Dedicated to Professor Nicu Boboc on the occasion of his 80^{th} birthday

DIMENSION-FREE EULER ESTIMATES OF ROUGH DIFFERENTIAL EQUATIONS

YOUNESS BOUTAIB, LAJOS GERGELY GYURKÓ, TERRY LYONS
and DANYU YANG

We extend the result in [6, 7] and [8], and give a dimension-free Euler estimation of solution of rough differential equations in terms of the driving rough path. In the meanwhile, we prove that, the solution of rough differential equation is close to the exponential of a Lie series, with a concrete error bound.

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1. INTRODUCTION

Suppose that X is a continuous bounded variation path defined on some interval I and taking its values in a Banach space $\mathcal{V}$. We view this path as a stream of information and allow it to be highly oscillatory on normal scales. The theory of rough paths considers streams of information, such as X, for their effect on other systems and provides quantitative tools to model this interaction. Consider the stream as the input to an automata or controlled differential equation and so impacting on the evolution of the state Y in some controlled system:

$$\mathrm{d}Y = f(Y)\,\mathrm{d}X,\ Y_0 = \xi. \tag{1}$$

A key contribution of the theory is the development of quantitative tools and estimates that allow one to analyze the response Y from a top down analysis of X, and in particular provides a mechanism for directly quantifying the effects of the oscillatory components of X without a detailed analysis of the trajectory of X. As a result, the methods apply to equations where X does not have finite length. Differential equations driven by Brownian motion can be treated deterministically.

The interest in modelling and understanding such interactions is rather wide. This paper is intended to create a useful interface by stating and proving one of the main results in a way that appears to the authors particularly useful for moving out into applications. It deliberately sets out to hide the machinery and implementation of the main proofs in rough path theory and to provide only a useful and rigorous statement of a result that captures the essence of what the machinery delivers and is valid across all Banach spaces (including finite dimensional ones) so that the methods can be used more widely without great initial intellectual investment.

Davie [6] established some high order Euler estimates of solution of rough differential equations, driven by p-rough paths, $1 \leq p < 3$. By using geodesic approximations, Friz and Victoir [7, 8] extended Davie's results to rough differential equations driven by weak geometric p-rough paths, $p \geq 3$. The formulation and proofs in [6, 7], and [8] are dimension-dependent, and the error bound may explode as the dimension increases.

By modifying the method used in [6, 7] and [8], we give a dimension-free high order Euler estimation of solution of rough differential equations (*i.e.* both the driving rough path and the solution path live in infinite dimensional spaces). Our estimates are first developed for ordinary differential equations. Then by passing to limit and using universal limit theorem (see [12, 14], similar estimates hold for rough differential equations.

The main idea of our proof is to compare the solution of (1) (on small interval $[s,t]$) with the solution of another ordinary differential equation (on $[0,1]$) whose vector field is invariant with time although it varies with s,t. We do this by contracting the logarithm of the signature with the vector fields of the differential equation, through the canonical map of the free Lie algebra to the space of vector fields. Chen [5] first observed that the logarithm of the signature is a Lie element in the free Lie algebra. Magnus [15] also gave a formula for the solution of linear ordinary differential equations as an exponential of a Lie series but it is not as transparent. Mielnik, Plebański [16] and Strichartz [20], gave the explicit expression of Magnus's formula. Arous [1], Hu [11] and Castell [3] proved that similar results hold for stochastic differential equations. However, none of these early references seem to have separated the expression into its two components – the free Lie element (or log signature) and the canonical image of the free Lie element as a vector field. However, it was well known that one could do this and that it was an effective numerical approach by the time of the paper [9]. See also Baudoin [2] for detailed treatment of the exponential Lie series for differential equations. Since the truncated exponential can be treated as the solution of an ordinary differential equation (see [4]), we use this ordinary differential equation to compute the truncated exponential of the original flow, and give a concrete error bound.

2. BACKGROUND AND NOTATIONS

Let $\mathcal{U}$ and $\mathcal{V}$ be two Banach spaces.

2.1. Algebraic Structure

Following Def 1.25 [13], we define admissible tensor norm on tensor products.

Definition 1 (admissible norm). Suppose $\mathcal{V}$ is a Banach space. Denote by $Sym(n)$ the symmetric group of degree n. We say a norm on tensor products of (elements in) $\mathcal{V}$ is admissible, if it satisfies that,

$$\left\|v^1\otimes\cdots\otimes v^n\right\| = \left\|v^{\sigma(1)}\otimes\cdots\otimes v^{\sigma(n)}\right\| \leq \Pi_{i=1}^n \left\|v^i\right\|, \ \forall\left\{v^i\right\}_i\subset\mathcal{V}, \quad \forall\sigma\in Sym(n), \forall n\geq 1. \tag{2}$$

For example, inequality (2) is satisfied by injective and projective tensor norms (Prop 2.1 and Prop 3.1 in [18]).

Definition 2 ($\mathcal{V}^{\otimes n}$ and $[\mathcal{V}]^n$). We select an admissible norm on tensor products of $\mathcal{V}$, and for integer $n\geq 1$, define $\mathcal{V}^{\otimes n}$ and $[\mathcal{V}]^n$ as the closure of

$$\left\{\sum_{k=1}^m v_k^1\otimes\cdots\otimes v_k^{n-1}\otimes v_k^n,\ \left\{v_k^i\right\}_{i,k}\subset\mathcal{V},\ m\geq 1\right\},$$

$$\left\{\sum_{k=1}^m \left[v_k^1,\cdots\left[v_k^{n-1},v_k^n\right]\right],\ \left\{v_k^i\right\}_{i,k}\subset\mathcal{V},\ m\geq 1\right\},$$

w.r.t. the norm selected, where $[u,v]:=u\otimes v-v\otimes u$.

Definition 3. For integers $n\geq k\geq 1$, let π_k denote the projection of $1\oplus\mathcal{V}\oplus\cdots\oplus\mathcal{V}^{\otimes n}$ to $\mathcal{V}^{\otimes k}$. Define $\exp_n:\mathcal{V}\oplus\cdots\oplus\mathcal{V}^{\otimes n}\to 1\oplus\mathcal{V}\oplus\cdots\oplus\mathcal{V}^{\otimes n}$ by

$$\exp_n(a):=1+\sum_{k=1}^n\pi_k\left(\sum_{j=1}^n\frac{a^{\otimes j}}{j!}\right), \forall a\in\mathcal{V}\oplus\cdots\oplus\mathcal{V}^{\otimes n}.$$

Define $\log_n:1\oplus\mathcal{V}\oplus\cdots\oplus\mathcal{V}^{\otimes n}\to\mathcal{V}\oplus\cdots\oplus\mathcal{V}^{\otimes n}$ by

$$\log_n(g):=\sum_{k=1}^n\pi_k\left(\sum_{j=1}^n\frac{(-1)^{j+1}}{j}(g-1)^{\otimes j}\right), \forall g\in 1\oplus\mathcal{V}\oplus\cdots\oplus\mathcal{V}^{\otimes n}.$$

Definition 4 ($G^n(\mathcal{V})$). Suppose $\mathcal{V}$ is a Banach space. Then we define recursively

$$[\mathcal{V}]^{k+1}:=\left[\mathcal{V},[\mathcal{V}]^k\right]\quad\text{with}\quad[\mathcal{V}]^1:=\mathcal{V},$$

and define

$$G^n(\mathcal{V}) := \left\{ \exp_n(a) \,|\, a \in [\mathcal{V}]^1 \oplus \cdots \oplus [\mathcal{V}]^n \right\}.$$

For $g, h \in G^n(\mathcal{V})$, we define product and inverse as

$$g \otimes h := \sum_{k=0}^{n} \left(\sum_{j=0}^{k} \pi_j(g) \otimes \pi_{k-j}(h) \right) \quad \text{and}$$

$$g^{-1} := 1 + \sum_{k=1}^{n} \pi_k \left(\sum_{j=1}^{n} (-1)^j (g-1)^{\otimes j} \right).$$

We equip $G^n(\mathcal{V})$ with $\|\cdot\|$ which is defined as

$$\|g\| := \sum_{k=1}^{n} \|\pi_k(g)\|^{\frac{1}{k}}, \forall g \in G^n(\mathcal{V}). \tag{3}$$

Then $G^n(\mathcal{V})$ is a topological group, called the step-n nilpotent Lie group over $\mathcal{V}$.

$G^n(\mathcal{V})$ is nilpotent because $\left[t^n, \ldots, \left[t^2, t^1\right]\right] = 0, \forall \{t^i\}_{i-1}^n \subset \mathcal{V} \oplus \cdots \oplus \mathcal{V}^{\otimes n}$. The $\|\cdot\|$ defined at (3) is not a norm because it is not sub-additive, but $\|\cdot\|$ is equivalent to a norm up to a constant depending on n (Exer 7.38 [8], which is valid for truncated Lie group over Banach space $\mathcal{V}$).

For more about Lie algebra and Lie group, please refer to [**?**].

2.2. **Rough Paths**

Definition 5 ($S_n(x)$). Suppose $x : [0,T] \to \mathcal{V}$ is a continuous bounded variation path. For integer $n \geq 1$, define the step-n signature of x, $S_n(x) : [0,T] \to (G^n(\mathcal{V}), \|\cdot\|)$, as

$$S_n(x)_t = \left(1, x_{0,t}^1, x_{0,t}^2, \ldots, x_{0,t}^n\right),$$

$$\text{with} \quad x_{0,t}^h = \int \cdots \int_{0<u_1<\cdots<u_h<t} \mathrm{d}x_{u_1} \otimes \cdots \otimes \mathrm{d}x_{u_k} \Big).$$

Definition 6 (d_p metric and p-variation). For $p \geq 1$, denote $[p]$ as the integer part of p. Suppose X and Y are continuous paths defined on $[0,T]$ taking value in $G^{[p]}(\mathcal{V})$. Define

$$d_p(X,Y) := \max_{k=1,2,\ldots,[p]} \sup_{D \subset [0,T]} \left(\sum_{j, t_j \in D} \left\| \pi_k\left(X_{t_j,t_{j+1}}\right) - \pi_k\left(Y_{t_j,t_{j+1}}\right) \right\|^{\frac{p}{k}} \right)^{\frac{1}{p}},$$

where the supremum is taken over all finite partitions $D=\{t_j\}_{j=0}^n$, $0=t_0<t_1<\cdots<t_n=T$, $n\geq 1$.

With e denotes the identity path (*i.e.* $e_t=1\in G^{[p]}(\mathcal{V})$, $t\in[0,T]$), we define the p-variation of X on $[0,T]$ as

$$\|X\|_{p-var,[0,T]}:=d_p(X,e).$$

Definition 7 (geometric p-rough path). $X:[0,T]\to\left(G^{[p]}(\mathcal{V}),\|\cdot\|\right)$ is called a geometric p-rough path, if there exists a sequence of continuous bounded variation paths $x_l:[0,T]\to\mathcal{V}$, $l\geq 1$, such that

$$\lim_{l\to\infty}d_p\left(S_{[p]}(x_l),X\right)=0.$$

Definition 8 ($C^\gamma(\mathcal{V},\mathcal{U})$). For $\gamma>0$, we say $r:\mathcal{V}\to\mathcal{U}$ is $Lip(\gamma)$ and denote $r\in C^\gamma(\mathcal{V},\mathcal{U})$, if and only if r is $\lfloor\gamma\rfloor$-times Fréchet differentiable ($\lfloor\gamma\rfloor$ denotes the largest integer which is strictly less than γ), and

$$|r|_{Lip(\gamma)}:=\left(\max_{k=0,1,\ldots,\lfloor\gamma\rfloor}\left\|D^k r\right\|_\infty\right)\vee\left\|D^{\lfloor\gamma\rfloor}r\right\|_{(\gamma-\lfloor\gamma\rfloor)-H\ddot{o}l}<\infty,$$

where $\|\cdot\|_\infty$ denotes the uniform norm and $\|\cdot\|_{(\gamma-\lfloor\gamma\rfloor)-H\ddot{o}l}$ denotes the $(\gamma-\lfloor\gamma\rfloor)$-Hölder norm.

Denote $C^0(\mathcal{V},\mathcal{U})$ as the space of bounded measurable mappings from $\mathcal{V}$ to $\mathcal{U}$.

Definition 9 ($L(\mathcal{W},C^\gamma(\mathcal{V},\mathcal{U}))$). Suppose $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are Banach spaces. Denote $L(\mathcal{W},C^\gamma(\mathcal{V},\mathcal{U}))$ as the space of linear mappings from $\mathcal{W}$ to $C^\gamma(\mathcal{V},\mathcal{U})$, and denote

$$|f|_{Lip(\gamma)}:=\sup_{w\in\mathcal{W},\|w\|=1}|f(w)|_{Lip(\gamma)},\ \forall f\in L(\mathcal{W},C^\gamma(\mathcal{V},\mathcal{U})).$$

Before proceeding to the definition of solution of rough differential equation, we define rough integral as in Lyons [13]. The following definition can be found on pages 73, 74 in [13].

