Dedicated to Professor Şerban Strătilă on the occasion of his 70^{th} birthday

A LETTER ON THE KADISON-SINGER PROBLEM

BETÜL TANBAY

Let H be a separable Hilbert space with a fixed orthonormal basis $(e_n)_{n\geq 1}$ and $B(H)$ be the von Neumann algebra of all bounded linear operators on H. Identifying $\ell^\infty = C(\beta N)$ with the diagonal operators, we consider $C(\beta N)$ as a subalgebra of $B(H)$. In 1959, Kadison and Singer raised the following question: Does every pure state of ℓ^∞ extend in a unique way to a pure state of $B(H)$? Since then, a lot of mathematicians thought on this problem, contributing with partial results of different impact until the question was positively answered in 2013. This letter intends to trace one story among many, emphazising on the set theory and operator algebras connections. A proof in the language of operator algebras is yet to come.

AMS 2010 Subject Classification: 46L30, 46L05.

Key words: pure state extension, Kadison-Singer problem.

1. PRELUDE

Dear Şerban,

I congratulate you for your 70^{th} birthday. You were not even in your fifties, my age today, when we met at the Department of Mathematics of the Boğaziçi University. You already knew about the Kadison-Singer Problem (hereinafter KS), I had just received my thesis thanks to a minor partial result on KS (see [13]). Kadison-Singer had shown that pure states on continuous maximal abelian subalgebras (masas) of $B(H)$ do not extend uniquely to pure states on the full algebra and conjectured the same for discrete masas in [10]. In the next 54 years, the problem has expanded to a very large number of equivalent problems in various fields (see [8] for an extensive discussion) and partial results pointed mostly in the direction that extensions were unique for discrete masas. Identifying $\ell^\infty = C(\beta N)$ with the diagonal operators, the question had boiled down to the so-called "paving conjecture" on matrices after Anderson (see [6]) and Akemann-Anderson (see [1]) papers. A few months ago, Marcus, Spielman and Srivastava proved indeed that extensions were unique, working on

Weaver's discrepancy theoretic versions of the problem [16] and using characteristic polynomials of matrices. It was a beautiful present to the mathematics community, announced in blogs of many well-known mathematicians such as Gowers or Tao. There is still a lot of work to do for the experts on the problem. Understanding the proof, making the connections with equivalent versions and now that we know extensions are unique, writing an operator algebraic proof of the result. In this letter of birthday, I shall try to retrace the road from KS to today, from a set theoretic approach.

Happy birthday,

Betül

2. TO START THE ROAD

Let us introduce the vocabulary and notations involved to trace the road. A basic course in operator algebras should be enough to follow. If H denotes a separable infinite dimensional complex Hilbert space and A is a unital C*-subalgebra of $B(H)$, the algebra of all bounded linear operators, then the set of states of A is a convex subset of the dual of A and is compact in the w*-topology on the dual. By the Krein-Milman theorem, the set of states is the closed convex hull of its extreme points (which are the pure states of A). One can extend a state from a C*-subalgebra of A to $B(H)$ using the Hahn-Banach theorem. The set of extensions of such a state forms a convex compact subset of the dual. If the state of the C*-subalgebra is pure, then the set of extensions has extreme points, which are pure states of A. Hence, if a pure state has a unique pure state extension, then the closed convex hull of its extension (which consists of the extension itself), is the set of all state extensions of the given pure state. So, a pure state has a unique state extension if and only if it has a unique pure state extension. If A is abelian, the set of pure states, which is the set of non-zero multiplicative linear functionals, is compact. The Gelfand map $T \mapsto \hat{T}$, where $\hat{T}(\rho) = \rho(T)$ for any pure state ρ on A, is an isometric *-isomorphism from A onto the continuous functions on the set of pure states of A. Notice that any hope for unicity needs maximality, because by Zorn's lemma, A is contained in a maximal abelian C*-subalgebra (masa) of $B(H)$, and by the Stone-Weierstrass theorem, there must be two distinct pure states of the masa which agree on A. For example, if $< X, \mu >$ is a σ-finite measure space, then $L^\infty(X, \mu)$ represented as an algebra of multiplication operators on $L^2(X, \mu)$ is a maximal abelian C*-subalgebra of $B(L^2(X, \mu))$. In particular, when $X = [0, 1]$, and μ is the Lebesgue measure, the masa we obtain is called the "continuous masa". When $X = N$, with μ the counting measure, we get, ℓ^∞, the "discrete masa". In case H is of finite dimension n, one replaces N by

$\{1, ..., n\}$. Any masa on a separable Hilbert space is unitarily equivalent to one of these three, or to the direct sum of the continuous masa with one of the two discrete types. The problem of extensions of pure states from masas of $B(H)$ is therefore reduced to the study of extensions from continuous masas and from discrete masas.

