CHARACTERIZATION OF NON-MINIMAL TREE ACTIONS

LISA CARBONE and LAURA CIOBANU

Let Γ be a group acting without inversions on a tree X. If there is no proper Γ-invariant subtree, we call the action of Γ on X *minimal*. Here we give a characterization of *non-minimal* tree actions. For a non-minimal action of Γ on X, we give structure theorems for the quotient graph of groups for Γ on X, its associated edge-indexed graph and its group of deck transformations.

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1. INTRODUCTION

Let Γ be a group acting without inversions on a locally finite tree X. We say that Γ acts *minimally* on X if there is no proper Γ-invariant subtree. As is the case with well known results concerning group actions on $\mathbb{R}$-trees, when the hyperbolic length function $l(\Gamma)$ of Γ is non-zero, there is a unique minimal Γ-invariant subtree $X_0 \subset X$ (see for example [1]). Now set $G = \mathrm{Aut}(X)$. If X is uniform, that is, X is the universal cover of a finite connected graph, then by ([2], (9.7)), $l(G) \neq 0$ and hence there is a unique minimal G-invariant subtree $X_0 \subset X$.

Here we give a characterization of *non-minimal* actions in terms of the quotient graph of groups for Γ on X, its associated edge-indexed graph (Section 2) and its group of deck transformations, that is, the group of automorphisms of X that commute with the quotient morphism (Section 5). In each case, we give a 'decomposition' of the action into 'minimal and non-minimal components' (Section 4). This gives rise to a structure theorem for the group of deck transformations as a semi-direct product of its minimal component and a product of finite groups (Section 5).

When the edge-indexed quotient graph is finite, we show that it contains a unique minimal invariant subgraph which we describe geometrically (Section 4). Moreover, we give a procedure to find the minimal invariant subgraph. It follows that the minimal Γ-invariant subtree X_0 of X is the universal covering of the minimal component of the quotient (Section 4).

Our motivation for this work is two-fold. Primarily, the results here of Section 2 and Section 4 are an important ingredient for the proof in [3] of Bass-Lubotzky's conjecture for the existence of non-uniform lattices on uniform trees ([2], Ch 8). Let X be a uniform tree and let $G = \mathrm{Aut}(X)$. Let X_0 be the unique minimal G-invariant subtree of X. Let $G_0 = \mathrm{Aut}(X) \mid_{X_0}$. Here we show that that $G_0 = \mathrm{Aut}(X_0)$ (Section 5). In [3], the first author showed how to extend non-uniform lattice subgroups $\Gamma_0 \leq G_0 = \mathrm{Aut}(X_0)$ to non-uniform lattices $\Gamma \leq G = \mathrm{Aut}(X)$. An important ingredient in this proof is the geometric characterization of an edge-indexed graph in terms of its unique minimal subgraph obtained here in Section 4.

Secondly, this work fills a gap in the literature concerning non-minimal actions on simplicial trees. Our results in Sections 3 and 4 are essentially a reworking, in terms of edge-indexed graphs, of known results (see for example [1] and [2]). Our structure theorem (Section 5) for the deck transformation group of a non-minimal action is a straightforward consequence of [2], Section 6, but does not seem to appear in the literature.

2. GROUP ACTIONS, EDGE-INDEXED QUOTIENTS AND MINIMALITY

Let Γ be a group acting without inversions on a tree X. The fundamental theorem of Bass and Serre states that Γ is encoded (up to isomorphism) in a quotient graph of groups $\mathbb{A} = \Gamma \backslash\backslash X$ ([1], [5]). Conversely a graph of groups $\mathbb{A}$ gives rise to a group $\Gamma = \pi_1(\mathbb{A}, a)$, $a \in V\mathbb{A}$, acting on a tree $X = \widetilde{(\mathbb{A}, a)}$ without inversions.

