of the corresponding Lie algebroid. The Chern character is defined by

$$\begin{aligned} \mathrm{ch}^{\mathcal{A}}(\nabla) &:= \mathrm{Tr}\Big(\exp\Big(\frac{1}{2\pi i}R(\nabla)\Big)\Big) \\ &= \mathrm{rk}(E) + c_1^{\mathcal{A}}(\nabla) + \frac{1}{2}\big(c_1^{\mathcal{A}}(\nabla)^2 - 2c_2^{\mathcal{A}}(\nabla)\big) + \cdots \end{aligned} \tag{22}$$

Here, the Chern classes are defined via

$$\det\Big(tI+\frac{R(\nabla)}{2\pi i}\Big)=\sum_i c_i^{\mathcal{A}}(\nabla)t^i. \tag{23}$$

We can also use the notation $\mathrm{ch}^{\mathcal{A}}(E)$ and $c_i^{\mathcal{A}}(E)$. We have

$$c_i^{\mathcal{A}}(E\oplus F)=c_i^{\mathcal{A}}(E)\cdot c_i^{\mathcal{A}}(F)$$

The Chern character has the formal properties

$$\begin{aligned}\mathrm{ch}^{\mathcal{A}}(E\oplus F)&=\mathrm{ch}^{\mathcal{A}}(E)+\mathrm{ch}^{\mathcal{A}}(F),\\ \mathrm{ch}^{\mathcal{A}}(E\otimes F)&=\mathrm{ch}^{\mathcal{A}}(E)\cdot\mathrm{ch}^{\mathcal{A}}(F).\end{aligned}$$

Thereby, the Chern character extends to a ring homomorphism

$$^{\mathcal{V}}\mathrm{ch}\colon K_c^0(\mathcal{A}^*)\to H^{\mathrm{ev}}(\mathcal{A}):=\bigoplus_{i\geq 0}H^{2i}(\mathcal{A}). \tag{24}$$

We also define the Todd class

$$\begin{aligned}\mathrm{Td}^{\mathcal{A}}(\nabla)&:=\det\left(\frac{\frac{1}{2\pi i}R(\nabla)}{1-\exp(-\frac{1}{2\pi i}R(\nabla))}\right)\\ &=1+\frac{1}{2}c_1^{\mathcal{A}}(\nabla)+\frac{1}{12}(c_2^{\mathcal{A}}(\nabla)+c_1^{\mathcal{A}}(\nabla)^2)+\cdots\end{aligned} \tag{25}$$

By multiplicativity of the $\widehat{A}$-class and the definition of the vector bundle ${}^{\varphi}\mathcal{A}$, combined with [30, Proposition 11.14], we have

$$\widehat{A}({}^{\varphi}\mathcal{A})=\widehat{A}(\varphi^*\mathcal{A})^2=\mathrm{Td}(\varphi^*\mathcal{A}\otimes\mathbb{C})=\varphi^*\mathrm{Td}(\mathcal{A}\otimes\mathbb{C}). \tag{26}$$

5.3. **Index via pairing**

We address the question of how to obtain a suitable generalization of the Atiyah–Patodi–Singer index formula for the fully elliptic operator P, via an application of the representation of the K-homology class of P in terms of tamed Dirac operators. More concretely, by an adaptation of a Getzler rescaling argument to Lie groupoids, we have an index formula [13]

$$\mathrm{ind}(D+R)={}^{\varphi\mathcal{V}}\!\!\!\!\!\!-\!\!\!\!\!\!\int_{\Sigma_{\mathcal{A}}}\mathrm{ch}(W/S)\wedge\widehat{A}({}^{\varphi}\nabla)+{}^{\varphi\mathcal{V}}\eta(D+R). \tag{27}$$