We mentioned that the set of states on a C*-algebra provided with the w*-topology is a compact Hausdorff space. Hence if $\{f_n\}_{n\in N}$ is a family of states on the algebra, the map $n \mapsto f_n$ from N, endowed with the discrete topology, into the set of states has a unique continuous extension to $\beta(N)$, the Stone-Čech compactification of N. The elements of $\beta(N)$ can be regarded as the set of ultrafilters on N, where N is identified with the fixed ultrafilters. For ℓ^∞, there is a 1-1 correspondence between pure states (as said, the non-zero multiplicative linear functionals) and ultrafilters on N, *i.e.* $\ell^\infty = C(\beta N)$. The vector states correspond to fixed ultrafilters.

When we fix an orthonormal basis $(e_n)_{n\geq 1}$ on H, we can identify each bounded sequence with a diagonal operator and see the discrete masa ℓ^∞ as the C*-subalgebra D of diagonal operators with respect to the fixed basis. For each operator T in $B(H)$, denote by D_T the diagonal matrix obtained from T considering only its diagonal entries. With a fixed basis, we can also say something more precise about the above mentioned connection between pure states on D and ultrafilters on N.

For any subset σ of N, let P_σ be the projection onto the span of $\{e_i : i \in \sigma\}$. Notice that a diagonal operator is a projection iff it is of the form P_σ, for some $\sigma \subseteq N$.

Given a pure state f on D, since f is multiplicative, every projection in D is mapped either to 0 or to 1. Now let $U = \{\sigma \subseteq N : f(P_\sigma) = 1\}$. It is easy to check that U is an ultrafilter on N.

On the other hand, given an ultrafilter U on N, let f be the map defined on the diagonal projections by $f(P_\sigma) = 1$ iff $\sigma \in U$; The linear span of diagonal projections is norm-dense in D by the Stone-Weierstrass theorem applied to the function representation of D , so extend f by linearity and norm-density to D. It is again easy to check that f is a pure state on D (by showing it is multiplicative).

Also, we can look at pure vector states on D, which are exactly the vector states given by the basis vectors. The set $S(D)$ of states on D is compact, so the map $f : n \mapsto f_n$ from N into $S(D)$, where $f_n(D) =< Te_n, e_n >$ for all T in D, has a unique continuous extension g to $\beta(N)$, with $g(U)(T) = lim_U < Te_n, e_n >$. Hence the pure vector states correspond to fixed ultrafilters, and all other pure states are given by free ultrafilters. Notice that every pure vector state on D has a unique pure state extension to $B(H)$.

3. THOSE WERE THE KADISON-SINGER DAYS, MY FRIEND

"I received an early copy of Heisenberg's first work a little before publication and I studied it for a while and within a week or two I saw that the noncommutation was really the dominant characteristic of Heisenberg's new theory... So I was led to concentrate on the idea of noncommutation and to see how the ordinary dynamics which people had been using until then should be modified to include it." (P. Dirac)

In his fundamental work on quantum mechanics [9], Dirac wants to find a representation for a family of observables (a commutative family of self-adjoint operators) which is completely determined by the observables that are diagonal. What Dirac called a complete commuting set, is a masa in today's language. In other words, we obtain a maximal set of observables that can be measured simultaneously. Physicists who are familiar with the Heisenberg's Uncertainty principle would not be surprised to hear that "randomness" is involved in the final solution! Observables are operators and pure states are probability distributions. For an observable T, the average of values measured for T, with the system in the state corresponding to a unit vector ξ, is the expectation value of T, given by the vector pure state w_ξ with $w_\xi(T) = < T\xi, \xi >$. When ξ is a basis vector e_n, the pure state is completely determined by its values on D, hence has a unique extension to $B(H)$. Dirac claims that any pure state on any masa will behave so.