Definition 10. Denote the symmetric group of degree N by $Sym(N)$. For integers $k_i\in\{1,2,\ldots,N\}$, $\sum_{i=1}^n k_i=N$, we denote $N_0:=0$, $N_{i+1}:=N_i+k_i$, $i=0,\ldots,n-1$, and define $OS(k_1,\ldots,k_n)$ by the set of $\sigma\in Sym(N)$ which satisfies

$$\text{(4)}\qquad \begin{gathered}\sigma(N_i+1)<\sigma(N_i+2)<\cdots<\sigma(N_{i+1}),\quad i=0,1,\ldots,n-1,\\ \text{and}\quad \sigma(N_1)<\sigma(N_2)<\cdots<\sigma(N_n).\end{gathered}$$

For $f\in L(\mathcal{V},C^\gamma(\mathcal{V},\mathcal{U}))$ and integer $k=1,\ldots,\lfloor\gamma\rfloor+1$, $D^{k-1}f\in L\left(\mathcal{V}^{\otimes k},C^{\gamma+1-k}(\mathcal{V},\mathcal{U})\right)$. For integer $n\geq 1$ and $k_1,\ldots,k_n\in\{1,2,\ldots,\lfloor\gamma\rfloor+1\}$, we denote $N_0:=0$, $N_{i+1}:=N_i+k_{i+1}$, $i=0,\ldots,n-1$, and define

$$\left(D^{k_1-1}f\right)\otimes\cdots\otimes\left(D^{k_n-1}f\right)\in L\left(\mathcal{V}^{\otimes N_n},C^{\gamma+1-\max_{i=1,\ldots,n}k_i}\left(\mathcal{V},\mathcal{U}^{\otimes n}\right)\right)$$

as the unique continuous linear operator, which satisfies, $\forall v_i \in \mathcal{V}$, $i = 1, 2, \ldots, N_n$, $\forall v \in \mathcal{V}$,

$$\begin{aligned} &\left(\left(D^{k_1-1}f\right) \otimes \cdots \otimes \left(D^{k_n-1}f\right)\right)\left(v_1 \otimes \cdots \otimes v_{N_n}\right)(v) \\ = &\left(\left(D^{k_1-1}f\right)\left(v_1 \otimes \cdots \otimes v_{N_1}\right)(v)\right) \otimes \cdots \otimes \\ &\left(\left(D^{k_n-1}f\right)\left(v_{N_{n-1}+1} \otimes \cdots \otimes v_{N_n}\right)(v)\right). \end{aligned}$$

Then based on Def 4.9 in [13], we define rough integral.

Definition 11 (rough integral). Suppose $X : [0,T] \to \left(G^{[p]}(\mathcal{V}), \|\cdot\|\right)$ is a geometric p-rough path, and $f \in L(\mathcal{V}, C^\gamma(\mathcal{V},\mathcal{U}))$ for some $\gamma \in (p-1, [p]]$. Then $Y : [0,T] \to \left(G^{[p]}(\mathcal{V}), \|\cdot\|\right)$ is called the rough integral of f against X, and denoted by $\int f(X)\, dX$, if for any $[s,t] \subseteq [0,T]$ satisfying $\|X\|_{p-var,[s,t]} \leq 1$ and any $n = 1, 2, \ldots, [p]$, we have (with $OS(k_1, \ldots, k_n)$ defined at (4) and $\sigma(v_1 \otimes \cdots \otimes v_N) := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)}$)

$$\begin{aligned} &\Bigg\| \pi_n(Y_{s,t}) - \sum_{k_i \geq 1, k_1+\cdots+k_n \leq [p]} \left(\left(D^{k_1-1}f\right) \otimes \cdots \otimes \left(D^{k_n-1}f\right)\right) \\ &\left(\sum_{\sigma \in OS(k_1,\ldots,k_n)} \sigma^{-1}\pi_{k_1+\cdots+k_n}(X_{s,t})\right)(\pi_1(X_s)) \Bigg\| \\ &\leq C_{p,\gamma} |f|^n_{Lip(\gamma)} \|X\|^{\gamma+1}_{p-var,[s,t]}. \end{aligned}$$

Then based on Thm 4.3, Thm 4.6 and Thm 4.12 in [13], we have

THEOREM 12. *For $p \geq 1$, if f in Definition 11 is in $L(\mathcal{V}, C^\gamma(\mathcal{V},\mathcal{U}))$ for $\gamma > p-1$, then the rough integral $\int f(X)\, dX$ exists uniquely, and is continuous in d_p-metric w.r.t. the driving rough path X.*

We define the solution of rough differential equations as in Def 5.1 in Lyons [13].

Definition 13 (solution of RDE). Suppose that $\mathcal{U}$ and $\mathcal{V}$ are two Banach spaces, $X : [0,T] \to \left(G^{[p]}(\mathcal{V}), \|\cdot\|\right)$ is a geometric p-rough path, $f \in L(\mathcal{V}, C^\gamma(\mathcal{U},\mathcal{U}))$ for some $\gamma > p-1$ and $\xi \in \mathcal{U}$. Define $h : \mathcal{V} \oplus \mathcal{U} \to L(\mathcal{V} \oplus \mathcal{U}, \mathcal{V} \oplus \mathcal{U})$ as

$$h(v_1, u_1)(v_2, u_2) = (v_2, f(v_2)(u_1 + \xi)), \quad \forall v_1, v_2 \in \mathcal{V}, \forall u_1, u_2 \in \mathcal{U}. \tag{5}$$

Then geometric p-rough path $Z : [0,T] \to \left(G^{[p]}(\mathcal{V} \oplus \mathcal{U}), \|\cdot\|\right)$ is said to be a solution to the rough differential equation

$$\mathrm{d}Y = f(Y)\,\mathrm{d}X,\ Y_0 = \xi, \tag{6}$$

if $\pi_{G^{[p]}(\mathcal{V})}(Z) = X$, and Z satisfies the rough integral equation (in sense of Definition 11):

$$Z_t = \int_0^t h(Z_u)\,\mathrm{d}Z_u, \ \ t \in [0,T].$$

For $g \in G^n(\mathcal{V})$ and $\lambda > 0$, we define $\delta_\lambda g \in G^n(\mathcal{V})$ by

$$\delta_\lambda g := 1 + \sum_{k=1}^{n} \lambda^k \pi_k(g). \tag{7}$$

THEOREM 14 (Lyons). *When f in* (6) *is in $L(\mathcal{V}, C^\gamma(\mathcal{U},\mathcal{U}))$ for $\gamma > p$, the solution of* (6) *exists uniquely. Moreover, there exists a constant $C_{p,\gamma}$, which only depends on p and γ, such that, for any interval $[s,t] \subseteq [0,T]$ satisfying $|f|_{Lip(\gamma)} \|X\|_{p-var,[s,t]} \leq 1$, we have (after rescaling f, X and Y)*

$$\left\| \left(\delta_{|f|_{Lip(\gamma)}} X, Y \right) \right\|_{p-var,[s,t]} \leq C_{p,\gamma} |f|_{Lip(\gamma)} \|X\|_{p-var,[s,t]}. \tag{8}$$

Based on the universal limit theorem (Thm 5.3 in [13]), when $|f|_{Lip(\gamma)} = 1$, there exists constant $C_{p,\gamma}$ (which only depends on p and γ), such that for interval $[s,t]$ satisfying $\|X\|_{p-var,[s,t]} \leq 1$, we have

$$\|(X,Y)\|_{p-var,[s,t]} \leq C_{p,\gamma}. \tag{9}$$

When $|f|_{Lip(\gamma)} \neq 1$, we rescale f, X and Y, and can get (8). Indeed, for $\mu > 0$, $\epsilon > 0$, (with δ_λ defined at (7)) we rewrite the rough differential equation

$$\mathrm{d}Y = f(Y)\,\mathrm{d}X \ \text{ as } \ \mathrm{d}(\delta_\epsilon Y) = \mu^{-1} f\left(\epsilon^{-1}(\epsilon Y)\right) \mathrm{d}\delta_{\mu\epsilon} X\ .$$

For interval $[s,t] \subseteq [0,T]$, we select μ and ϵ such that

$$\begin{aligned} \|\delta_{\mu\epsilon} X\|_{p-var,[s,t]} = \mu\epsilon \|X\|_{p-var,[s,t]} = 1,\ & \left|\mu^{-1} f\left(\epsilon^{-1}\cdot\right)\right|_{Lip(\gamma)} \\ & = \mu^{-1}\left(1 \vee \epsilon^{-\gamma}\right) |f|_{Lip(\gamma)} = 1. \end{aligned} \tag{10}$$

Then, based on (9), we have

$$\|(\delta_{\mu\epsilon} X, \delta_\epsilon Y)\|_{p-var,[s,t]} = \epsilon \|(\delta_\mu X, Y)\|_{p-var,[s,t]} \leq C_{p,\gamma}. \tag{11}$$

Thus, by solving (10), and substituting the value of μ and ϵ into (11), we get (8).

Remark 15. The unique solution of (6) is recovered by a sequence of rough integrals. Then based on Thm 4.12 and Prop 5.9 in [13] and by using lower semi-continuity of p-variation, the constant $C_{p,\gamma}$ in (8) is an absolute constant which only depends on p and γ, and can be chosen to be finite whenever $\gamma > p - 1$.

When $\mathcal{U}$ and $\mathcal{V}$ are finite dimensional spaces, for any $f \in L(\mathcal{V}, C^{\gamma}(\mathcal{U},\mathcal{U}))$, $\gamma > p-1$, there exists a solution to (6) which satisfies (8). Indeed, based on Prop 5.9 [13], when f is $Lip(\gamma)$ for $\gamma > p-1$, the sequence of Picard iterations $\{Z^n\}_n : [0,T] \to G^{[p]}(\mathcal{V} \oplus \mathcal{U})$, define recursively as rough integrals: (with h defined at (5))

$$\begin{aligned} Z_t^0 &= (X_t, 0), \ t \in [0,T], \\ Z_t^{n+1} &= \int_0^t h(Z_u^n) \mathrm{d}Z_u^n, \ t \in [0,T], \ n \geq 0, \end{aligned}$$

are uniformly bounded and equi-continuous. When $\mathcal{V}$ and $\mathcal{U}$ are finite-dimensional spaces, bounded sets in $G^{[p]}(\mathcal{V} \oplus \mathcal{U})$ are relatively compact. Thus, based on Arzelà-Ascoli theorem, there exists a subsequence $\{Z^{n_k}\}_k$ which converge uniformly (denoted the limit as Z). Then by spelling out the almost-multiplicative functional (associated with the Picard iteration) and letting k tend to infinity, one can prove that Z is a solution to the rough differential equation (6). Then based on Thm 4.12 and Prop 5.9 in [13] and by using lower semi-continuity of p-variation, the estimate (8) holds for Z.

When $\mathcal{U}$ is a Banach space and when f in (6) is $Lip(\gamma)$ for $\gamma \in (p-1, p)$, there does not always exist a solution to (6). Godunov [10] proved that, "each Banach space in which Peano's theorem is true is finite-dimensional". Shkarin [19] (in Cor 1.5) proved that, for any real infinite dimensional Banach space (denoted as $\mathcal{V}$), which has a complemented subspace with an unconditional Schauder basis, and for any $\alpha \in (0,1)$, there exists α-Hölder continuous function $f : \mathcal{V} \to \mathcal{V}$, such that the equation $\dot{x} = f(x)$ has no solution in any interval of the real line. Based on Shkarin [19] (Rrk 1.4), $L_p[0,1]$ ($1 \leq p < \infty$) and $C[0,1]$ are examples of such Banach spaces, and "roughly speaking, all infinite dimensional Banach spaces, which naturally appear in analysis" fall into this category.

2.3. **Differential Operator**

For $\gamma \geq 0$, recall $C^{\gamma}(\mathcal{U},\mathcal{U})$ in Definition 8, and that $L(\mathcal{V}, C^{\gamma}(\mathcal{U},\mathcal{U}))$ denotes the space of linear mappings from $\mathcal{V}$ to $C^{\gamma}(\mathcal{U},\mathcal{U})$. For $f \in L(\mathcal{V}, C^{\gamma}(\mathcal{U},\mathcal{U}))$ and integer $k \leq \gamma + 1$, we clarify in this subsection the meaning of differential operator $f^{\circ k}(v)$ for $v \in \mathcal{V}^{\otimes k}$. For $r \in C^k(\mathcal{U},\mathcal{U})$ and $j = 0, 1, \ldots, k$, $D^j r \in L\left(\mathcal{U}^{\otimes j}, C^{k-j}(\mathcal{U},\mathcal{U})\right)$.

Notation 16 ($\mathcal{D}^k(\mathcal{U})$). For integer $k \geq 0$, denote with $\mathcal{D}^k(\mathcal{U})$ the set of bounded kth order differential operators (on $C^k(\mathcal{U},\mathcal{U})$). More specifically, $p \in \mathcal{D}^k(\mathcal{U})$ if and only if $p : C^k(\mathcal{U},\mathcal{U}) \to C^0(\mathcal{U},\mathcal{U})$ and there exist bounded

$p_j : \mathcal{U} \to \mathcal{U}^{\otimes j}$, $j = 0, 1, \ldots, k$, with $p_k \not\equiv 0$, such that

$$p(r)(u) = \sum_{j=0}^{k} \left(D^j r\right)(p_j(u))(u), \ \forall u \in \mathcal{U}, \ \forall r \in C^k(\mathcal{U},\mathcal{U}).$$

We define the following norm $|\cdot|_k$ on $\mathcal{D}^k(\mathcal{U})$ as

$$|p|_k := \max_{j=0,1,\ldots,k} \sup_{u\in\mathcal{U}} \|p_j(u)\|, \ \forall p \in \mathcal{D}^k(\mathcal{U}).$$

Then $\mathcal{D}^k(\mathcal{U})$ can be extended to a Banach space $\left(\mathcal{D}^k(\mathcal{U}), |\cdot|_k\right)$ (with the natural addition and scalar multiplication).