Now, assume that X is locally finite, and that Γ acts on X with quotient graph of groups $\mathbb{A} = \Gamma \backslash\backslash X$. Then $\mathbb{A}$ naturally gives rise to an 'edge-indexed' graph (A, i), defined as follows. The graph A is the underlying graph of $\mathbb{A}$ with vertex set VA, edge set EA, initial and terminal functions $\partial_0, \partial_1 : EA \mapsto VA$ which pick out the endpoints of an edge, and with fixed point free involution $- : EA \mapsto EA$ which reverses the orientation. The indexing $i : EA \mapsto \mathbb{Z}_{>0}$ of (A, i) is defined to be the group theoretic index

$$i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e(\mathcal{A}_e)],$$

where

$$(\mathcal{A}_a)_{a \in VA} \quad \text{and} \quad (\mathcal{A}_e = \mathcal{A}_{\overline{e}})_{e \in EA}$$

are the vertex and edge groups of $\mathbb{A}$, and $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}$ are the boundary monomorphisms of $\mathbb{A}$. We write $(A, i) = I(\mathbb{A})$ when $i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e(\mathcal{A}_e)]$ for data

$$\{\mathcal{A}_a,\ \mathcal{A}_e = \mathcal{A}_{\overline{e}},\ \alpha_e :\ \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}\}$$

from $\mathbb{A}$. Conversely, an *edge-indexed* graph (A, i) is defined to be a graph A and an assignment $i(e)$ ($\neq i(\overline{e})$ in general) of a positive integer to each oriented edge. Then (A, i) determines a universal covering tree $X = \widetilde{(A, i)}$ up to isomorphism, and every edge-indexed graph arises from a tree action [2]. Here we assume $i(e)$ is finite for each $e \in EA$. Under this assumption the universal covering tree $X = \widetilde{(A, i)}$ is locally finite [2].

Let (A, i) be an edge-indexed graph. We say that (A, i) is *minimal* if it is the edge-indexed quotient of a minimal tree action. A vertex $a \in VA$ is a *terminal vertex* of (A, i) if $\deg_{(A,i)}(a) = 1$, where

$$\deg_{(A,i)}(a) = \sum_{e \in E_0(a)} i(e),$$

and $E_0(a) = \{e \in EA \mid \partial_0 e = a\}$. A terminal vertex in (A, i) is then a geometrically terminal vertex in the graph A. That is, there i s a unique edge e with $\partial_0 e = a$. If (A, i) consists of a single vertex v, we set $\deg_{(A,i)}(v) = 0$ and so v is not a terminal vertex.

Now, let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups. If $A' \subset A$ is a connected subgraph of A, we write $\mathbb{A}\,|_{A'}$ for the subgraph of groups obtained by restriction to A'. Following ([1], (7.10)) we call $\mathbb{A}$ a *minimal graph of groups* if

$$\pi_1(\mathbb{A}\,|_{A'}, a_0) = \pi_1(\mathbb{A}, a_0)$$

only when $A' = A$, where $a_0 \in A'$. We call a vertex $a \in VA$ a *terminal vertex* of $\mathbb{A}$ if there is a unique edge e with $\partial_0 e = a$ and $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_a$ is an isomorphism.

Lemma 1. *Let $\mathbb{A}$ be a graph of groups and let $(A, i) = I(\mathbb{A})$. Then $\mathbb{A}$ has no terminal vertices if and only if (A, i) has no terminal vertices.*

Proof. Let a be a terminal vertex of $\mathbb{A}$. Then there is a unique edge e with $\partial_0 e = a$ and $i(e) = [\mathcal{A}_a : \alpha_e(\mathcal{A}_e)] = 1$. That is, a is a terminal vertex of (A, i).

Conversely, suppose (A, i) has a terminal vertex $a \in VA$, i.e. $\deg_{(A,i)}(a) = 1$ and a is a geometrically terminal vertex. This forces $[\mathcal{A}_a : \alpha_e(\mathcal{A}_e)] = 1$, which occurs if and only if $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_a$ is an isomorphism. Thus a is a terminal vertex of $\mathbb{A}$. □

3. CHARACTERIZING MINIMAL ACTIONS

Our applications in [3] and Section 5 require a detailed description of a non-minimal action on a simplicial tree. We use the formulation of Bass [1] of a minimal graph of groups to describe minimality of a tree action $\Gamma \times X \to X$

in terms of the edge-indexed quotient graph $(A, i) = I(\mathbb{A})$ of the graph of groups $\mathbb{A} = \Gamma\backslash\backslash X$.

Let $\mathbb{A}$ be a graph of groups. The notion of a minimal graph of groups of Bass ([1], (7.10)) is given in terms of intrinsic properties of $\mathbb{A}$. Bass then showed that minimal graphs of groups encode minimal tree actions. His result is as follows.