In order to gain insight to the question on how to obtain a corresponding index formula for P from the one given above for the taming $D+R$, we present a representation of the index in terms of a pairing: between relative K-theory and cyclic cohomology. Fix the notation

$$\mathfrak{J}_M:=\Psi_{\mathcal{G},\mathrm{tr}}^{-\infty},\ \mathfrak{A}_M:=\Psi_{\mathcal{G}}^{-\infty,+},\ \mathfrak{B}_M:=\Psi_{\mathcal{G}}^{-\infty,+}/\Psi_{\mathcal{G},\mathrm{tr}}^{-\infty}$$

as well as

$$\mathfrak{J}_\Sigma := \Psi^{-\infty}_{\varphi\mathcal{G},\mathrm{tr}},\ \mathfrak{A}_\Sigma := \Psi^{-\infty,+}_{\varphi\mathcal{G}},\ \mathfrak{B}_\Sigma := \Psi^{-\infty,+}_{\varphi\mathcal{G}}/\Psi^{-\infty}_{\varphi\mathcal{G},\mathrm{tr}}.$$

Denote by τ_M the trace and by $\overline{\tau}_M$ the renormalized trace. We obtain a class $[\tau_M]$ in cyclic cohomology $HC^0(\mathfrak{J}_M)$ and a cochain $\overline{\tau}_M$ over $\mathfrak{A}_M$. Similarly, we can define

$$\mu_M\big(Q(a_0), Q(a_1)\big) = \overline{\tau}\big([a_0, a_1]\big),\ a_0, a_1 \in \mathfrak{A}_M.$$

Since $\overline{\tau}_M$ is a linear extension of τ_M, we observe that μ_M indeed only depends on $Q(a_0), Q(a_1)$. In addition, μ furnishes a cyclic 1-cocycle, i.e. we obtain a class $[\mu_M] \in HC^1(\mathfrak{B}_M)$. Together, the pair $(\overline{\tau}_M, \mu_M)$ yields a relative cyclic 0-cocycle in $HC^0(\mathfrak{A}_M, \mathfrak{B}_M)$. By an application of the suspension and linear extension, we obtain

$$\tau_M^+ := S^p\overline{\tau}_M \in HC^{2p}(\mathfrak{A}_M),\ \mu_M^+ := S^p\mu_M \in HC^{2p+1}(\mathfrak{B}_M)$$

which yields a $2p$-relative class $[(\tau_M^+, \mu_M^+)] \in HC^{2p}(\mathfrak{A}_M, \mathfrak{B}_M)$, via the procedure described in [41, Section 8.2]. We express the Fredholm index as a pairing $\langle -, - \rangle_K \colon K_0(\mathfrak{A}_M, \mathfrak{B}_M) \times \mathrm{HC}^{2p}(\mathfrak{A}_M, \mathfrak{B}_M) \to \mathbb{C}$. As observed in [41, Section 6.1], to obtain a bilinear continuous pairing, we need to complete the algebras $\mathfrak{J}_M$, $\mathfrak{A}_M$ and $\mathfrak{B}_M$ in Schatten class norms. The resulting algebras are still holomorphically closed. By an abuse of notation, we keep our designation for such completed algebras in what follows. The group $K_0(\mathfrak{A}_M, \mathfrak{B}_M)$ is represented by a triple (p, q, p_t) where $p, q \in M_N(\mathfrak{A}_M)$ are idempotents and p_t is a continuous path of idempotents connecting $Q_M(p)$ with $Q_M(q)$. Denote by $\alpha_{\mathrm{ex}} \colon K_0(\mathfrak{J}_M) \xrightarrow{\sim} K_0(\mathfrak{A}_M, \mathfrak{B}_M)$ the excision isomorphism, $[p, q] \mapsto [p, q, c]$, where by c, we denote the constant path. Note that by definition of the ideal $\mathfrak{J}_M$, we have an inclusion $*$-homomorphism $i^M : \mathfrak{J}_M \to \mathcal{K}_{M_0}$ which induces an isomorphism

$$i^M_* \colon K_0(\mathfrak{J}_M) \xrightarrow{\sim} K_0(\mathcal{K}_{M_0}).$$

We construct a relative K-cycle $(\mathbb{B}_G, e_1, p_t)$ where

$$e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\mathbb{B}_G$ denotes the graph projection of the unbounded representative G. Setting $\tau(x^2)^2x^2 = e^{-x^2}(1 - e^{-x^2})$, this projection is given by

$$\mathbb{B}_G = \begin{pmatrix} 1 - e^{-G_-G_+} & \tau(G_-G_+)G_- \\ \tau(G_+G_-)G_+ & e^{-G_+G_-} \end{pmatrix}$$