Definition 17 (composition). Suppose $p^1 \in \mathcal{D}^{j_1}(\mathcal{U})$ and $p^2 \in \mathcal{D}^{j_2}(\mathcal{U})$ for integers $j_1 \geq 0$, $j_2 \geq 0$. Then when the components of p^2 are j_1-times differentiable, we define the composition of $p^1 \circ p^2 \in \mathcal{D}^{j_1+j_2}(\mathcal{U})$ as

$$\left(p^1 \circ p^2\right)(r) := p^1\left(p^2(r)\right), \ \forall r \in C^{j_1+j_2}(\mathcal{U},\mathcal{U}).$$

For $p \in \mathcal{D}^j(\mathcal{U})$, $j \geq 0$, when the components of p satisfy the required smoothness condition, we define the differential operator $p^{\circ k} \in \mathcal{D}^{k\times j}(\mathcal{U})$ for integer $k \geq 1$ by

$$p^{\circ 1} := p \ \text{ and } \ p^{\circ k} := p \circ p^{\circ(k-1)}, \ k \geq 2. \tag{12}$$

Composition of differential operators is associative, *i.e.* $\left(p^1 \circ p^2\right) \circ p^3 = p^1 \circ \left(p^2 \circ p^3\right)$.

Definition 18 ($f^{\circ k}$). Suppose $f \in L(\mathcal{V}, C^\gamma(\mathcal{U},\mathcal{U}))$ for some $\gamma \geq 0$. Then for any $v \in \mathcal{V}$, we treat $f(v)$ as a first order differential operator (*i.e.* in $\mathcal{D}^1(\mathcal{U})$), and define

$$(f(v))(r)(u) := (Dr)(f(v)(u))(u), \ \forall u \in \mathcal{U}, \ \forall r \in C^1(\mathcal{U},\mathcal{U}).$$

For integer $k \in 1, 2, \ldots, [\gamma]+1$ and $\{v_j\}_{j=1}^k \subset \mathcal{V}$, we define $f^{\circ k}(v_1 \otimes \cdots \otimes v_k) \in \mathcal{D}^k(\mathcal{U})$ as

$$f^{\circ k}(v_1 \otimes \cdots \otimes v_k) := (f(v_1)) \circ (f(v_2)) \circ \cdots \circ (f(v_k)). \tag{13}$$

Then we denote with $f^{\circ k} \in L\left(\mathcal{V}^{\otimes k}, \left(\mathcal{D}^k(\mathcal{U}), |\cdot|_k\right)\right)$ the unique continuous linear operator satisfying (13).

3. MAIN RESULT

We work with the first level (or “path” level) solution of rough differential equations.

Firstly, we prove a lemma for ordinary differential equations. Then after applying universal limit theorem (Thm 5.3 [13]), this lemma leads to similar estimates of rough differential equations. The proof of this lemma is in the same spirit as Lemma 2.4(a) in [6], Lemma 16 in [7] and Lemma 10.7 in [8], only that we use the ordinary differential equation (16) to compute the truncated exponential of the original flow.

Let $\mathcal{U}$ and $\mathcal{V}$ be two Banach spaces. For $p \geq 1$, denote $[p]$ the integer part of p. For $\gamma > 0$, denote by $\lfloor \gamma \rfloor$ the largest integer which is strictly less than γ. Denote $I_d : \mathcal{U} \to \mathcal{U}$ as the identity function, *i.e.* $I_d(u) = u$, $\forall u \in \mathcal{U}$. For $f \in L(\mathcal{V}, C^\gamma(\mathcal{U},\mathcal{U}))$, integer $k \leq \gamma + 1$ and $v \in \mathcal{V}^{\otimes k}$, recall the differential operator $f^{\circ k}(v)$ in Definition 18.

Lemma 19. *Suppose $x : [0,T] \to \mathcal{V}$ is a continuous bounded variation path, $f \in L(\mathcal{V}, C^\gamma(\mathcal{U},\mathcal{U}))$ for $\gamma > 1$, and $\xi \in \mathcal{U}$. Let $y : [0,T] \to \mathcal{U}$ be the unique solution to the ordinary differential equation*

$$\mathrm{d}y = f(y)\,\mathrm{d}x, \quad y_0 = \xi \in \mathcal{U}. \tag{14}$$

Then for any $p \in [1, \gamma+1)$, there exists a constant $C_{p,\gamma}$, which only depends on p and γ, such that, for any $0 \leq s < t \leq T$,

$$\begin{aligned}(1),\ & \left\| y_t - y_1^{s,t} \right\| \leq C_{p,\gamma} |f|_{Lip(\gamma)}^{\gamma+1} \left\| S_{[p]}(x) \right\|_{p-var,[s,t]}^{\gamma+1}, \\ (2),\ & \left\| y_t - y_s - \sum_{k=1}^{\lfloor\gamma\rfloor+1} f^{\circ k} \pi_k \left(S_{\lfloor\gamma\rfloor+1}(x)_{s,t} \right)(I_d)(y_s) \right\| \\ & \qquad \leq C_{p,\gamma} |f|_{Lip(\gamma)}^{\gamma+1} \left\| S_{[p]}(x) \right\|_{p-var,[s,t]}^{\gamma+1},\end{aligned} \tag{15}$$

where $y^{s,t} : [0,1] \to \mathcal{U}$ in (15) is the unique solution of the ordinary differential equation (with y_s denotes the value of y in (14) at point s):

$$\begin{aligned}\mathrm{d}y_u^{s,t} &= \left(\sum_{k=1}^{\lfloor\gamma\rfloor} f^{\circ k} \pi_k \Big(\log_{\lfloor\gamma\rfloor+1} \Big(S_{\lfloor\gamma\rfloor+1}(x)_{s,t} \Big) \Big)(I_d)\left(y_u^{s,t}\right) \right) \mathrm{d}u,\ u \in [0,1], \\ y_0^{s,t} &= y_s + f^{\circ(\lfloor\gamma\rfloor+1)} \pi_{\lfloor\gamma\rfloor+1} \left(\log_{\lfloor\gamma\rfloor+1} \left(S_{\lfloor\gamma\rfloor+1}(x)_{s,t} \right) \right)(I_d)(y_s).\end{aligned} \tag{16}$$

The proof of Lemma 19 starts from page 23. Since $f \in L(\mathcal{V}, C^\gamma(\mathcal{U},\mathcal{U}))$, it might be more appropriate to write (14) as $\mathrm{d}y = f(\mathrm{d}x)(y)$. We keep it in the current form so that it is consistent with the classical notation of ordinary differential equations.

Remark 20. Consider the ordinary differential equation:

$$\begin{aligned}\mathrm{d}\widetilde{y}_u^{s,t} &= \left(\sum_{k=1}^{\lfloor\gamma\rfloor+1} f^{\circ k} \pi_k \left(\log_{\lfloor\gamma\rfloor+1} \left(S_{\lfloor\gamma\rfloor+1}(x)_{s,t} \right) \right)(I_d)\left(\widetilde{y}_u^{s,t}\right) \right) \mathrm{d}u,\ u \in [0,1], \\ \widetilde{y}_0^{s,t} &= y_s.\end{aligned} \tag{17}$$

(The summation of k in (17) is from 1 to $\lfloor\gamma\rfloor+1$.) In Lemma 19, we used (16) instead of (17), because based on Cor 1.5 in Shkarin [19], (17) may not have a solution. If (17) has a solution (e.g. when $\mathcal{U}$ is finite dimensional), then (15) holds with $y_1^{s,t}$ replaced by $\widetilde{y}_1^{s,t}$.

Remark 21. When $\mathcal{V}$ is $\mathbb{R}^d$ and $\mathcal{U}$ is $\mathbb{R}^e$ in Lemma 19, suppose $f=\left(f^1,\ldots,f^d\right)\in L\left(\mathbb{R}^d,C^\gamma\left(\mathbb{R}^e,\mathbb{R}^e\right)\right)$. For $i=1,\ldots,d$, we treat $f^i=\left(f_1^i,\ldots,f_e^i\right)$ as a first order differential operator: $\sum_{j=1}^e f_j^i\frac{\partial}{\partial y_j}$. Then it can be checked that (with $x=\left(x^1,\ldots,x^d\right):[0,T]\to\mathbb{R}^d$)

$$
\begin{aligned}
&f^{\circ k}\pi_k\left(S_{\lfloor\gamma\rfloor+1}\left(x\right)_{s,t}\right)\left(I_d\right)\\
&\quad=\sum_{i_1,\ldots,i_k\in\{1,\ldots,d\}}\left(f^{i_1}\circ\cdots\circ f^{i_k}\right)\left(I_d\right)\int\cdots\int_{s<u_1<\cdots<u_k<t}\mathrm{d}x_{u_1}^{i_1}\cdots\mathrm{d}x_{u_k}^{i_k},
\end{aligned}
$$

and our formulation coincides with [6, 7] and [8].

The theorem below follows from the universal limit theorem (Thm 5.3 [13]) and Lemma 19. Suppose $X:[0,T]\to G^{[p]}\left(\mathcal{V}\right)$ is a geometric p-rough path. For integer $n\geq[p]$, we denote with $S_n\left(X\right)$ the unique enhancement of X to a continuous path with finite p-variation taking values in $G^n\left(\mathcal{V}\right)$ (Thm 3.7 [13]).

THEOREM 22. *Suppose $f\in L\left(\mathcal{V},C^\gamma\left(\mathcal{U},\mathcal{U}\right)\right)$ for $\gamma>1$, $X:[0,T]\to G^{[p]}\left(\mathcal{V}\right)$ is a geometric p-rough path for some $p\in[1,\gamma)$, and $\xi\in\mathcal{U}$. Denote by Z the unique solution (in the sense of Definition 13) of the rough differential equation*

$$
\mathrm{d}Y=f\left(Y\right)\mathrm{d}X,\quad Y_0=\xi. \tag{18}
$$

Denote $Y:=\pi_{G^{[p]}(\mathcal{U})}\left(Z\right)$. Then there exists a constant $C_{p,\gamma}$, which only depends on p and γ, such that, for any $0\leq s\leq t\leq T$,

$$
(1)\,,\left\|\pi_1\left(Y_t\right)-y_1^{s,t}\right\|\leq C_{p,\gamma}\left|f\right|_{Lip(\gamma)}^{\gamma+1}\left\|X\right\|_{p-var,[s,t]}^{\gamma+1},
$$

$$
\begin{aligned}
(2)\,,&\left\|\pi_1\left(Y_{s,t}\right)-\sum_{k=1}^{\lfloor\gamma\rfloor+1}f^{\circ k}\pi_k\left(S_{\lfloor\gamma\rfloor+1}\left(X\right)_{s,t}\right)\left(I_d\right)\left(\pi_1\left(Y_s\right)\right)\right\|\\
&\qquad\leq C_{p,\gamma}\left|f\right|_{Lip(\gamma)}^{\gamma+1}\left\|X\right\|_{p-var,[s,t]}^{\gamma+1},
\end{aligned}
$$

where $y^{s,t}:[0,1]\to\mathcal{U}$ is the unique solution of the ordinary differential equation:

$$
\begin{aligned}
\mathrm{d}y_u^{s,t}&=\left(\sum_{k=1}^{\lfloor\gamma\rfloor}f^{\circ k}\pi_k\left(\log_{\lfloor\gamma\rfloor+1}\left(S_{\lfloor\gamma\rfloor+1}\left(X\right)_{s,t}\right)\right)\left(I_d\right)\left(y_u^{s,t}\right)\right)\mathrm{d}u,\ u\in[0,1],\\
y_0^{s,t}&=\pi_1\left(Y_s\right)+f^{\circ(\lfloor\gamma\rfloor+1)}\pi_{\lfloor\gamma\rfloor+1}\left(\log_{\lfloor\gamma\rfloor+1}\left(S_{\lfloor\gamma\rfloor+1}\left(X\right)_{s,t}\right)\right)\left(I_d\right)\left(\pi_1\left(Y_s\right)\right).
\end{aligned}
$$

The proof of Theorem 22 is on page 27.

Remark 23. Suppose $f \in L(\mathcal{V}, C^{\gamma}(\mathcal{U},\mathcal{U}))$ for $\gamma > 1$, $X : [0,T] \to G^{[p]}(\mathcal{V})$ is a weak geometric p-rough path[1] for some $p \in [1, \gamma+1)$, and $\xi \in \mathcal{U}$. Then by following similar arguments as in the proof of Theorem 22, one can prove that, any solution, in the sense of Def 10.17 in [8], to the rough differential equation

$$\mathrm{d}y = f(y)\,\mathrm{d}X,\ y_0 = \xi,$$

satisfies the estimates in Theorem 22.