THEOREM 1 ([1], (7.12)). *Let $\mathbb{A}$ be a graph of groups, let $a_0 \in VA$, $\Gamma = \pi_1(\mathbb{A}, a_0)$, and $X = (\widetilde{\mathbb{A}, a_0})$. The following conditions are equivalent.*

(a) *The action of Γ on X is minimal.*

(b) *$\mathbb{A}$ is minimal.*

These conditions imply

(c) *$\mathbb{A}$ has no terminal vertices.*

If A is finite then (c) is equivalent to the other conditions. The following proposition gives a restatement of Bass' theorem in terms of the edge-indexed graph arising from $\mathbb{A}$.

PROPOSITION 1. *Let Γ be a group acting without inversions on a tree X with quotient graph of groups $\mathbb{A} = \Gamma\backslash\backslash X$ and edge-indexed quotient $(A, i) = I(\mathbb{A})$.*

(1) *If (A, i) is minimal then (A, i) has no terminal vertices.*

(2) *If (A, i) is finite and has no terminal vertices then (A, i) is minimal.*

Proof. (1) Suppose that (A, i) is minimal. Then (A, i) is the edge-indexed quotient graph of a minimal action Γ' x $X' \mapsto X'$ which corresponds to the action of Γ on X up to isomorphism. Without loss of generality we take $\Gamma' = \Gamma, X' = X$. Then by Theorem 1, $\mathbb{A} = \Gamma\backslash\backslash X$ has no terminal vertices, which by Lemma 1 is equivalent to (A, i) having no terminal vertices.

(2) Since (A, i) has no terminal vertices, $\mathbb{A}$ has no terminal vertices by Lemma 1. Assume that (A, i) is finite. Then by Theorem 1, Γ acts minimally on X. That is, $(A, i) = I(\Gamma\backslash\backslash X)$ is minimal. □

We remark that (2) in Proposition 1 is false without the assumption of finiteness of (A, i). This can be seen from the following example. Let (A, i) be the graph $A = \mathbb{Z}$ of the integer line with $i(e) = 1$ for all edges except for a finite number of $e_1, e_2, \ldots, e_n \in EA$ with $\partial_1 e_j = \partial_0 e_{j+1}$, $j = 1, \ldots, n-1$, so that $i(e_j) > 1$, $j = 1, \ldots, n$. If $\mathbb{A}$ is any graph of groups with $(A, i) = I(\mathbb{A})$, then $\mathbb{A}$ cannot be minimal, since $\pi_1(\mathbb{A}\,|_\gamma, a_0) = \pi_1(\mathbb{A}, a_0)$, where γ is the path $\gamma = (e_1, e_2, \ldots, e_n)$ and $a_0 \in \gamma$.

In summary, we see that the property of minimality can be detected equivalently from a graph of groups $\mathbb{A}$ or its edge-indexed graph (A, i) with $(A, i) = I(\mathbb{A})$. Combining the results of Lemma 1, Theorem 1 and Proposition 1, we obtain

COROLLARY 1. *Let $\mathbb{A}$ be a graph of groups and let $(A,i) = I(\mathbb{A})$. Assume that A is finite. Then the following conditions are equivalent.*

(1) *$\mathbb{A}$ is minimal.*

(2) *$\mathbb{A}$ has no terminal vertices.*

(3) *(A,i) is minimal.*

(4) *(A,i) has no terminal vertices.*

4. STRUCTURE OF NON-MINIMAL ACTIONS

In this section we show that a finite edge-indexed quotient graph (A,i) of a non-minimal action has a unique minimal subgraph $(A_0,i_0) \subset (A,i)$. We describe (A,i) and $X = \widetilde{(A,i)}$ geometrically in terms of their 'minimal components'. We first recall the following results of [1] and [2] which give criteria for the existence of minimal subtrees.

PROPOSITION 2 ([1], (7.5)). *Let Γ be a group acting on a tree X and let $l(\Gamma)$ be the hyperbolic length function of the action. If $l(\Gamma) \neq 0$ then there is a unique minimal Γ-invariant subtree X_Γ of X. We have*

$$X_\Gamma = \bigcup_{\gamma\in\Gamma,\ l(\gamma)>0} X_\gamma,$$

where X_γ is the axis of translation of the hyperbolic element $\gamma \in \Gamma$.

If we consider subgroups of $\mathrm{Aut}(X)$ rather than general group actions, we obtain the following criterion of Bass and Lubotzky for the existence of a unique minimal invariant subtree of X.