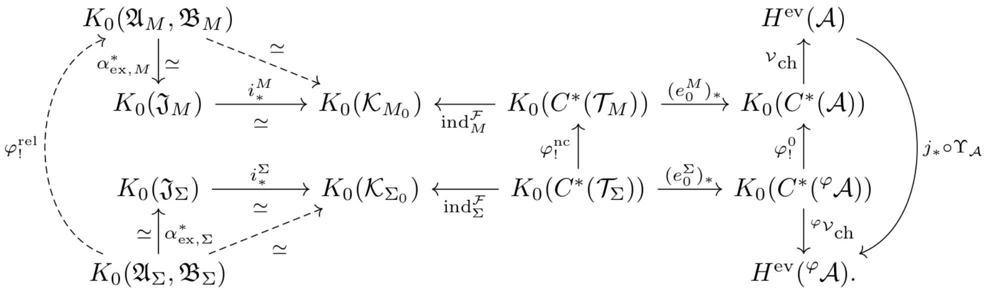
with the path of projections $p \colon [0,1] \to \mathcal{P}_\infty(\mathfrak{A}_M)$ given by

$$p_t = \begin{cases} \mathbb{B}_{tQ_M(G)},\ t \in (0,1] \\ e_1,\ t = 0. \end{cases}$$

Now, the Fredholm index of P can be expressed in terms of the pairing (cf. [41]):

$$\operatorname{ind}(P) = \big\langle \big[(\mathbb{B}_G, e_1, p_t)\big], \big[(\tau^+, \mu^+)\big] \big\rangle_K. \tag{28}$$

The situation at the level of K-theory and cohomology is summarized in the following diagram (cf. Figure 6):



In the case of fully elliptic pseudodifferential operators, the strategy is to first establish a Radul cocycle index formula in terms of renormalized trace functionals. This is based on a construction of a full parabolic calculus and an intricate analysis of an asymptotic expansion of the heat semigroup, subject to growth conditions on the underlying Lie groupoid, cf. [12]. However, the resulting formula over the base manifold M is not very explicit, since we face the problem that fitting explicit representatives of cohomology classes that are defined by the Chern characters may not exist in general. One can instead obtain a representative for relative cohomology which includes a transgression form, cf. the remarks of D. Quillen in [47]. At least for Dirac operators, convergence as $t \to \infty$ to an explicit representative for cohomology can be demonstrated, cf. [5]. In the general case, convergence as $t \to \infty$ is not guaranteed. However, assuming a Radul cocycle trace formula can be established, then the relation (17) in combination with the functoriality properties of the Chern character maps, as well as a comparison of the pairing formula (28) with the analogous formula for the index of the taming, via a suitable limit argument as $t \to 0^+$ (adapted to our case from [41]), furnishes an index formula for P, formulated herein on the clutching manifold Σ. Based on a homotopy argument and cobordism invariance, over the clutching manifold Σ, we therefore obtain the Atiyah–Bott–Patodi type index formula

$$\operatorname{ind}(P) = {}^{\varphi\mathcal{V}}\!\!\!\!\!\!\!\!\int_\Sigma \operatorname{ch}(E_\sigma) \wedge \operatorname{Td}(\varphi^*\mathcal{A} \otimes \mathbb{C}) + {}^{\varphi\mathcal{V}}\!\!\!\!\!\!\!\!\int_{\partial\Sigma} \omega_\partial + {}^{\varphi\mathcal{V}}\eta(\widetilde{G})^0, \tag{29}$$

where ${}^{\varphi\mathcal{V}}\eta(\widetilde{G})^0$ denotes the constant term in the asymptotic expansion of ${}^{\varphi\mathcal{V}}\eta_{\widetilde{G}}(t)$ as $t \to 0^+$ and where the notation $\widetilde{G}$ indicates a change in choice