THEOREM 24. *Suppose $f \in L(\mathcal{V}, C^{\gamma}(\mathcal{V},\mathcal{U}))$ for $\gamma > 1$, and X is a geometric p-rough path for some $p \in [1, \gamma+1)$. Let $Y : [0,T] \to G^{[p]}(\mathcal{U})$ denote the rough integral (in the sense of Definition 11):*

$$Y_t = \int_0^t f(X)\,\mathrm{d}X,\ t \in [0,T].$$

Then there exists constant $C_{p,\gamma}$, which only depends on p and γ and is finite whenever $\gamma > p-1$, such that, for any $0 \le s \le t \le T$,

$$(1),\left\|\pi_1(Y_t) - y_1^{s,t}\right\| \le C_{p,\gamma}|f|_{Lip(\gamma)}\|X\|_{p-var,[s,t]}^{\gamma+1}, \tag{19}$$

$$(2),\left\|\pi_1(Y_{s,t}) - \sum_{k=1}^{\lfloor\gamma\rfloor+1}\left(D^{k-1}f\right)\pi_k\left(S_{\lfloor\gamma\rfloor+1}(X)_{s,t}\right)(\pi_1(Y_s))\right\| \le C_{p,\gamma}|f|_{Lip(\gamma)}\|X\|_{p-var,[s,t]}^{\gamma+1}, \tag{20}$$

where $y^{s,t} : [0,1] \to \mathcal{U}$ satisfies the ordinary differential equation:

$$\mathrm{d}y_u^{s,t} = \sum_{k=1}^{\lfloor\gamma\rfloor}\left(D^{k-1}f\right)\pi_k\left(\log_{\lfloor\gamma\rfloor+1}\left(S_{\lfloor\gamma\rfloor+1}(X)_{s,t}\right)\right)(\pi_1(X_s) + u(\pi_1(X_{s,t})))\,\mathrm{d}u,$$

$$y_0^{s,t} = \pi_1(Y_s) + \left(D^{\lfloor\gamma\rfloor}f\right)\pi_{\lfloor\gamma\rfloor+1}\left(\log_{\lfloor\gamma\rfloor+1}\left(S_{\lfloor\gamma\rfloor+1}(X)_{s,t}\right)\right)(\pi_1(X_s)).$$

Actually, (20) is part of the definition of rough integral as seen in Definition 11 (on page 6), and (19) can be obtained by combining (20) and Lemma 26 (on page 15).

Alternatively, Theorem 24 can be proved similarly as Theorem 22, the only difference is that we consider the rough differential equation

$$\mathrm{d}\left(X, \delta_{|f|_{Lip(\gamma)}^{-1}}Y\right) = \left(1_{\mathcal{V}}, \frac{f}{|f|_{Lip(\gamma)}}(X)\right)\mathrm{d}X,\quad (X,Y)_0 = (X_0, 0) \in \mathcal{V}\oplus\mathcal{U},$$

and uses Thm 4.12 [13] instead of the universal limit theorem.

[1] A continuous path $X : [0,T] \to G^{[p]}(\mathcal{V})$ is called a weak geometric p-rough path, if $\|X\|_{p-var,[0,T]} < \infty$.

4. PROOF

We made explicit the dependence of constants (*e.g.* $C_{p,\gamma}$), but the exact value of constants may change from line to line. Recall $C^{\gamma}(\mathcal{U},\mathcal{U})$ in Definition 8 on page 5.

LEMMA 25. *Suppose $k\geq 1$ is an integer, and $f\in L(\mathcal{V},C^{k-1}(\mathcal{U},\mathcal{U}))$. Recall $f^{\circ k}\in L(\mathcal{V}^{\otimes k},(\mathcal{D}^k(\mathcal{U}),|\cdot|_k))$ defined in Definition 18 (on page 9). Then for any $v\in[\mathcal{V}]^k$ (defined in Definition 2 on page 3), $f^{\circ k}(v)$ is a first order differential operator, which satisfies (with $I_d:\mathcal{U}\to\mathcal{U}$ denotes the identity function)*

$$f^{\circ k}(v)(r)=(Dr)\left(f^{\circ k}(v)(I_d)\right),\quad \forall r\in C^1(\mathcal{U},\mathcal{U}).$$

Proof. To prove that $f^{\circ k}(v)$ is a first order differential operator, we define another first order differential operator, and will prove that they coincide.

Suppose $\mathcal{V}^1$ and $\mathcal{V}^2$ are two Banach spaces, and $p^i\in L(\mathcal{V}^i,(\mathcal{D}^{k_i}(\mathcal{U}),|\cdot|_{k_i}))$, $i=1,2$. Define $[p^1,p^2]\in L(\mathcal{V}^1\otimes\mathcal{V}^2,(\mathcal{D}^1(\mathcal{U}),|\cdot|_1))$ as the unique continuous linear operator, which satisfies that, for any $v^1\in\mathcal{V}^1$, any $v^2\in\mathcal{V}^2$, and any $r\in C^1(\mathcal{U},\mathcal{U})$,

$$[p^1,p^2](v^1\otimes v^2)(r)=(Dr)\left(p^1(v^1)\circ p^2(v^2)-p^2(v^2)\circ p^1(v^1)\right)(I_d). \tag{21}$$

For integer $k\geq 1$ and $f\in L(\mathcal{V},C^{k-1}(\mathcal{U},\mathcal{U}))$, define $[f]^{\circ k}\in L(\mathcal{V}^{\otimes k},(\mathcal{D}^1(\mathcal{U}),|\cdot|_1))$ as (with $f^{\circ 1}$ defined in Definition 18)

$$[f]^{\circ 1}(v):=f^{\circ 1}(v),\ \forall v\in\mathcal{V},\text{ and }[f]^{\circ k}:=\left[f^{\circ 1},[f]^{\circ(k-1)}\right]\text{ for }k\geq 2. \tag{22}$$

Then by definition, for any $k\geq 1$ and any $v\in\mathcal{V}^{\otimes k}$, $[f]^{\circ k}(v)$ is a first order differential operator.

If we can prove that $f^{\circ k}(v^k)$ is a first order differential operator for any v^k in the form

$$v^k=\begin{cases} v_1, & \text{if } k=1\\ [v_1,\dots,[v_{k-1},v_k]], & \text{if } k\geq 2\end{cases},\quad\text{with } \{v_j\}_{j=1}^k\subset\mathcal{V}, \tag{23}$$

then since any $v\in[\mathcal{V}]^k$ can be approximated by linear combinations of v^k in the form of (23), by using that $f^{\circ k}:\mathcal{V}^{\otimes k}\to(\mathcal{D}^k(\mathcal{U}),|\cdot|_k)$ is linear and continuous (Definition 18), and that $[\mathcal{V}]^k$ is a closed subspace of $\mathcal{V}^{\otimes k}$, we can prove that $f^{\circ k}(v)$ is a first order differential operator for any $v\in[\mathcal{V}]^k$.

We define the linear map $\sigma:[\mathcal{V}]^k\to\mathcal{V}^{\otimes k}$ by assigning

$$\begin{aligned}\sigma(v_1)\ &:=v_1,\ \forall v_1\in\mathcal{V},\text{ if }k=1,\\ \sigma([v_1,\dots,[v_{k-1},v_k]])\ &:=v_1\otimes\cdots\otimes v_{k-1}\otimes v_k,\ \forall\{v_j\}_{j=1}^k\subset\mathcal{V},\ \text{ if }k\geq 2.\end{aligned} \tag{24}$$

For any v^k in the form of (23) with σ defined at (24), we want to prove

$$f^{\circ k}\left(v^k\right)(r) = [f]^{\circ k}\left(\sigma\left(v^k\right)\right)(r), \quad \forall r \in C^1(\mathcal{U},\mathcal{U}). \tag{25}$$

By definition, $[f]^{\circ k}\left(\sigma\left(v^k\right)\right)$ is a first order differential operator, and

$$[f]^{\circ k}\left(\sigma\left(v^k\right)\right)(r) = (Dr)\left([f]^{\circ k}\left(\sigma\left(v^k\right)\right)(I_d)\right), \quad \forall r \in C^1(\mathcal{U},\mathcal{U}).$$

If we can prove (25), then

$$f^{\circ k}\left(v^k\right)(r) = (Dr)\left([f]^{\circ k}\left(\sigma\left(v^k\right)\right)(I_d)\right) = (Dr)\left(f^{\circ k}\left(v^k\right)(I_d)\right), \quad \forall r \in C^1(\mathcal{U},\mathcal{U}).$$

Thus, in the following, we concentrate on proving (25).

It is clear that, (25) is true when $k = 1$. Indeed, for any $v^1 \in \mathcal{V}$, since $[f]^{\circ 1}\left(v^1\right) := f^{\circ 1}\left(v^1\right)$ (see (22)) and $\sigma\left(v^1\right) := v^1$ (see (24)), we have

$$[f]^{\circ 1}\left(v^1\right) = f^{\circ 1}\left(v^1\right) = f^{\circ 1}\left(\sigma\left(v^1\right)\right), \forall v^1 \in \mathcal{V}.$$

Then we prove (25) by using mathematical induction. Suppose that for integer $K \geq 1$, $k = 1, 2, \ldots, K$ and any v^k in the form of (23), we have

$$f^{\circ k}\left(v^k\right) = [f]^{\circ k}\left(\sigma\left(v^k\right)\right). \tag{26}$$

We want to prove that, for any $v_0 \in \mathcal{V}$, and any v^K in the form of (23),

$$f^{\circ(K+1)}\left(\left[v_0, v^K\right]\right) = [f]^{\circ(K+1)}\left(v_0 \otimes \sigma\left(v^K\right)\right).$$

Based on the definitions in (21) and (22), we have, for any $r \in C^1(\mathcal{U},\mathcal{U})$,

$$\begin{aligned} [f]^{\circ(K+1)}\left(v_0 \otimes \sigma\left(v^K\right)\right)(r) &= \left[f^{\circ 1}(v_0), [f]^{\circ K}\left(\sigma\left(v^K\right)\right)\right](r) \\ &= (Dr)\left(f^{\circ 1}(v_0) \circ [f]^{\circ K}\left(\sigma\left(v^K\right)\right) - [f]^{\circ K}\left(\sigma\left(v^K\right)\right) \circ f^{\circ 1}(v_0)\right)(I_d). \end{aligned} \tag{27}$$

If we assume in addition that $r \in C^2(\mathcal{U},\mathcal{U})$, then by using that $f^{\circ 1}(v_0)$ and $[f]^{\circ K}\left(\sigma\left(v^K\right)\right)$ are first order differential operators, we have

$$\begin{aligned} &(Dr)\left(f^{\circ 1}(v_0) \circ [f]^{\circ K}\left(\sigma\left(v^K\right)\right)\right)(I_d) \\ &= \left(f^{\circ 1}(v_0) \circ [f]^{\circ K}\left(\sigma\left(v^K\right)\right)\right)(r) - \left(D^2 r\right)\left([f]^{\circ K}\left(\sigma\left(v^K\right)\right)(I_d)\right)\left(f^{\circ 1}(v_0)(I_d)\right), \end{aligned} \tag{28}$$

and

$$\begin{aligned} &(Dr)\left([f]^{\circ K}\left(\sigma\left(v^K\right)\right) \circ f^{\circ 1}(v_0)\right)(I_d) \\ &= \left([f]^{\circ K}\left(\sigma\left(v^K\right)\right) \circ f^{\circ 1}(v_0)\right)(r) - \left(D^2 r\right)\left(f^{\circ 1}(v_0)(I_d)\right)\left([f]^{\circ K}\left(\sigma\left(v^K\right)\right)(I_d)\right). \end{aligned} \tag{29}$$

Using inductive hypothesis (26) and the definition of $f^{\circ(K+1)}$ in Definition 18, we have

$$f^{\circ 1}(v_0) \circ [f]^{\circ K}\left(\sigma\left(v^K\right)\right) - [f]^{\circ K}\left(\sigma\left(v^K\right)\right) \circ f^{\circ 1}(v_0) \tag{30}$$

$$= f^{\circ 1}(v_0) \circ f^{\circ K}(v^K) - f^{\circ K}(v^K) \circ f^{\circ 1}(v_0) = f^{\circ (K+1)}(v_0 \otimes v^K - v^K \otimes v_0)$$
$$= f^{\circ (K+1)}([v_0, v^K]).$$

Since our differentiability is in Fréchet's sense and $r \in C^2(\mathcal{U},\mathcal{U})$, we have

$$(D^2 r)(u_1 \otimes u_2) = (D^2 r)(u_2 \otimes u_1), \forall u_1 \in \mathcal{U}, \forall u_2 \in \mathcal{U}. \tag{31}$$

Thus, combining (27), (28), (29), (30) and (31), we have,

$$[f]^{\circ (K+1)}(v_0 \otimes \sigma(v^K))(r) = f^{\circ (K+1)}([v_0, v^K])(r), \ \ \forall r \in C^2(\mathcal{U},\mathcal{U}). \tag{32}$$