PROPOSITION 3 ([2], (9.7)). *Let X be a tree and let $G = \mathrm{Aut}(X)$. If X is uniform, then $l(G) \neq 0$ and there is a unique minimal G-invariant subtree $X_0 \subset X$.*

Our aim is to obtain results analogous to Propositions 2 and 3 in terms of edge-indexed quotient graphs.

Let (T,i) be an edge-indexed graph. As in [2] and [4] we say that (T,i) is a *dominant edge-indexed tree* if T is a tree, $|VT| > 1$ and there is a vertex $a \in VT$ such that

$$d(\partial_0 e, a) > d(\partial_1 e, a) \text{ implies } i(e) = 1$$

for all $e \in ET$. We call such a vertex $a \in VT$ a *dominant root* of (T,i) and we write (T,i,a) when (T,i) is a dominant edge-indexed tree, rooted at $a \in VT$.

If (T,i,a) is a dominant rooted edge-indexed tree, then it is clear from the definition that (T,i,a) contains a terminal vertex.

Let (A,i) be any finite edge-indexed graph. We may view (A,i) as the edge-indexed quotient of a tree action $\Gamma \times X \to X$, with $X = \widetilde{(A,i)}$ $I(\Gamma\backslash\backslash X) =$

(A, i), that is in general non-minimal. The following theorem establishes the existence of a minimal subgraph $(A_0, i_0) \subset (A, i)$. The edge-indexed graph (A_0, i_0) will then coincide (up to isomorphism) with the edge-indexed quotient graph of any action $\Gamma_0 \times X_0 \to X_0$ on the minimal subtree $X_0 \subset X$. Our assumption that (A, i) is finite guarantees that $X = \widetilde{(A, i)}$ is uniform and hence by Proposition 3 there is a unique minimal subtree $X_0 \subset X$.

Moreover, we will show that the minimal subgraph (A_0, i_0) is essesntially unique. That is, (A_0, i_0) is unique unless (A, i) is a tree, in which case (A_0, i_0) is unique up to isomorphism (Theorem 3 below).

THEOREM 2. *Let (A, i) be a finite edge-indexed graph. Then*

(1) *There is a connected subgraph (A_0, i_0) of (A, i), maximal with respect to the property of having no terminal vertices, such that (A_0, i_0) contains the union of all closed paths in (A, i) and $(A, i) \setminus (A_0, i_0)$ is a disjoint union of edge-indexed trees.*

(2) *Let (A_0, i_0) be the subgraph of (A, i) as in (1). Then (A, i) is a union of (A_0, i_0) and dominant rooted edge-indexed trees with roots in (A_0, i_0). That is,*

$$(A, i) = (A_0, i_0) \amalg \coprod_{a_j \in \Delta} (T_j, i_j, a_j),$$

where $\Delta = \{a_1, \ldots, a_n\}$ is a subset of VA_0, (T_j, i_j, a_j) are finite dominant-rooted edge-indexed trees with root vertices $a_j \in VA_0$, $j = 1, \ldots, n$.

Proof. (1) If (A, i) has no terminal vertices, then (A, i) is minimal and we take $(A_0, i_0) = (A, i)$. Now, suppose (A, i) has terminal vertices. In order to obtain (A_0, i_0), we will use a *minimalization procedure* which we now describe.

Suppose (A, i) has terminal vertices $v_{11}, \ldots, v_{1n_1}$, where n_1 is some positive integer. Then for each v_{1j} there is a unique edge e_{1j} with $\partial_0 e_{1j} = v_{1j}$, $j = 1, \ldots, n_1$. Since v_{1j} is terminal, e_{1j} cannot belong to a closed path in (A, i). Let (A_2, i_2) be the subgraph of $(A_1, i_1) = (A, i)$ obtained by removing v_{1j} and e_{1j} but not $\partial_1 e_{1j}$, $j = 1, \ldots, n_1$. If (A_2, i_2) has no terminal vertices then set $(A_0, i_0) = (A_2, i_2)$ and we are done. Note that removing v_{1j} and e_{1j} may give rise to terminal vertices in (A_2, i_2). Suppose that (A_2, i_2) has terminal vertices. Let (A_3, i_3) be the subgraph of (A_2, i_2) obtained by removing the terminal vertices of (A_2, i_2) and their unique incident edges.