$t=1$	$M_0 \times M_0$	$\cong \mathcal{K}_{M_0} \ni \mathrm{ind}_M^{\mathcal{F}}([\mathbb{B}_G] - [e_1]) = [(\mathbb{B}_G, e_1, p_t)]$
$0 < t < 1$	$\mathcal{G}$	$\cong K_1(C^*(\mathcal{G}))$
$t = 0$	$\mathcal{A}(\mathcal{G})$	$\cong K_c^0(\mathcal{A}^*) \ni [P]_{\mathrm{pr}}$

Figure 6 – Schematic for $K_0(C^*(\mathcal{G}_M^{\mathcal{F}}))$.

of smoothing operator R in the taming, resulting from the application of a homotopy between the principal symbols of P and $D + R$. In cases where a variant of Stokes' theorem holds, the Chern–Weyl representative, that is given up to an exact form, gives rise to the renormalized integral of ω_∂ in the formula. Note that ω_∂ is a direct sum of forms on the boundary strata of Σ, since the boundary is stratified in general.

6. APPLICATIONS AND FUTURE DIRECTIONS

We summarize some of the consequences, as well as open problems in the study of index theory questions on Lie structures.

• The index theorem for geometric Dirac operators gives rise to obstructions for the existence of positive scalar curvature metrics on Lie manifolds. In addition, open problems arise in the study of concordance classes of metrics, relative to positive scalar curvature metrics, as detailed in [8].

• In some cases, Lie groupoid geometries give rise to approximate solutions of index problems associated to certain non-local operators with operator-valued principal symbols, e.g. [9].

• The problem of finding index theoretic formulas for Shapiro–Lopatinskij elliptic boundary value problems on manifolds with polycylindrical ends is a potentially very important area of study. These boundary value problems differ significantly from the ones in the regular case of compact manifolds with boundary. Calculi along the lines of Boutet de Monvel's calculus for boundary value problems have been studied in the setting of Lie groupoids [7, 19]; see also [6].

• It is an open problem to show that the comparison homomorphism λ between our variant of geometric K-homology and relative K-theory is injective and therefore an isomorphism. A positive answer to this question has significant implications for the study of universal secondary invariants [10].

Acknowledgments. I thank Jean-Marie Lescure, Victor Nistor, Elmar Schrohe and Georges Skandalis for stimulating discussions over the years, regarding the groupoid approach to index theory. In addition, the referee made several helpful suggestions which lead to an improvement of the paper. This research was conducted with support by the DFG project "Geometric operators on a class of manifolds with bounded geometry", as a member of the SPP 2026 "Geometry at Infinity" priority program.

REFERENCES

[1] M.F. Atiyah and I.M. Singer, *The index of elliptic operators on compact manifolds.* Bull. Amer. Math. Soc. **69** (1963), 422–433.

[2] M.F. Atiyah and I.M. Singer, *The index of elliptic operators.* III. Ann. of Math. (2) **87** (1968), 546–604.

[3] B. Ammann, R. Lauter, and V. Nistor, *Pseudodifferential operators on manifolds with a Lie structure at infinity.* Ann. of Math. (2) **165** (2007), *3*, 717–747.

[4] I. Androulidakis and G. Skandalis, *The holonomy groupoid of a singular foliation.* J. Reine Angew. Math. **626** (2009), 1–37.

[5] N. Berline, E. Getzler, amd M. Vergne, *Heat kernels and Dirac operators.* Grundlehren Math. Wiss., 298, Springer, Berlin, 1992.

[6] K. Bohlen, *A splitting index theorem on manifolds with corners.* Rev. Roumaine Math. Pures Appl. **67** (2022), *3-4*, 95–125.