Since $[f]^{\circ (K+1)}(v_0 \otimes \sigma(v^K)) := \left[f^{\circ 1}(v_0), [f]^{\circ K}(\sigma(v^K))\right]$ has the explicit form (21), by comparing the "coefficients" of $\{D^j r\}_{j=0}^{K+1}$ for any $r \in C^{K+1}(\mathcal{U},\mathcal{U})$, we get that (32) holds for any $r \in C^1(\mathcal{U},\mathcal{U})$. □

LEMMA 26. *Suppose $f \in L(\mathcal{V}, C^\gamma(\mathcal{U},\mathcal{U}))$ for some $\gamma > 1$ and $\xi \in \mathcal{U}$. Denote $\lfloor\gamma\rfloor$ as the largest integer which is strictly less than γ. Suppose $g \in G^{\lfloor\gamma\rfloor+1}(\mathcal{V})$. Then, there exists a constant C_γ, which only depends on γ, such that, the unique solution of the ordinary differential equation*

$$\begin{aligned} \mathrm{d}y_u &= \sum_{k=1}^{\lfloor\gamma\rfloor} f^{\circ k}\pi_k\left(\log_{\lfloor\gamma\rfloor+1} g\right)(I_d)(y_u)\,\mathrm{d}u, \ u \in [0,1], \\ y_0 &= \xi + f^{\circ(\lfloor\gamma\rfloor+1)}\pi_{\lfloor\gamma\rfloor+1}\left(\log_{\lfloor\gamma\rfloor+1}(g)\right)(I_d)(\xi), \end{aligned} \tag{33}$$

satisfies

$$\left\| y_1 - \xi - \sum_{k=1}^{\lfloor\gamma\rfloor+1} f^{\circ k}\pi_k(g)(I_d)(\xi) \right\| \le C_\gamma \left(|f|_{Lip(\gamma)} \|g\|\right)^{\gamma+1}.$$

Proof. Let $\{\gamma\} := \gamma - \lfloor\gamma\rfloor$, and $N := \lfloor\gamma\rfloor + 1$. We assume $|f|_{Lip(\gamma)} = 1$. Otherwise, we replace f by $|f|_{Lip(\gamma)}^{-1} f$ and replace g by $\delta_{|f|_{Lip(\gamma)}} g$ (with $\delta_\lambda g := 1 + \sum_{k=1}^N \lambda^k \pi_k(g)$).

For integer $k = 1, 2, \ldots, \lfloor\gamma\rfloor + 1$ and $v \in \mathcal{V}^{\otimes k}$, based on Definition of differential operator $f^{\circ k}(v)$ in Definition 18, $f^{\circ k}(v)(I_d) \in C^{\gamma-k+1}(\mathcal{U},\mathcal{U})$ and (by using $|f|_{Lip(\gamma)} = 1$)

$$\left| f^{\circ k}(v)(I_d) \right|_{Lip(\gamma-k+1)} = \|v\| \left| f^{\circ k}\left(\frac{v}{\|v\|}\right)(I_d) \right|_{Lip(\gamma-k+1)} \le C_\gamma \|v\|. \tag{34}$$

When $\|g\| > 1$, it can be computed that ($|f|_{Lip(\gamma)} = 1$)

$$\left\| \sum_{k=1}^{N-1} f^{\circ k}(I_d)(\xi)\,\pi_k(g) \right\| \le C_\gamma \|g\|^{N-1}$$

$$\text{and } \|y_1 - \xi\| \leq \|y_0 - \xi\| + \|y\|_{1-var,[0,1]} \leq C_\gamma \|g\|^N .$$

Thus, ($\|g\| > 1$, $N \leq \gamma + 1$)

$$\left\| y_1 - \xi - \sum_{k=1}^{N} f^{\circ k} (I_d) (\xi) \pi_k (g) \right\| \leq C_\gamma \|g\|^N \leq C_\gamma \|g\|^{\gamma+1} ,$$

and lemma holds. In the following, we assume $|f|_{Lip(\gamma)} = 1$ and $\|g\| \leq 1$.

It is clear that, the solution y of the ordinary differential equation (33) satisfies (since $|f|_{Lip(\gamma)} = 1$ and $\|g\| \leq 1$)

$$\sup_{u\in[0,1]} \|y_u - \xi\| \leq \|y_0 - \xi\| + \|y\|_{1-var,[0,1]} \leq C_\gamma \|g\| . \tag{35}$$

For $k = 1, \ldots, N$, denote the differential operator F^k by

$$F^k := f^{\circ k} \pi_k (\log_N (g)) .$$

Based on Lemma 25, $\left\{F^k\right\}_{k=1}^{N}$ are first order differential operators, and satisfy

$$F^k (r) = (Dr) F^k (I_d) , \quad \forall r \in C^1 (\mathcal{U}, \mathcal{U}) .$$

Similarly to (34), since we assumed $|f|_{Lip(\gamma)} = 1$, for $k = 1, 2, \ldots, N - 1$, we have

$$\left| DF^k (I_d) \right|_{Lip(\gamma-k)} \leq C_\gamma \|g\|^k ; \tag{36}$$

for $k_i \geq 1$, $\sum_{i=1}^{j} k_i = k \leq N$, we have

$$\left| \left(F^{k_j} \circ \cdots \circ F^{k_1} \right) (I_d) \right|_{Lip(\gamma+1-k)} \leq C_\gamma \|g\|^k . \tag{37}$$

By using the fact that y satisfies (33), we have

$$\begin{aligned} & y_1 - \xi - f^{\circ N} \pi_N (\log_N (g)) (I_d) (\xi) - \sum_{k=1}^{N-1} F^k (I_d) (\xi) \\ & = \sum_{k=1}^{N-1} \left(F^k (I_d) (y_0) - F^k (I_d) (\xi) \right) \\ & + \sum_{1\leq k_i \leq N-1, i=1,2} \iint_{0\leq u_1 \leq u_2 \leq 1} DF^{k_1} (I_d) F^{k_2} (I_d) (y_{u_1}) \, \mathrm{d}u_1 \mathrm{d}u_2. \end{aligned} \tag{38}$$

Since $y_0 = \xi + f^{\circ N} \pi_N (\log_N g) (I_d) (\xi)$, by using (37), we have ($|f|_{Lip(\gamma)} = 1$, $\|g\| \leq 1$ and $\gamma \leq N$)

$$\sum_{k=1}^{N-1} \left\| F^k (I_d) (y_0) - F^k (I_d) (\xi) \right\| \leq C_\gamma \|g\|^{1+N} \leq C_\gamma \|g\|^{1+\gamma} . \tag{39}$$

When $k_1 \geq 1$, $k_2 \geq 1$, $k_1 + k_2 \leq N$, using that F^{k_2} is a first order differential operator, we have $DF^{k_1}(I_d)\,F^{k_2}(I_d) = \left(F^{k_2} \circ F^{k_1}\right)(I_d)$. Thus,

$$\iint_{0\leq u_1\leq u_2\leq 1} DF^{k_1}(I_d)\,F^{k_2}(I_d)\,(y_{u_1})\,\mathrm{d}u_1\mathrm{d}u_2 = \iint_{0\leq u_1\leq u_2\leq 1} \left(F^{k_2} \circ F^{k_1}\right)(I_d)\,(y_{u_1})\,\mathrm{d}u_1\mathrm{d}u_2. \tag{40}$$

When $1 \leq k_1 \leq N-1$, $1 \leq k_2 \leq N-1$, $k_1 + k_2 \geq N+1$, by combining (36) and (37), we get

$$\left\| \iint_{0\leq u_1\leq u_2\leq 1} DF^{k_2}(I_d)\,F^{k_1}(I_d)\,(y_{u_1})\,\mathrm{d}u_1\mathrm{d}u_2 \right\| \leq C_\gamma \|g\|^{k_1+k_2} \leq C_\gamma \|g\|^{N+1} \leq C_\gamma \|g\|^{\gamma+1}. \tag{41}$$

Therefore, by combining (38), (39), (40) and (41), we get

$$\left\| y_1 - \xi - \sum_{k=1}^{N} F^k(I_d)(\xi) - \sum_{1\leq k_i\leq N-1, k_1+k_2\leq N} \iint_{0\leq u_1\leq u_2\leq 1} \left(F^{k_2}\circ F^{k_1}\right)(I_d)(y_{u_1})\mathrm{d}u_1\mathrm{d}u_2 \right\| \leq C_\gamma \|g\|^{\gamma+1}.$$

Then we continue to estimate

$$\sum_{1\leq k_i\leq N-1, k_1+k_2\leq N} \iint_{0\leq u_1\leq u_2\leq 1} \left(F^{k_2} \circ F^{k_1}\right)(I_d)\,(y_{u_1})\,\mathrm{d}u_1\mathrm{d}u_2.$$

When $1 \leq k_1 \leq N-1$, $1 \leq k_2 \leq N-1$, $k_1 + k_2 = N$, by using (37) and (35), we have

$$\left\| \iint_{0\leq u_1\leq u_2\leq 1} \left(\left(F^{k_2} \circ F^{k_1}\right)(I_d)\,(y_{u_1}) - \left(F^{k_2} \circ F^{k_1}\right)(I_d)\,(\xi) \right) \mathrm{d}u_1\mathrm{d}u_2 \right\| \leq C_\gamma \|g\|^{N} \sup_{u\in[0,1]} \|y_u - \xi\|^{\{\gamma\}} \leq C_\gamma \|g\|^{\gamma+1}.$$

When $k_1 \geq 1$, $k_2 \geq 1$ and $k_1 + k_2 \leq N-1$, we have

$$\begin{aligned}
&\sum_{k_i\geq 1, k_1+k_2\leq N-1} \iint_{0\leq u_1\leq u_2\leq 1} \left(\left(F^{k_2}\circ F^{k_1}\right)(I_d)(y_{u_1}) - \left(F^{k_2}\circ F^{k_1}\right)(I_d)(\xi) \right) \mathrm{d}u_1\mathrm{d}u_2 \\
&\quad = \sum_{k_i\geq 1, k_1+k_2\leq N-1} \frac{1}{2} \left(\left(F^{k_2} \circ F^{k_1}\right)(I_d)\,(y_0) - \left(F^{k_2} \circ F^{k_1}\right)(I_d)\,(\xi) \right) \\
&+ \sum_{k_i\geq 1, k_1+k_2\leq N-1, k_3\leq N-1} \iiint_{0<u_1<u_2<u_3} D\left(F^{k_2}\circ F^{k_1}\right)(I_d)F^{k_3}(I_d)(y_{u_1})\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3.
\end{aligned}$$

Then since $y_0 = \xi + f^{\circ N}\pi_N(\log_N g)(I_d)(\xi)$, by using (37), we have

$$\sum_{k_i \geq 1, k_1+k_2 \leq N} \left\| \left(F^{k_2} \circ F^{k_1}\right)(I_d)(y_0) - \left(F^{k_2} \circ F^{k_1}\right)(I_d)(\xi) \right\| \\ \leq C_\gamma \|g\|^2 \|y_0 - \xi\| \leq C_\gamma \|g\|^{\gamma+1}.$$

Then similar as in (40) and (41), when $k_1 + k_2 + k_3 \leq N$, we have

$$\iiint_{0<u_1<u_2<u_3} D\left(F^{k_2} \circ F^{k_1}\right)(I_d) F^{k_3}(I_d)(y_{u_1}) \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \\ = \iiint_{0<u_1<u_2<u_3} \left(F^{k_3} \circ F^{k_2} \circ F^{k_1}\right)(I_d)(y_{u_1}) \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3;$$

when $k_1 + k_2 + k_3 \geq N + 1$, we have

$$\iiint_{0<u_1<u_2<u_3} D\left(F^{k_2} \circ F^{k_1}\right)(I_d) F^{k_3}(I_d)(y_{u_1}) \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \leq C_\gamma \|g\|^{\gamma+1}.$$

Repeating this "subtraction and estimation" process for N times, we get

$$\left\| y_1 - \xi - \sum_{j=1}^{N} \frac{1}{j!} \sum_{k_i \geq 1, k_1+\cdots+k_j \leq N} \left(F^{k_j} \circ \cdots \circ F^{k_1}\right)(I_d)(\xi) \right\| \leq C_\gamma \|g\|^{\gamma+1}.$$

Since $f^{\circ k}$ is linear in $\mathcal{V}^{\otimes k}$ (Definition 18 on page 9), we have

$$\sum_{j=1}^{N} \frac{1}{j!} \sum_{k_i \geq 1, k_1+\cdots+k_j \leq N} \left(F^{k_j} \circ \cdots \circ F^{k_1}\right)(I_d)(\xi)$$
$$= \sum_{j=1}^{N} \frac{1}{j!} \sum_{k_i \geq 1, k_1+\cdots+k_j \leq N} f^{\circ(k_1+\cdots+k_j)} \pi_{k_j}(\log_N(g)) \otimes \cdots \otimes \pi_{k_1}(\log_N(g))(I_d)(\xi)$$
$$= \sum_{k=1}^{N} f^{\circ k} \left(\sum_{k_i \geq 1, k_1+\cdots+k_j = k} \frac{1}{j!} \pi_{k_j}(\log_N(g)) \otimes \cdots \otimes \pi_{k_1}(\log_N(g)) \right)(I_d)(\xi)$$
$$= \sum_{k=1}^{N} f^{\circ k} \pi_k(g)(I_d)(\xi).$$