If at some stage of the procedure, the subgraph (A_n, i_n) consists of exactly two vertices connected by an edge, we can remove either of the two vertices and obtain a subgraph consisting of only one vertex.

Since (A, i) is finite, the successive 'pruning' of terminal vertices eventually produces a non-empty, connected subgraph (A_0, i_0) of (A, i) that is maximal with respect to the property of having no terminal vertices. Moreover, (A_0, i_0) contains the union of all closed paths in (A, i).

We remark that this minimalization procedure is not unique. For example, one can 'prune a branch at a time'. That is, we can successively remove all vertices on a path from a given terminal vertex that are not terminal in (A, i) but become terminal at some stage of the procedure.

(2) Let (A_0, i_0) be the minimal subgraph of (A, i) as in (1), obtained by a minimalization procedure. We claim that (A, i) is a union of (A_0, i_0) and a finite number of dominant rooted trees (T_j, i_j), $j = 1, \ldots, n$, rooted at $a_j \in VA_0$, $j = 1, \ldots, n$. From (1) we have that $(A, i) \setminus (A_0, i_0)$ is a disjoint union of edge-indexed trees, say (T_j, i_j), $j = 1, \ldots, n$. It remains to show that each (T_j, i_j) is a dominant rooted edge-indexed tree with root vertices in VA_0.

Let $e \in EA \setminus EA_0$ be such that $d(\partial_0 e, A_0) = 1$ and e points 'towards' (A_0, i_0) where for $a \in VA$

$$d(a, A_0) = \min_{x \in VA} d(a, x),$$

and $d : VA \to \mathbb{Z}_{>0}$ is the edge-path distance. Then we must have $i(e) = 1$, otherwise $(A_0, i_0) \cup \{e\}$ is minimal, contradicting the fact that (A_0, i_0) is maximal with respect to having no terminal vertices. Let

$$S(A_0, r) = \{e \in EA \setminus EA_0 \mid d(\partial_0 e, A_0) = r \text{ and } e \text{ points towards } A_0\}$$

be the sphere of radius r in (A, i) about (A_0, i_0). Let m be the maximum radius of any sphere in (A, i) about (A_0, i_0). For each $k = 1, \ldots, m$, let $(A_k, i_k) = (A_0, i_0) \cup S(A_0, k)$. Then for each $e \in (A_k, i_k) \setminus (A_0, i_0)$ we must have $i(e) = 1$. Otherwise,

$$(A_0, i_0) \cup \gamma \cup \{e\}$$

is minimal, where γ is the unique path from $\partial_1 e$ to (A_0, i_0), contradicting the maximality of (A_0, i_0). Now, let (T_j, i_j) denote the connected components of $(A, i) \setminus (A_0, i_0)$, $j = 1, \ldots, n$. For each $j = 1, \ldots, n$, let $a_j \in VA_0$ be such that $a_j = \partial_1 e$ for some $e \in S(A_0, 1)$. Then for each $e \in ET_j$ we have shown that $d(\partial_0 e, a_j) > d(\partial_1 e, a_j)$ implies $i(e) = 1$. That is, (T_j, i_j) is a dominant rooted tree, rooted at $a_j \in VA_0$. In general,

$$(A, i) = (A_0, i_0) \amalg \coprod_{a_j \in \Delta} (T_j, i_j, a_j),$$

where $\Delta = \{a_1, \ldots, a_n\}$ is a subset of VA_0. □

The proof of Theorem 2 shows that any dominant rooted edge-indexed tree (T, i, a) in (A, i) is removed during minimalizatiation while the root a remains in (A_0, i_0). We obtain the following result.

Corollary 2. *Let (A, i) be a finite edge-indexed graph. Let (A_0, i_0) be as in Theorem 2. If $v \in VA$ is the root of a dominant rooted edge-indexed tree in (A, i), v is a vertex of (A_0, i_0).*

THEOREM 3. *Let (A,i) be a finite edge-indexed graph. Let (A_0,i_0) be a connected subgraph, maximal with respect to having no terminal vertices, as in* (1) *of Theorem* 2. *If (A,i) is a tree then (A_0,i_0) may not be unique, but is unique up to isomorphism. Otherwise, (A_0,i_0) is unique.*

Proof. Suppose that $|V_{A_0}|=1$. Then either (A,i) is a tree in which case A_0 consists of a single vertex and is unique up to isomorphism, or (A,i) is not a tree in which case (A_0,i_0) is a bouquet of loops attached at a single vertex, and is unique. This completes the proof when $|V_{A_0}|=1$.