[7] K. Bohlen, *Boutet de Monvel operators on Lie manifolds with boundary.* Adv. Math. **312** (2017), 234–285.

[8] K. Bohlen, *Positive scalar curvature metrics on manifolds with controlled geometry at infinity.* Oberwolfach Reports, 1732, pp. 2275–2278, EMS Press, 2017.

[9] K. Bohlen, *The Fredholm index for operators of tensor product type.* J. Operator Theory **90** (2023), *2*, 425–451.

[10] K. Bohlen, *Topological quantum field theory over Lie structures.* In preparation.

[11] K. Bohlen and J.-M. Lescure, *A geometric approach to K-homology for Lie manifolds.* Ann. Sci. Éc. Norm. Supér. (4) **56** (2023), *6*, 1747–1776.

[12] K. Bohlen, J.-M. Lescure, and E. Schrohe, *The Atiyah–Patodi–Singer index formula for pseudodifferential operators on Lie structures.* In preparation.

[13] K. Bohlen and E. Schrohe, *Getzler rescaling via adiabatic deformation and a renormalized index formula.* J. Math. Pures Appl. (9) **120** (2018), 220–252.

[14] C. Carvalho, V. Nistor, and Y. Qiao, *Fredholm conditions on non-compact manifolds: theory and examples.* In: *Operator Theory, Operator Algebras, and Matrix Theory.* Oper. Theory Adv. Appl., 267, pp. 79–122. Birkhäuser/Springer, Cham, 2018.

[15] A. Connes, *Noncommutative Geometry.* Academic Press, San Diego, CA, 1994.

[16] C. Debord, *Holonomy Groupoids of singular Foliations*. J. Differential Geom. **58** (2001), *3*, 467–500.

[17] C. Debord and J.M. Lescure, *K-duality for stratified pseudomanifolds*. Geom. Topol. **13** (2009), *1*, 49–86.

[18] C. Debord, J.M. Lescure, and F. Rochon, *Pseudodifferential operators on manifolds with fibred corners*. Ann. Inst. Fourier (Grenoble) **65** (2015), *4*, 1799–1880.

[19] C. Debord and G. Skandalis, *Lie groupoids, pseudodifferential calculus, and index theory*. In: *Advances in Noncommutative Geometry*, pp. 245–289. Springer, Cham, 2019.

[20] C.L. Epstein and R. Mazzeo, *Degenerate diffusion operators arising in population biology*. Ann. of Math. Stud., 185. Princeton Univ. Press, Princeton, NJ, 2013.

[21] E. van Erp and R. Yuncken, *On the tangent groupoid of a filtered manifold*. Bull. Lond. Math. Soc **49** (2017), *6*, 1000–1012.

[22] L. Guillaume, *Géométrie non-commutative et calcul pseudodifféntiel sur les variétés à coins fibrés*. Ph.D. Thesis, Université Paul Sabatier Toulouse 3, 2012.

[23] P. Hagan, A. Lesniewski, and D. Woodward, *Probability distribution in the SABR model of stochastic volatility*. In: *Large Deviations and Asymptotic Methods in Finance*. Springer Proc. Math. Stat., 110, pp. 1–35. Springer, Cham, 2015.

[24] M. Hilsum and G. Skandalis, *Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov (d'après une conjecture d'A. Connes)*. Ann. Sci. École Norm. Sup. (4) **20** (1987), *3*, 325–390.

[25] A.D. Ionescu and S. Klainerman, *On the uniqueness of smooth, stationary black holes in vacuum*. Invent. Math. **175** (2009), *1*, 35–102.

[26] V.A. Kondratiev, *Boundary value problems for elliptic equations in domains with conical or angular points*. Trans. Moscow Math. Soc. **16** (1967), 227–313.

[27] R. Lauter, B. Monthubert, and V. Nistor, *Pseudodifferential analysis on continuous family groupoids*. Doc. Math. **5** (2000), 625–655.