Therefore, we have

$$\left\| y_1 - \xi - \left(\sum_{k=1}^{N} f^{\circ k} \pi_k(g)(I_d)(\xi) \right) \right\| \leq C_\gamma \|g\|^{\gamma+1}. \quad \square$$

LEMMA 27. *Suppose* $f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))$ *for some* $\gamma > 1$ *and* $\xi \in \mathcal{U}$. *Suppose* $g \in G^{\lfloor \gamma \rfloor + 1}(\mathcal{V})$. *Then, the unique solution of the ordinary differential* *equation*

$$\begin{aligned} \mathrm{d}y_u &= \sum_{k=1}^{\lfloor\gamma\rfloor} f^{\circ k}\pi_k\left(\log_{\lfloor\gamma\rfloor+1}(g)\right)(I_d)(y_u)\,\mathrm{d}u,\ u\in[0,1],\\ y_0 &= \xi + f^{\circ(\lfloor\gamma\rfloor+1)}\pi_{\lfloor\gamma\rfloor+1}\left(\log_{\lfloor\gamma\rfloor+1}(g)\right)(I_d)(\xi), \end{aligned}$$

satisfies, for $k=1,2,\dots,\lfloor\gamma\rfloor+1$, *and any* $v\in\mathcal{V}^{\otimes k}$,

(42)

$$\left\| f^{\circ k}(v)(I_d)(y_1) - \sum_{j=0}^{\lfloor\gamma\rfloor+1-k} f^{\circ(j+k)}(\pi_j(g)\otimes v)(I_d)(\xi)\right\| \le C_\gamma \|v\|\, |f|_{Lip(\gamma)}^{\gamma+1}\, \|g\|^{\gamma+1-k}.$$

Proof. This lemma can be proved similarly as Lemma 26. □

LEMMA 28. *Suppose* $f\in L(\mathcal{V}, C^\gamma(\mathcal{U},\mathcal{U}))$ *for some* $\gamma>1$ *and* $\xi\in\mathcal{U}$. *Denote* $N:=\lfloor\gamma\rfloor+1$, *and suppose* $g\in G^N(\mathcal{V})$. *Let* y^g *be the solution of the ordinary differential equation:*

$$\begin{aligned} \mathrm{d}y_u^g &= \left(\sum_{k=1}^{N-1} f^{\circ k}\pi_k(\log_N(g))(I_d)(y_u^g)\right)\mathrm{d}u,\ u\in[0,1],\\ y_0^g &= \xi + f^{\circ N}\pi_N(\log_N(g))(I_d)(\xi). \end{aligned}$$

For $g,h\in G^N(\mathcal{V})$, *denote by* $y^{g,h}$ *the unique solution to the integral equation:*

$$y_t^{g,h} = \begin{cases} \xi + f^{\circ N}\pi_N(\log_N(g))(I_d)(\xi) \\ \qquad + \int_0^t \left(\sum_{k=1}^{N-1} f^{\circ k}\pi_k(\log_N(g))(I_d)\left(y_u^{g,h}\right)\right)\mathrm{d}u, & t\in[0,1] \\ y_1^{g,h} + f^{\circ N}\pi_N(\log_N(h))(I_d)\left(y_1^{g,h}\right) \\ \qquad + \int_1^t \left(\sum_{k=1}^{N-1} f^{\circ k}\pi_k(\log_N(h))(I_d)\left(y_u^{g,h}\right)\right)\mathrm{d}u, & t\in(1,2] \end{cases}.$$

Then we have

$$\left\|y_2^{g,h} - y_1^{g\otimes h}\right\| \le C_\gamma |f|_{Lip(\gamma)}^{\gamma+1}(\|g\|\vee\|h\|\vee\|g\otimes h\|)^{\gamma+1}.$$

Proof. We only prove the Lemma when $|f|_{Lip(\gamma)}=1$. Otherwise, we replace f by $|f|_{Lip(\gamma)}^{-1} f$, and replace g and h by $\delta_{|f|_{Lip(\gamma)}} g$ and $\delta_{|f|_{Lip(\gamma)}} h$, respectively.

Since $\sum_{k=1}^{N-1} f^{\circ k}\pi_k(\log_N(g))(I_d)\in C^1(\mathcal{U},\mathcal{U})$, based on the definition of $y^{g,h}$ and y^g, we have $y_t^{g,h}=y_t^g$, $t\in[0,1]$. For $g,h\in G^N(\mathcal{V})$, by using Lemma 26, we get

$$\begin{aligned}
&\left\|y_2^{g,h}-y_1^{g\otimes h}\right\| \\
&=\left\|y_1^g-\xi+y_2^{g,h}-y_1^g-\left(y_1^{g\otimes h}-\xi\right)\right\| \\
&\leq\left\|\sum_{k=1}^N f^{\circ k}\pi_k\left(g\right)\left(I_d\right)\left(\xi\right)+\sum_{k=1}^N f^{\circ k}\pi_k\left(h\right)\left(I_d\right)\left(y_1^g\right)-\sum_{k=1}^N f^{\circ k}\pi_k\left(g\otimes h\right)\left(I_d\right)\left(\xi\right)\right\| \\
&\quad+C_\gamma\left(\|g\|\vee\|h\|\vee\|g\otimes h\|\right)^{\gamma+1}.
\end{aligned}$$

Based on Lemma 27, for $k=1,2,\ldots,N$,

$$\left\|f^{\circ k}\pi_k\left(h\right)\left(I_d\right)\left(y_1^g\right)-\sum_{j=0}^{N-k}f^{\circ(j+k)}\left(\pi_j\left(g\right)\otimes\pi_k\left(h\right)\right)\left(I_d\right)\left(\xi\right)\right\|\leq C_\gamma\|h\|^k\|g\|^{\gamma+1-k}.$$

As a result,

$$\begin{aligned}
&\left\|y_2^{g,h}-y_1^{g\otimes h}\right\| \\
&\leq\left\|\sum_{k=1}^N f^{\circ k}\pi_k\left(g\right)\left(I_d\right)\left(\xi\right)+\sum_{k=1}^N\sum_{j=0}^{N-k}f^{\circ(j+k)}\left(\pi_j\left(g\right)\otimes\pi_k\left(h\right)\right)\left(I_d\right)\left(\xi\right)\right. \\
&\quad\left.-\sum_{k=1}^N f^{\circ k}\pi_k\left(g\otimes h\right)\left(I_d\right)\left(\xi\right)\right\|+C_\gamma\left(\|g\|\vee\|h\|\vee\|g\otimes h\|\right)^{\gamma+1} \\
&\quad+C_\gamma\sum_{k=1}^N\|h\|^k\|g\|^{\gamma+1-k}\leq C_\gamma\left(\|g\|\vee\|h\|\vee\|g\otimes h\|\right)^{\gamma+1}. \quad\square
\end{aligned}$$

LEMMA 29. *Suppose $x:[0,T]\to\mathcal{V}$ is a continuous bounded variation path, and $f\in L\left(\mathcal{V},C^\gamma\left(\mathcal{U},\mathcal{U}\right)\right)$ for $\gamma\geq1$. Let $y:[0,T]\to\mathcal{U}$ denote as the unique solution of the ordinary differential equation*

$$\mathrm{d}y=f\left(y\right)\mathrm{d}x,\ y_0=\xi\in\mathcal{U}. \tag{43}$$

Then for any $p\in[1,\gamma+1)$, there exists a constant $C_{p,\gamma}$ (which only depends on p and γ), such that for any interval $[s,t]\subset[0,T]$ satisfying $|f|_{Lip(\gamma)}\left\|S_{[p]}\left(x\right)\right\|_{p-var,[s,t]}\leq1$, we have

$$\left\|S_{[p]}\left(y\right)\right\|_{p-var,[s,t]}\leq C_{p,\gamma}|f|_{Lip(\gamma)}\left\|S_{[p]}\left(x\right)\right\|_{p-var,[s,t]}. \tag{44}$$

Proof. Define $h:\mathcal{V}\oplus\mathcal{U}\to L\left(\mathcal{V}\oplus\mathcal{U},\mathcal{V}\oplus\mathcal{U}\right)$ as

$$h\left(v_1,u_1\right)\left(v_2,u_2\right)=\left(v_2,f\left(v_2\right)\left(u_1+\xi\right)\right),\ \ \forall v_1,v_2\in\mathcal{V},\forall u_1,u_2\in\mathcal{U}.$$

We define geometric p-rough paths $Z\left(n\right):[0,T]\to G^{[p]}\left(\mathcal{V}\oplus\mathcal{U}\right)$, $n\geq0$, recursively as the rough integral (in the sense of Definition 11 on page 6):

$$Z\left(0\right)_t\ :=\left(S_{[p]}\left(x\right)_t,0\right)\in G^{[p]}\left(\mathcal{V}\oplus\mathcal{U}\right),\ t\in[0,T], \tag{45}$$

$$Z(n+1)_t \quad : \quad = \int_0^t h(Z(n))\,\mathrm{d}Z(n)\,, \ \ t\in[0,T]\,,\, n\geq 0,$$

and define $Y(n):[0,T]\to G^{[p]}(\mathcal{U})$ as $Y(n):=\pi_{G^{[p]}(\mathcal{U})}Z(n)$. Then based on Prop 5.9 [13], there exists constant $C_{p,\gamma}$, which only depends on p and γ and is finite whenever $\gamma>p-1$, such that, for any interval $[s,t]$ satisfying $|f|_{Lip(\gamma)}\left\|S_{[p]}(x)\right\|_{p-var,[s,t]}\leq 1$, we have

$$\sup_n \|Y(n)\|_{p-var,[s,t]}\leq C_{p,\gamma}\,|f|_{Lip(\gamma)}\left\|S_{[p]}(x)\right\|_{p-var,[s,t]}. \tag{46}$$

(Indeed, by properly scaling f and $S_{[p]}(x)$, the constant $C_{p,\gamma}$ in (46) can be chosen to be independent of $|f|_{Lip(\gamma)}$ and $\left\|S_{[p]}(x)\right\|_{p-var,[0,T]}$, as we did after Theorem 14.) On the other hand, since x is continuous with bounded variation, it can be checked that, if we define continuous bounded variation paths $y(n):[0,T]\to\mathcal{U}$, $n\geq 1$, recursively as

$$\begin{aligned} y(0)_t &\equiv 0\in\mathcal{U},\, t\in[0,T]\,, \\ y(n+1)_t &= \int_0^t f(y(n)+\xi)\,\mathrm{d}x,\, t\in[0,T]\,, \end{aligned} \tag{47}$$

then based on the definition of rough integral in Definition 11 on page 6, it can be checked that,

$$Y(n)=S_{[p]}(y(n))\,,\ \forall n\geq 0. \tag{48}$$

Combined with (46), for interval $[s,t]$ satisfying $|f|_{Lip(\gamma)}\left\|S_{[p]}(x)\right\|_{p-var,[s,t]}\leq 1$, we have

$$\sup_n \left\|S_{[p]}(y(n))\right\|_{p-var,[s,t]}\leq C_{p,\gamma}\,|f|_{Lip(\gamma)}\left\|S_{[p]}(x)\right\|_{p-var,[s,t]}. \tag{49}$$

On the other hand, since f is $Lip(\gamma)$ for $\gamma\geq 1$, by using (47), we have, for any $[s,t]\subset[0,T]$ and any integers $n,m\geq 0$,

$$\begin{aligned} &\|y(n+m+1)-y(n+1)\|_{1-var,[s,t]} \\ &\leq |f|_{Lip(\gamma)}\|x\|_{1-var,[s,t]}\Big(\|y(n+m)-y(n)\|_{1-var,[s,t]}+\|y(n+m)_s-y(n)_s\|\Big). \end{aligned} \tag{50}$$

Then we divide $[0,T]:=\cup_{j=0}^{l-1}[t_j,t_{j+1}]$ in such a way that

$$|f|_{Lip(\gamma)}\|x\|_{1-var,[t_j,t_{j+1}]}\leq c<1,\ j=0,1,\ldots,l-1.$$

Then, for $[t_j,t_{j+1}]$, $j=0,1,\ldots,l-1$, we take supremum (in (50)) over $m\geq 1$, let n tend to infinity, then

$$\begin{aligned} &\overline{\lim}_{n\to\infty}\sup_{m\geq 1}\|y(n+m+1)-y(n+1)\|_{1-var,[t_j,t_{j+1}]} \\ &\qquad\leq c\,\overline{\lim}_{n\to\infty}\sup_{m\geq 1}\|y(n+m)-y(n)\|_{1-var,[t_j,t_{j+1}]} \end{aligned}$$

$$+ c \overline{\lim}_{n\to\infty} \sup_{m\geq 1} \left\| y(n+m)_{t_j} - y(n)_{t_j} \right\|.$$

Since $y(n)_0 \equiv 0$, $\forall n \geq 0$, and $c \in (0,1)$, we can prove inductively that $\overline{\lim}_{n\to\infty} \sup_{m\geq 1} \| y(n+m) - y(n) \|_{1-var,[t_j,t_{j+1}]} = 0$, $j = 0, 1, \ldots, l-1$.