Now, suppose that $|V_{A_0}|\geq 2$, and that (B_0,j_0) is another minimal subgraph of (A,i). Suppose that $(B_0,j_0)\cap(A_0,i_0)=\emptyset$. This implies that (A,i) is a tree, since by Theorem 2 both (B_0,j_0) and (A_0,i_0) contain the union of all closed paths in (A,i). If (A,i) is a tree, and $(B_0,j_0)\cap(A_0,i_0)=\emptyset$, then (B_0,j_0) and (A_0,i_0) are joined by a separating path γ that is removed during minimalization, since $|V_{A_0}|\geq 2$ and $|V_{B_0}|\geq 2$. This is impossible as no vertex of γ is a terminal vertex at any stage of minimalization. Hence we must have $(B_0,j_0)\cap(A_0,i_0)\neq\emptyset$. We no longer assume that (A,i) is a tree.

Let $x_0\in VB_0\backslash VA_0$. Then x_0 belongs to a dominant rooted edge-indexed tree (T,i,a) in (A,i) removed during a minimalization procedure that produces (A_0,i_0). In particular, x_0 is *not* the root vertex a since, by Corollary 2, (A_0,i_0) contains all dominant roots of (A,i). But x_0 *is* the root vertex of the subtree of (T,i,a) consisting of all paths from x_0 to terminal vertices of (A,i). Hence x_0 is removed during a minimalization procedure that produces (B_0,j_0), which is a contradiction. Hence $(B_0,j_0)=(A_0,i_0)$ and the minimal subgraph is unique. □

We call (A_0,i_0) in Theorems 2 and 3 the *unique minimal subgraph* of (A,i). Let (A,i) be finite and $X=\widetilde{(A,i)}$. Suppose we apply the above minimalization procedure to X and (A,i) simultaneously. That is, when we remove an edge e from A, we remove all edges in X in the fiber above e. Once we obtain the unique minimal subgraph (A_0,i_0), we have also obtained its covering tree $\widetilde{(A_0,i_0)}$. It can be easily shown that $\widetilde{(A_0,i_0)}$ coincides with the unique minimal subtree $X_0\subset X$. That is,

$$(A,i)=(A_0,i_0)\amalg\coprod_{a_j\in\Delta\ \subseteq\ VA_0}(T_j,i_j,a_j),$$

and X has the form

$$X=X_0\quad\amalg\coprod_{x_{j,k}\in p^{-1}(a_j),\ a_j\in VA_0}(Y_{j,k},x_{j,k}),$$

where $X_0=\widetilde{(A_0,i_0)}$, $(Y_{j,k},x_{j,k})=\widetilde{(T_j,i_j,a_j)}$, and p is the covering map $p(x_{j,k})=a_j$, $j=1,\ldots,n$, $k\geq 1$. Since T_j is a dominant rooted edge-indexed

tree and X is uniform, $Y_{j,k}$ is a finite rooted tree, rooted at $x_{j,k} \in p^{-1}(a_j)$, $a_j \in VA_0$, for $j = 1, \ldots, n$, $k \geq 1$. Combining the results in this section we obtain the following result.

COROLLARY 3. *Let X be a uniform tree, let $G = \mathrm{Aut}(X)$, and let $(A, i) = I(G\backslash\backslash X)$. Then there is a unique minimal G-invariant subtree $X_0 \subset X$ and a unique minimal subgraph $(A_0, i_0) \subset (A, i)$. Moreover, $X_0 = \widetilde{(A_0, i_0)}$, and $(A_0, i_0) = I(G\backslash\backslash X_0)$. Conversely, for any finite edge-indexed graph (A, i) and universal covering tree $X = \widetilde{(A, i)}$, there is a unique minimal subgraph (A_0, i_0) of (A, i) unique minimal subtree X_0 of X, and $X_0 = \widetilde{(A_0, i_0)}$.*