In [10], in 1959, Kadison and Singer showed that there are pure states on the continuous masa that do not extend uniquely. In 1979, [6], Anderson strengthened the result by showing that no pure state extends uniquely in the continuous case.

The discrete case remained open and was to live more than half a century under the name "Kadison-Singer problem (KS)". More and more people touched "KS", leaving many "dead corpses" as Dick Kadison jokingly called them (us!). He also emphasized that the improvements to come were announced in their famous "lemma five" in [10]:

Lemma 3.1. *If A is a maximal abelian algebra then there exists a sequence of projections $\{E_n\}$ in A such that $B^{|E_1|\ldots|E_n}$ converges to an operator of A in the uniform topology if and only if $\rho_1(B) = \rho_2(B)$ for each pair of states, ρ_l, ρ_2 of all bounded operators such that $\rho_1|A = \rho_2|A$ is a pure state of A.*

($B^{|E}$ denotes $EBE + (I - E)B(I - E)$)

Indeed, a significant step was taken using this Lemma, by Anderson ([6]) when he introduced the notion of "compressible" operators, today known as "paveable" operators.

4. ANDERSON AND AKEMANN HIT THE ROAD

Joel Anderson had set-theorist friends (Alexander Kechris was one of them, I think), and he was not scared of ultrafilters. In 1979, he introduces the notion of "compressibility" (paving) and states an equivalent version of KS [6]. Small world, he exposes them at the 5^{th} International Conference on Operator Theory, held in Timisoara in 1980.

For an operator T in $B(H)$, let us denote the operator $P_\sigma(T - D_T)P_\sigma$ by T_σ, the "block" reduction of the $N \times N$ matrix $T - D_T$ to the subset $\sigma \times \sigma$.

The first version said that **extensions are unique iff for any operator T, for any ultrafilter U, for any $\epsilon > 0$, there is a subset σ of N in U such that $||T_\sigma|| < \epsilon$.**

A triple universal statement! Here is a great source for partial results! In the years to come, some took basic operators and tried, some looked at "touchable" ultrafilters even if one had to assume not only the axiom of choice but also the continuum hypothesis (P-points, Q-points, Ramsey ultrafilters..), some played with ϵ... For those who hated ultrafilters a user-friendly version was ready with a so-called elegant proof [1, 14]:

Extensions are unique iff for any operator T, for any $\epsilon > 0$, there is a partition $\sigma_1, ..., \sigma_m$ of N such that $||T_{\sigma_j}|| < \epsilon$ for all $j = 1, ..., m$.

A more descriptive word turned out to be "paving", mostly expressed in terms of projections instead of partitions:

Definition 4.1. An operator T in $B(H)$ is said to be **paveable** if for all $\epsilon > 0$, there exist a natural number m and projections $P_{\sigma_1}, ..., P_{\sigma_m} \in D$ with $\sum P_{\sigma_j} = 1$ such that $||T_{\sigma_j}|| < \epsilon$ for all $j = 1, ..., m$.

So, KS became equivalent to the statement "every operator is paveable". It was shown quickly that the class of paveable operators were a closed subspace of $B(H)$ [13]. The finite version was even more irresistible and also had quite a nice proof using König's Infinity lemma which beautifully states that any finitely branching infinite tree has an infinite branch [13]:

Extensions are unique iff there is a natural number m such that every finite-dimensional matrix is paveable into m blocks.

Notice of course that this natural number m must be independent of the dimension of the matrix. The term "m-paved" is also used, and everyone knew that m must be greater than two. To the surprise of many, Marcus, Spielman and Srivastasva show in the recent solution that two is sufficient for their projections.

From the eighties on, Charles Akemann (hit the road, Chuck!) is supposed to have given this version to his graduate students as a week-end homework!

Beginner's luck did not occur, and the problem outlived many many weekends... It was hard not to be tempted! The land was free, paving one matrix could turn into a paper, a reasonable bunch into a thesis!