Thus,

$$\lim_{n\to\infty} \sup_{m\geq 1} \| y(n+m) - y(n) \|_{1-var,[0,T]} \leq \sum_{j=0}^{l-1} \lim_{n\to\infty} \sup_{m\geq 1} \| y(n+m) - y(n) \|_{1-var,[t_j,t_{j+1}]} = 0.$$

As a result, $y(n)$ converge in 1-variation as n tends to infinity (denote the limit by $\widetilde{y}$), and we have

$$\lim_{n\to\infty} \max_{k=1,2,\ldots,[p]} \sup_{0\leq s\leq t\leq T} \left\| \pi_k \left(S_{[p]} (y(n))_{s,t} \right) - \pi_k \left(S_{[p]} (\widetilde{y})_{s,t} \right) \right\| = 0. \tag{51}$$

Based on (47) and let n tends to infinity, we have

$$\widetilde{y}_t = \int_0^t f(\widetilde{y}_u + \xi) \, \mathrm{d}x_u.$$

As a result, if y denotes the unique solution of the ordinary differential equation (43), then we have

$$y = \widetilde{y} + \xi. \tag{52}$$

Therefore, by combining (49), (51), (52) and using lower semi-continuity of p-variation, we get, for interval $[s,t]$ satisfying $|f|_{Lip(\gamma)} \left\| S_{[p]}(x) \right\|_{p-var,[s,t]} \leq 1$,

$$\begin{aligned} \left\| S_{[p]}(y) \right\|_{p-var,[s,t]} &= \left\| S_{[p]}(\widetilde{y} + \xi) \right\|_{p-var,[s,t]} = \left\| S_{[p]}(\widetilde{y}) \right\|_{p-var,[s,t]} \\ &\leq \underline{\lim}_{n\to\infty} \left\| S_{[p]}(y(n)) \right\|_{p-var,[s,t]} \leq C_{p,\gamma} |f|_{Lip(\gamma)} \left\| S_{[p]}(x) \right\|_{p-var,[s,t]}. \quad \square \end{aligned}$$

Then, we state Lemma 19 (on page 10) and give a proof.

LEMMA 19. *Suppose $x : [0,T] \to \mathcal{V}$ is a continuous bounded variation path, $f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))$ for $\gamma > 1$, and $\xi \in \mathcal{U}$. Suppose $y : [0,T] \to \mathcal{U}$ is the unique solution of the ordinary differential equation:*

$$\mathrm{d}y = f(y)\,\mathrm{d}x, \quad y_0 = \xi \in \mathcal{U}. \tag{53}$$

Then, for any $p \in [1, \gamma+1)$, there exists a constant $C_{p,\gamma}$, which only depends on p and γ, such that, for any $0 \leq s < t \leq T$,

$$(1), \left\| y_t - y_1^{s,t} \right\| \leq C_{p,\gamma} |f|_{Lip(\gamma)}^{\gamma+1} \left\| S_{[p]}(x) \right\|_{p-var,[s,t]}^{\gamma+1}, \tag{54}$$

(55)

$$(2)\,,\left\|y_t-y_s-\sum_{k=1}^{\lfloor\gamma\rfloor+1}f^{\circ k}\pi_k\Big(S_{\lfloor\gamma\rfloor+1}(x)_{s,t}\Big)(I_d)(y_s)\right\|\le C_{p,\gamma}\left|f\right|_{Lip(\gamma)}^{\gamma+1}\left\|S_{[p]}(x)\right\|_{p-var,[s,t]}^{\gamma+1},$$

where $y^{s,t}:[0,1]\to\mathcal{U}$ *denotes the unique solution of the ordinary differential equation (with* y_s *denotes the value of* y *in* (53) *at point* s*):*

$$(56)\;\mathrm{d}y_u^{s,t}=\left(\sum_{k=1}^{\lfloor\gamma\rfloor}f^{\circ k}\pi_k\left(\log_{\lfloor\gamma\rfloor+1}\Big(S_{\lfloor\gamma\rfloor+1}\left(x\right)_{s,t}\Big)\right)(I_d)\left(y_u^{s,t}\right)\right)\mathrm{d}u,\ u\in[0,1]\,,$$

$$y_0^{s,t}=y_s+f^{\circ(\lfloor\gamma\rfloor+1)}\pi_{\lfloor\gamma\rfloor+1}\left(\log_{\lfloor\gamma\rfloor+1}\left(S_{\lfloor\gamma\rfloor+1}\left(x\right)_{s,t}\right)\right)(I_d)\left(y_s\right).$$

Proof. We only prove (54); (55) follows from (54) and Lemma 26 on page 15. We prove the result when $|f|_{Lip(\gamma)}=1$. The general case can be proved by replacing f by $|f|_{Lip(\gamma)}^{-1}f$ and replacing $S_{[p]}(x)$ by $\delta_{|f|_{Lip(\gamma)}}\left(S_{[p]}(x)\right)$ (in which case both y in (53) and $y^{s,t}$ in (56) would stay unchanged).

Denote $N:=\lfloor\gamma\rfloor+1$. Define $\omega:\{(s,t)\,|0\le s\le t\le T\}\to\overline{\mathbb{R}^{+}}$ as

$$\omega\left(s,t\right):=\left\|S_{[p]}\left(x\right)\right\|_{p-var,[s,t]}^{p}.$$

Then it can be checked that, ω is continuous and is super-additive, *i.e.*

$$\omega\left(s,u\right)+\omega\left(u,t\right)\le\omega\left(s,t\right),\ \ \forall 0\le s\le u\le t\le T. \tag{57}$$

With $y^{s,t}$ defined at (56), we define $\Gamma:\{(s,t)\,|0\le s\le t\le T\}\to\mathcal{U}$ as

$$\Gamma_{s,t}:=y_t-y_1^{s,t}=y_t-y_s-\left(y_1^{s,t}-y_s\right).$$

For $0\le s\le u\le t\le T$, with $y^{s,u}$ defined at (56) and x in (53), we denote $\widetilde{y}^{u,t}$ as the unique solution of the ordinary differential equation:

$$\begin{aligned}\mathrm{d}\widetilde{y}_r^{u,t}&=\left(\sum_{k=1}^{N-1}f^{\circ k}\left(I_d\right)\left(\widetilde{y}_r^{u,t}\right)\pi_k\left(\log_N\left(S_N\left(x\right)_{u,t}\right)\right)\right)\mathrm{d}r,\ r\in[0,1]\,,\\ \widetilde{y}_0^{u,t}&=y_1^{s,u}+f^{\circ N}\left(I_d\right)\left(y_1^{s,u}\right)\pi_N\left(\log_N\left(S_N\left(x\right)_{u,t}\right)\right).\end{aligned}$$

For $0\le s\le u\le t\le T$, we denote piecewise continuous path $y^{s,u,t}:[0,2]\to\mathcal{U}$ by assigning

$$y_r^{s,u,t}:=y_r^{s,u}\text{ when }r\in[0,1]\ \text{ and }y_r^{s,u,t}:=\widetilde{y}_{r-1}^{u,t}\text{ when }r\in(1,2]. \tag{58}$$

Firstly, suppose that $[s,t]$ is an interval satisfying $\omega\left(s,t\right)\le1$ and $u\in(s,t)$. It can be computed that,

$$\left\|\Gamma_{s,u}+\Gamma_{u,t}-\Gamma_{s,t}\right\|=\left\|y_1^{s,u}-y_s+y_1^{u,t}-y_u-\left(y_1^{s,t}-y_s\right)\right\|$$

$$\leq \quad \left\| y_2^{s,u,t} - y_1^{s,t} \right\| + \left\| \left(y_2^{s,u,t} - y_1^{s,u} \right) - \left(y_1^{u,t} - y_u \right) \right\| .$$

Then, Lemma 28 and Lemma 26 imply that, ($|f|_{Lip(\gamma)} = 1$ and $\omega(s,t) \leq 1$, $\{\gamma\} := \gamma - \lfloor \gamma \rfloor$)

$$\|\Gamma_{s,u} + \Gamma_{u,t} - \Gamma_{s,t}\| \tag{59}$$

$$\leq C_\gamma \omega(s,t)^{\frac{\gamma+1}{p}} + \left\| \sum_{k=1}^{N} f^{\circ k} \pi_k \left(S_{[p]}(x)_{u,t} \right) \left((I_d)(y_1^{s,u}) - (I_d)(y_u) \right) \right\|$$

$$\leq C_\gamma \omega(s,t)^{\frac{\gamma+1}{p}} + C_\gamma \omega(s,t)^{\frac{1}{p}} \|\Gamma_{s,u}\| + C_\gamma \|y_u - y_s - (y_1^{s,u} - y_s)\|^{\{\gamma\}} \omega(s,t)^{\frac{N}{p}} .$$

Based on the definition of $y^{s,u}$ (at (56)), when $\omega(s,t) \leq 1$, we have

$$\|y_1^{s,u} - y_s\| \leq \|y^{s,u}\|_{1-var,[0,1]} + \|y_0^{s,u} - y_s\| \leq C_\gamma \omega(s,t)^{\frac{1}{p}} . \tag{60}$$

On the other hand, according to Lemma 29, there exists constant $C_{p,\gamma}$ (which only depends on p and γ, and is finite whenever $\gamma > p-1$), such that for any interval $[s,t]$ that satisfies $\omega(s,t) \leq 1$, we have

$$\|y_u - y_s\| \leq C_{p,\gamma} \omega(s,t)^{\frac{1}{p}} . \tag{61}$$

As a result, by combining (60) and (61), we get, when $\omega(s,t) \leq 1$,

$$\|(y_u - y_s) - (y_1^{s,u} - y_s)\|^{\{\gamma\}} \omega(s,t)^{\frac{N}{p}} \leq C_{p,\gamma} \omega(s,t)^{\frac{\gamma+1}{p}} .$$

Then, continuing with (59), we get, for any interval $[s,t]$ satisfying $\omega(s,t) \leq 1$ and any $u \in (s,t)$,

$$\|\Gamma_{s,t}\| \leq \left(1 + C_\gamma \omega(s,t)^{\frac{1}{p}} \right) (\|\Gamma_{s,u}\| + \|\Gamma_{u,t}\|) + C_{p,\gamma} \omega(s,t)^{\frac{\gamma+1}{p}} . \tag{62}$$

With C_γ and $C_{p,\gamma}$ in (62), suppose $[s,t]$ is an interval satisfying $\omega(s,t) \leq 1$, denote

$$\delta := \left(C_\gamma^p \vee C_{p,\gamma}^{\frac{p}{\gamma+1}} \right) \omega(s,t) .$$

Then since ω is super-additive (*i.e.* (57)), by setting $[t_0^0, t_1^0) = [s,t)$ and recursively dividing $[t_j^n, t_{j+1}^n) = [t_{2j}^{n+1}, t_{2j+1}^{n+1}) \cup [t_{2j+1}^{n+1}, t_{2j+2}^{n+1})$ in such a way that

$$\omega\left(t_{2j}^{n+1}, t_{2j+1}^{n+1} \right) = \omega\left(t_{2j+1}^{n+1}, t_{2j+2}^{n+1} \right) \leq \frac{1}{2} \omega\left(t_j^n, t_{j+1}^n \right), \; j = 0, 1, \ldots, 2^n - 1, \; n \geq 0,$$

we have, based on (62),

$$\|\Gamma_{s,t}\| \leq \overline{\lim}_{n\to\infty} \left(\sum_{k=0}^{n} \left(\prod_{j=0}^{k} \left(1 + 2^{-\frac{j}{p}} \delta^{\frac{1}{p}} \right) \right) \left(\frac{1}{2} \right)^{\left(\frac{\gamma+1}{p} - 1 \right) k} \right) \delta^{\frac{\gamma+1}{p}} \tag{63}$$

$$+ \overline{\lim}_{n\to\infty} \left(\prod_{j=0}^{n} \left(1 + 2^{-\frac{j}{p}} \delta^{\frac{1}{p}} \right) \right) \left(\sum_{j=0}^{2^n - 1} \left\| \Gamma_{t_j^n, t_{j+1}^n} \right\| \right)$$

$$\leq \exp\left(\frac{2^{\frac{1}{p}}}{2^{\frac{1}{p}}-1}\delta^{\frac{1}{p}}\right)\left(\frac{2^{\frac{\gamma+1}{p}-1}}{2^{\frac{\gamma+1}{p}-1}-1}\delta^{\frac{\gamma+1}{p}}+\overline{\lim}_{n\to\infty}\sum_{j=0}^{2^n-1}\left\|\Gamma_{t_j^n,t_{j+1}^n}\right\|\right).$$