[28] R. Lauter, B. Monthubert, and V. Nistor, *Spectral invariance for certain algebras of pseudodifferential operators*. J. Inst. Math. Jussieu **4** (2005), *3*, 405–442.

[29] R. Lauter and S. Moroianu, *The index of cusp operators on manifolds with corners*. Ann. Glob. Anal. Geom. **21** (2002), *1*, 31–49.

[30] B.H. Lawson and M.-L. Michelsohn, *Spin Geometry*. Princeton Math. Ser., 38, Princeton Univ. Press, 1989.

[31] E. Leichtnam, R. Mazzeo, and P. Piazza, *The index of Dirac operators on manifolds with fibered boundaries*. Bull. Belg. Math. Soc. Simon Stevin **13** (2006), *5*, 845–855.

[32] M. Lesch, H. Moscovici, and M. Pflaum, *Connes–Chern character for manifolds with boundary and eta cochains*. Mem. Amer. Math. Soc. **220** (2012), *1036*.

[33] J.-M. Lescure and S. Vassout, *On evolution equations for Lie groupoids*. 2020, arXiv:2010.00227.

[34] R. Louis, *Nash resolutions of singular foliations with a view towards affine varieties*. 2023, arXiv:2301.08706.

[35] R. Mazzeo and R. Melrose, *Pseudodifferential operators on manifolds with fibred boundary*. Asian J. Math. **2** (1999), *4*, 833–866.

[36] H. McKean Jr. and I. Singer, *Curvature and the eigenvalues of the Laplacian.* J. Differential Geometry **1** (1967), *1*, 43–69.

[37] R.B. Melrose, *Analysis on Manifolds with Corners.* In preparation.

[38] R. Melrose, *Pseudodifferential operators, corners and singular limits.* ICM, 90, American Mathematical Society, Providence, RI, 1990.

[39] R.B. Melrose, *The Atiyah–Patodi–Singer Index Theorem.* Research Notes in Mathematics, 4, AK Peters, Ltd., Wellesley, MA, 1993.

[40] S. Minakshisundaram and A. Pleijel, *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds.* Canadian J. Math. **1** (1949), 242–256.

[41] H. Moriyoshi and P. Piazza, *Eta cocycles, relative pairings and the Godbillon-Vey index theorem.* Geom. Funct. Anal. **22** (2012), *6*, 1708–1813.

[42] V. Nistor, *Analysis on singular spaces: Lie manifolds and operator algebras.* J. Geom. Phys. **105** (2016), 75–101.

[43] V. Nistor, A. Weinstein, and P. Xu, *Pseudodifferential operators on differential groupoids.* Pacific J. Math. **189** (1999), *1*, 117–152.

[44] A. Paterson, *Continuous family groupoids.* Homology Homotopy Appl. **2** (2000), 89–104.

[45] A. Pazy, *Semigroups of linear operators and applications to partial differential equations.* Appl. Math. Sci., 44, Springer, New York, 1983.

[46] J. Pradines, *How to define the differentiable graph of a singular foliation.* Cahiers Topologie Géom. Différentielle Catég. **26** (1985), *4*, 339–380.

[47] D. Quillen, *Superconnections and the Chern character.* Topology **24** (1985), *1*, 89–95.

[48] F. Rochon, *Pseudodifferential operators on manifolds with foliated boundaries.* J. Funct. Anal. **262** (2012), *3*, 1309–1362.

[49] G. Skandalis, *C^*-algèbres, Algèbres de Von Neumann, Exemples.* Cours de M2, 2015.

[50] S.R.S. Varadhan, *Diffusion processes in a small time interval.* Comm. Pure Appl. Math. **20** (1967), 659–685.

[51] S. Vassout, *Unbounded pseudodifferential calculus on Lie groupoids.* J. Funct. Anal. **236** (2006), *1*, 161–200.

Received 28 June 2023

Princeton Research Forum
P.O.Box 264, Kingston NJ 08528-0264
kbohlen@gmail.com