5. STRUCTURE OF DECK TRANSFORMATION GROUPS FOR NON-MINIMAL ACTIONS

Let (A, i) be a finite edge-indexed graph, and let (A_0, i_0) be the unique minimal subgraph of (A, i). Let $X_0 = \widetilde{(A_0, i_0)}$ and $X = \widetilde{(A, i)}$ with corresponding covering maps $p_0 : X_0 \to (A_0, i_0)$ and $p : X \to (A, i)$. By Theorem 2 we have

$$(A, i) = (A_0, i_0) \amalg \coprod_{a_j \in \Delta \subseteq VA_0} (T_j, i_j, a_j),$$

and the universal covering tree $X = \widetilde{(A, i)}$ has the form

$$X = X_0 \quad \amalg \coprod_{x_{j,k} \in p^{-1}(a_j),\ a_j \in VA_0} (Y_{j,k}, x_{j,k}),$$

where $X_0 = \widetilde{(A_0, i_0)}$, $(Y_{j,k}, x_{j,k}) = \widetilde{(T_j, i_j, a_j)}$, and p is the covering map $p(x_{j,k}) = a_j$, $j = 1, \ldots, n$, $k \geq 1$. The subgroups of $\mathrm{Aut}(X)$

$$G_{(A_0,i_0)} = \{g \in \mathrm{Aut}(X_0) \mid g \circ p_0 = p_0\},$$

$$G_{(A,i)} = \{g \in \mathrm{Aut}(X) \mid g \circ p = p\}$$

are the groups of deck transformations of (A_0, i_0), and (A, i) respectively. The following lemma shows that deck transformations of the minimal subtree X_0 extend naturally to deck transformations of X.

LEMMA 2. *Let $g_0 \in G_{(A_0,i_0)}$. Then g_0 extends to $g \in G_{(A,i)}$.*

Proof. Let $a_0 \in VA_0$ and let (T_{a_0}, i, a_0) be a dominant rooted edge-indexed tree attached at a_0. Let $x_0 \in p_0^{-1}(a_0) \in VX_0$, and let $g_0 \in G_{(A_0,i_0)}$. Let $y_0 = g_0(x_0)$. Then

$$p_0(y_0) = p_0(g_0(x_0)) = p_0(x_0) = a_0.$$

Thus, $y_0 \in p_0^{-1}(a_0)$. Let $(Y_{x_0}, x_0) = (\widetilde{T_{a_0}, i, a_0})$. Then Y_{x_0} is a finite rooted tree, rooted at x_0. Moreover, since $y_0 \in p_0^{-1}(a_0)$, there is a finite rooted tree Y_{y_0} rooted at y_0 such that

$$(Y_{y_0}, y_0) = (\widetilde{T_{a_0}, i, a_0}) \cong (Y_{x_0}, x_0).$$

That is, we can extend $g_0 \in G_{(A_0,i_0)}$ to $g \in G_{(A,i)}$ in such a way that $g \mid_{X_0} = g_0$, and g carries Y_{x_0} isomorphically to $Y_{g(x_0)} = Y_{y_0}$. □

Corollary 4. *We have*

$$G_{(A,i)} \mid_{X_0} = G_{(A_0,i_0)}.$$

Proof. Let $g \in G_{(A,i)} \mid_{X_0}$. Then $g(X_0) = X_0$ and $p(g(X_0)) = A_0$. So $g \mid_{X_0} \in G_{(A_0,i_0)}$. Conversely, let $g_0 \in G_{(A_0,i_0)}$. By Lemma 2, g_0 extends to $g \in G_{(A,i)}$ in such a way that $g \mid_{X_0} = g_0$. □

Let $G_{X_0} = \mathrm{Stab}_G(X_0) = \{g \in G \mid gx = x,\ x \in VX_0\}$, and set

$$\mathrm{Stab}_{G_{(A,i)}}(X_0) = G_{(A,i)} \cap G_{X_0}.$$

Theorem 4. *There is a split short exact sequence*

$$1 \longrightarrow \mathrm{Stab}_{G_{(A,i)}}(X_0) \longrightarrow G_{(A,i)} \longrightarrow G_{(A,i)} \mid_{X_0} \longrightarrow 1\,.$$

Proof. Since X_0 is $G_{(A,i)}$-invariant, by ([2], (6.6)(6)) there is an exact sequence

$$1 \longrightarrow \mathrm{Stab}_{G_{(A,i)}}(X_0) \longrightarrow G_{(A,i)} \longrightarrow G_{(A,i)} \mid_{X_0} \longrightarrow 1\,.$$