Then we prove that $\overline{\lim}_{n\to\infty}\sum_{j=0}^{2^n-1}\left\|\Gamma_{t_j^n,t_{j+1}^n}\right\|=0$. Since $x:[0,T]\to\mathcal{V}$ is a continuous bounded variation path and $y:[0,T]\to\mathcal{U}$ is the solution of the ordinary differential equation

$$\mathrm{d}y=f\left(y\right)\mathrm{d}x,\ \ y_0=\xi,$$

we have that ($|f|_{Lip(\gamma)}=1$, $\gamma>1$)

$$\|y\|_{1-var,[s,t]}\leq\|x\|_{1-var,[s,t]},\ \forall 0\leq s\leq t\leq T.$$

Thus,

$$\left\|y_t-y_s-\sum_{k=1}^{N}f^{\circ k}\pi_k\left(S_{[p]}\left(x\right)_{s,t}\right)\left(I_d\right)\left(y_s\right)\right\| \tag{64}$$

$$\begin{aligned}
&= \left\|\int\cdots\int_{s<u_1<\cdots<u_N<t}f^{\circ N}\left(dx_{u_1}\otimes\cdots\otimes dx_{u_N}\right)\left(\left(I_d\right)\left(y_{u_1}\right)-\left(I_d\right)\left(y_s\right)\right)\right\|\\
&\leq C_\gamma\int\cdots\int_{s<u_1<\cdots<u_N<t}\left\|y_{u_1}-y_s\right\|^{\{\gamma\}}\left\|dx_{u_1}\right\|\cdots\left\|dx_{u_N}\right\|\\
&\leq C_\gamma\sup_{u\in[s,t]}\left\|y_u-y_s\right\|^{\{\gamma\}}\left\|x\right\|_{1-var,[s,t]}^{N}\leq C_\gamma\left\|x\right\|_{1-var,[s,t]}^{\gamma+1}.
\end{aligned}$$

On the other hand, based on Lemma 26,

(65)

$$\left\|y_1^{s,t}-y_s-\sum_{k=1}^{N}f^{\circ k}\pi_k\left(S_{[p]}\left(x\right)_{s,t}\right)(I_d)(y_s)\right\|\leq C_\gamma\left\|S_{[p]}\left(x\right)_{s,t}\right\|^{\gamma+1}\leq C_{p,\gamma}\left\|x\right\|_{1-var,[s,t]}^{\gamma+1}.$$

Thus, combining (64) and (65), we get

$$\left\|\Gamma_{t_j^n,t_{j+1}^n}\right\|\leq C_{p,\gamma}\left\|x\right\|_{1-var,\left[t_j^n,t_{j+1}^n\right]}^{\gamma+1},\ j=0,1,\ldots,2^n-1,\ n\geq 0,$$

and we have ($\gamma\geq 1$)

$$\overline{\lim}_{n\to\infty}\sum_{j=0}^{2^n-1}\left\|\Gamma_{t_j^n,t_{j+1}^n}\right\|=0.$$

Thus, continuing with (63), we get that, there exists constant $C_{p,\gamma}$, which only depends on p and γ, and is finite whenever $\gamma>p-1$, such that, for any interval $[s,t]$ satisfying $\omega\left(s,t\right)\leq 1$,

$$\left\|y_t-y_1^{s,t}\right\|=\left\|\Gamma_{s,t}\right\|\leq C_{p,\gamma}\omega\left(s,t\right)^{\frac{\gamma+1}{p}}. \tag{66}$$

For $[s,t]$ satisfying $\omega\left(s,t\right)>1$, as in Prop 5.10 [8], we decompose $[s,t)=\cup_{j=0}^{n-1}[t_j,t_{j+1})$ in such a way that $\omega\left(t_j,t_{j+1}\right)=1$, $j=0,1,\ldots,n-2$, and

$\omega(t_{n-1},t_n)\leq 1$. Then by using super-additivity of ω, we have $n-1\leq\omega(s,t)$, and

$$\begin{aligned}\|y_t-y_s\| &\leq \sum_{j=0}^{n-1}\left\|y_{t_{j+1}}-y_{t_j}\right\|\leq C_{p,\gamma}\left(n-1+\omega(t_{n-1},t_n)^{\frac{1}{p}}\right)\\ &\leq C_{p,\gamma}n\leq C_{p,\gamma}(\omega(s,t)+1)\leq 2C_{p,\gamma}\omega(s,t).\end{aligned}$$

On the other hand, when $\omega(s,t)\geq 1$,

$$\begin{aligned}\left\|y_1^{s,t}-y_s\right\| &\leq \left\|y_1^{s,t}-y_0^{s,t}\right\|+\left\|y_0^{s,t}-y_s\right\|\\ &\leq C_\gamma\left\|S_{[p]}(x)_{s,t}\right\|^{N-1}+C_\gamma\left\|S_{[p]}(x)_{s,t}\right\|^{N}\leq C_\gamma\omega(s,t)^{\frac{N}{p}}.\end{aligned}$$

Therefore, when $\omega(s,t)\geq 1$,

$$\begin{aligned}\left\|y_t-y_1^{s,t}\right\| &= \left\|y_t-y_s-\left(y_1^{s,t}-y_s\right)\right\|\\ &\leq C_{p,\gamma}\omega(s,t)+C_\gamma\omega(s,t)^{\frac{N}{p}}\leq C_{p,\gamma}\omega(s,t)^{\frac{\gamma+1}{p}}. \quad\square\end{aligned}$$

LEMMA 30. *Suppose* $f\in L(\mathcal{V},C^\gamma(\mathcal{U},\mathcal{U}))$ *for some* $\gamma>1$. *For* $g\in G^{\lfloor\gamma\rfloor+1}(\mathcal{V})$ *and* $\xi\in\mathcal{U}$, *define* $y(g,\xi):[0,1]\to\mathcal{U}$ *as the unique solution of the ordinary differential equation:*

$$\begin{aligned}\mathrm{d}y_u &= \sum_{k=1}^{\lfloor\gamma\rfloor} f^{\circ k}\pi_k\left(\log_{\lfloor\gamma\rfloor+1}(g)\right)(I_d)(y_u)\,\mathrm{d}u,\quad u\in[0,1],\\ y_0 &= \xi+f^{\circ(\lfloor\gamma\rfloor+1)}\pi_{\lfloor\gamma\rfloor+1}\left(\log_{\lfloor\gamma\rfloor+1}(g)\right)(I_d)(\xi)\in\mathcal{U},\end{aligned}$$

If there exist $\{g^l\}_{l\geq 1}\subset G^{\lfloor\gamma\rfloor+1}(\mathcal{V})$ *and* $\{\xi^l\}_{\geq 1}\subset\mathcal{U}$ *such that*

$$\lim_{l\to\infty}\max_{k=1,2,\ldots,\lfloor\gamma\rfloor+1}\left\|\pi_k\left(g^l\right)-\pi_k(g)\right\|=0 \text{ and } \lim_{l\to\infty}\left\|\xi^l-\xi\right\|=0,$$

then

$$\lim_{l\to\infty}\sup_{t\in[0,1]}\left\|y\left(g^l,\xi^l\right)_t-y(g,\xi)_t\right\|=0.$$

Proof. Since $\sum_{k=1}^{\lfloor\gamma\rfloor}f^{\circ k}(I_d)\in C^1(\mathcal{U},\mathcal{U})$, based on Thm 3.15 [8] (their result extends naturally to ordinary differential equations in Banach spaces), we get

$$\begin{aligned}&\sup_{t\in[0,1]}\left\|y\left(g^l,\xi^l\right)_t-y(g,\xi)_t\right\|\\ &\leq C_f\left(\left\|y\left(g^l,\xi^l\right)_0-y(g,\xi)_0\right\|+\sum_{k=1}^{\lfloor\gamma\rfloor}\left\|f^{\circ k}\left(\pi_k\left(\log_{\lfloor\gamma\rfloor+1}\left(g^l\right)\right)\right.\right.\right.\end{aligned}$$

$$
\begin{aligned}
&-\pi_k\left(\log_{\lfloor\gamma\rfloor+1}(g)\right)\right)(I_d)\Big\|_\infty\Big) \\
&\leq C_f\left(\left\|\xi^l-\xi\right\|+\left\|\pi_{\lfloor\gamma\rfloor+1}\left(\log_{\lfloor\gamma\rfloor+1}(g)\right)\right\|\left\|\xi^l-\xi\right\|^{\{\gamma\}}\right. \\
&\qquad\left.+\sum_{k=1}^{\lfloor\gamma\rfloor+1}\left\|\pi_k\left(\log_{\lfloor\gamma\rfloor+1}\left(g^l\right)\right)-\pi_k\left(\log_{\lfloor\gamma\rfloor+1}(g)\right)\right\|\right). \qquad \square
\end{aligned}
$$

Proof of Theorem 22. Based on the definition of geometric p-rough path (Definition 7 on page 5), there exists a sequence of continuous bounded variation paths $\{x^l\}:[0,T]\to\mathcal{V}$, such that

$$\lim_{l\to\infty} d_p\left(S_{[p]}\left(x^l\right),X\right)=0.$$

As a result, we have (combined with Thm 3.1.3 [14] when $\lfloor\gamma\rfloor\geq[p]$)

(67)
$$\lim_{l\to\infty}\max_{n=1,2,\ldots,\lfloor\gamma\rfloor+1}\left\|\pi_n\left(S_{\lfloor\gamma\rfloor+1}\left(x^l\right)_{s,t}\right)-\pi_n\left(S_{\lfloor\gamma\rfloor+1}(X)_{s,t}\right)\right\|=0,\ \forall 0\leq s\leq t\leq T,$$

and

(68)
$$\lim_{l\to\infty}\left\|S_{[p]}\left(x^l\right)\right\|_{p-var,[s,t]}=\|X\|_{p-var,[s,t]},\ \ \forall 0\leq s\leq t\leq T.$$

On the other hand, denote $y^l:[0,T]\to\mathcal{U}$ as the unique solution of the ordinary differential equation

(69)
$$\mathrm{d}y^l=f\left(y^l\right)\mathrm{d}x^l,\ y_0^l=\xi,$$

and denote $Y:=\pi_{G^{[p]}(\mathcal{U})}(Z)$ with Z denotes the unique solution (in the sense of Definition 13 on page 6) of the rough differential equation

(70)
$$\mathrm{d}Y=f(Y)\,\mathrm{d}X,\ Y_0=\xi.$$

Then, based on the universal limit theorem (Thm 5.3 [13]), we have

(71)
$$\lim_{l\to\infty}\left\|y_t^l-\pi_1(Y_t)\right\|=0,\ \ \forall t\in[0,T].$$

For $0\leq s\leq t\leq T$ and $l\geq 1$, with y^l in (69), denote by $y^{s,t,l}:[0,1]\to\mathcal{U}$ the unique solution of the ordinary differential equation:

$$
\begin{aligned}
\mathrm{d}y_u^{s,t,l}&=\sum_{k=1}^{\lfloor\gamma\rfloor}f^{\circ k}\pi_k\left(\log_{\lfloor\gamma\rfloor+1}\left(S_{\lfloor\gamma\rfloor+1}\left(x^l\right)_{s,t}\right)\right)(I_d)\left(y_u^{s,t,l}\right)\mathrm{d}u,\ \ u\in[0,1], \\
y_0^{s,t,l}&=y_s^l+f^{\circ(\lfloor\gamma\rfloor+1)}\pi_{\lfloor\gamma\rfloor+1}\left(\log_{\lfloor\gamma\rfloor+1}\left(S_{\lfloor\gamma\rfloor+1}\left(x^l\right)_{s,t}\right)\right)(I_d)\left(y_s^l\right)\in\mathcal{U}.
\end{aligned}
$$

For $0 \leq s \leq t \leq T$, with Y in (70), denote by $y^{s,t} : [0,1] \to \mathcal{U}$ the unique solution of the ordinary differential equation:

$$\begin{aligned} dy_u^{s,t} &= \sum_{k=1}^{\lfloor \gamma \rfloor} f^{\circ k} \pi_k \left(\log_{\lfloor \gamma \rfloor + 1} \left(S_{\lfloor \gamma \rfloor + 1} (X)_{s,t} \right) \right) (I_d) \left(y_u^{s,t} \right) du, \;\; u \in [0,1], \\ y_0^{s,t} &= \pi_1 (Y_s) + f^{\circ (\lfloor \gamma \rfloor + 1)} \pi_{\lfloor \gamma \rfloor + 1} \left(\log_{\lfloor \gamma \rfloor + 1} \left(S_{\lfloor \gamma \rfloor + 1} (X)_{s,t} \right) \right) (I_d) (\pi_1 (Y_s)) \in \mathcal{U}. \end{aligned}$$

Then, according to Lemma 30,

$$\lim_{l \to \infty} \left\| y_1^{s,t,l} - y_1^{s,t} \right\| = 0. \tag{72}$$

Based on Lemma 19, for each $l \geq 1$,

$$\left\| y_t^l - y_1^{s,t,l} \right\| \leq C_{p,\gamma} \left(|f|_{Lip(\gamma)} \left\| S_{[p]} \left(x^l \right) \right\|_{p-var,[s,t]} \right)^{\gamma+1}, \tag{73}$$

$$\begin{aligned} &\left\| y_t^l - y_s^l - \sum_{k=1}^{\lfloor \gamma \rfloor + 1} f^{\circ k} \pi_k \left(S_{\lfloor \gamma \rfloor + 1} \left(x^l \right)_{s,t} \right) (I_d) \left(y_s^l \right) \right\| \\ &\qquad \leq C_{p,\gamma} \left(|f|_{Lip(\gamma)} \left\| S_{[p]} \left(x^l \right) \right\|_{p-var,[s,t]} \right)^{\gamma+1}. \end{aligned} \tag{74}$$

By combining (67), (68), (71) and (72), we let $l \to \infty$ in (73) and (74), and get Theorem 22. ☐

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Oxford-Man Institute of Quantitative Finance
University of Oxford
Walton Well Road, Oxford
United Kingdom