By Lemma 2, given $g_0 \in G_{(A,i)} \mid_{X_0} = G_{(A_0,i_0)}$, g_0 extends to $g \in G_{(A,i)}$ in such a way that $g \mid_{X_0} = g_0$. Hence there is a map $\mu : G_{(A,i)} \mid_{X_0} \to G_{(A,i)}$ such that $\mu \circ \phi = 1$, where ϕ is the restriction map $\phi : G_{(A,i)} \to G_{(A,i)} \mid_{X_0}$ Thus the above is a split exact sequence. □

An immediate consequence of Theorem 4 is

Corollary 5. *With the notation in this section, we have*

$$G_{(A,i)} \cong \mathrm{Stab}_{G_{(A,i)}}(X_0) \rtimes G_{(A_0,i_0)}.$$

Using ([2], (6.6)) we obtain the following description of $G_{X_0} = \mathrm{Stab}_G(X_0)$ as the product of automorphism groups of rooted trees attached at the vertices of X_0.

Corollary 6. *With the notation in this section, by* ([2], (6.6)(3)) *we have*

$$G_{X_0} = \mathrm{Stab}_G(X_0) \cong \prod_{x_{j,k} \in p^{-1}(a_j),\ a_j \in VA_0} \mathrm{Aut}(Y_{j,k}, x_{j,k}),$$

where $j = 1, \dots, n$, $k \geq 1$, *and since each* $Y_{j,k}$ *is a finite tree,* $\mathrm{Aut}(Y_{j,k}, x_{j,k})$ *is a finite group. Furthermore, for* $j = 1, \dots, n$, $k \geq 1$ *we have*

$$\mathrm{Stab}_{G_{(A,i)}}(X_0) \cong \prod_{x_{j,k} \in p^{-1}(a_j),\ a_j \in VA_0} \mathrm{Aut}(Y_{j,k}, x_{j,k}) \cap G_{(A,i)}.$$

We illustrate the results of this section with the following example.

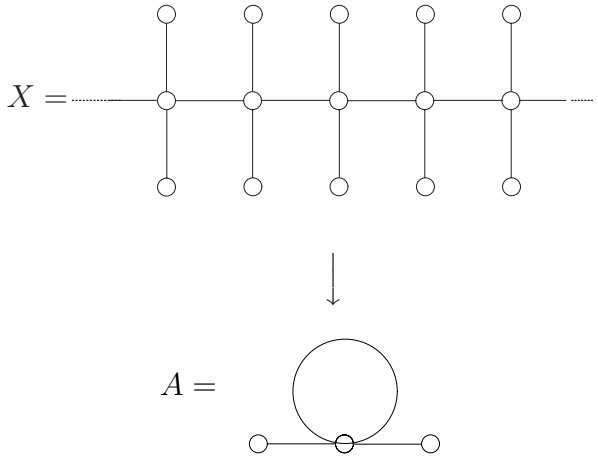


Fig. 1. The tree X and its quotient A.

Let $\Gamma = \mathrm{Aut}(X)$, where X is as in Figure 1. The action of Γ on X leaves the central axis invariant. In the edge-indexed quotient (A, i) we have $i(e) = 1$ for every $e \in EA$. The unique minimal subgraph is shown in Figure 2.

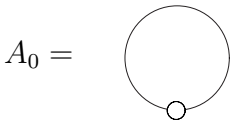


Fig. 2. The minimal subgraph.

We have $G_{(A,i)} = \mathrm{Aut}(X)$ and X_0 is (isomorphic to) the integer line $\mathbb{Z}$. Applying Corollary 5 and Corollary 6 we obtain

$$\mathrm{Stab}_{G_{(A,i)}}(X_0) = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \text{ and } G_{(A_0,i_0)} = D_\infty,$$

$$G_{(A,i)} \cong (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \rtimes D_\infty,$$

where D_∞ denotes the infinite dihedral group which acts as deck transformations on the central axis, and $\mathrm{Stab}_{G_{(A,i)}}(X_0) = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ acts as flips (of order 2) about the central axis.

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Rutgers, The State University of New Jersey
Department of Mathematics
Hill Center-Busch Campus
110 Frelinghuysen Rd
Piscataway, NJ 08854-8019, USA
carbonel@math.rutgers.edu

and

University of Auckland
Department of Mathematics
Private Bag 92019, Auckland, New Zealand
ciobanu@math.auckland.ac.nz