5. I DID IT MY WAY

What is a graduate student in set theory doing in this land? This can only be explained by the rich proliferic atmosphere of the last three floors of Evans Hall, Berkeley. Logicians were on the 7^{th} floor, but certainly closer to the sky than all the others, Solovay was pushing me towards very large cardinals, so when I heard the word C*-algebras at the tea time in the 10^{th} floor, I thought this was the most concrete thing I had touched in a long time! One question led to another and I found myself with KS, asking the first "natural" questions a set-theorist-to-be would ask: can I do the magic "forcing" on KS?? The initial statement was quantifying over ultrafilters, so the complexity of the task seemed high, but it did not take too long to bring the complexity down to a so-called Π^1_2-statement and feel proud about it, until I had to discover that Joel Anderson had gone this way ten years earlier!! (Today it is a question of minutes, but literature search in the eighties was a task to learn in the basement of libraries.) Later Anderson and Akemann were generous enough to like my proofs and promote them here and there. Given the low complexity, set theory techniques were probably unnecessary, due to what is called Shoenfield's Absoluteness Lemma and the big boss (Solovay) declared that the problem was "doable" (which meant to solve it in ZFC) and I found myself in C*-algebras knowing about them a little more than my sister-dancer!

What seemed approachable was almost untouchable: the operator norm! Even for a finite matrix! Instead of struggling with the operator norm, I tried "l_1-bounded" matrices and paved them using a method I called "splitting in two". (Does this have anything to do with the recent solution of KS where families interlace and split in two? Afterall, as we knew that general matrices cannot be two-paved, it wasn't so expected that projections with "small diagonal" could be two-paved) The result was a C*-subalgebra of $B(H)$ of reasonable size upto which extensions were unique. This algebra also contained my favourite projection baptized p_B by Chuck: if M_n denotes the complex $n \times n$ matrices, the projection p_B is defined on the direct sum of the M_n by the condition that $(p_B)_n$ has $1/n$ in each of its entries. This projection had been studied in [15], and motivated by KS, we had unsuccessfully tried to see how much the operator norm of p_B (namely 1) is affected by perturbing a "small" fraction of its entries. In a recent paper, appeared in 2012, [2], we showed that it can be affected more than we thought, thanks to a graph-theoretic technique due to Joel Friedman. We were still not tired of partial results, and some recent work is obsolete before being printed [3]!

So much tells why it is not a surprise to some "dead corpses" that KS had so many different analogues, in so many different branches of mathematics ending with random matrices. But the end of the road was not yet to come in the early nineties:

6. BOURGAIN AND TZAFRIRI BLOCK THE ROAD

As we said, lots of paved operators were swimming around until Bourgain and Tzafriri came up in 1991 with a paper [7] noone could understand, but swept almost all these operators into one bucket. They showed the existence of a constant $c > 0$ such that for any $\epsilon > 0, n \geq 1/c$ and for any $n \times n$-matrix T, there is a subset σ of $\{1, ..., n\}$ of cardinality greater or equal to $c\epsilon^2 n$ for which $||T_\sigma|| < \epsilon ||T||$. Also using their Random Paving Principle, they showed that matrices with "relatively small" entries were paveable.

This remained as the best partial result for the years to come, maybe due to an intimidation. Each time he gave a look at KS, Sorin Popa had something relevant to add, and his last paper [12] was heading for a major contribution. Rumors are that he was in contact with Bourgain and they were about to combine their techniques! On the more humble "my way", it was also clear that besides operator algebras and von Neumann algebras, Banach spaces and harmonic analysis expertise was important for KS. With the contribution of Ali Ülger, we had been once again convinced that extensions were unique when we showed that for each $t \in \beta(N)$, the set of states extending the Dirac measure δ_t is a finite-dimensional space or else it would contain a homeomorphic copy of $\beta(N)$ [4].

We hope not to be unfair to anyone by saying that after the Bourgain-Tzafriri blockage, the next interesting movement was the work of Nik Weaver more than 10 years after.

7. DISCREPANCY ON THE ROAD AND HAPPY END

For those familiar with the subject, wanting to approximate a continuous object by a discrete one, with respect to some measure of uniformity involves discrepancy theory. One tries to study deviations from uniformity, to measure how far a situation can be from where we would like it to be. In measure-theoretic or combinatorial settings, one studies the inevitable irregularities of distributions. In other words, one is concerned with coloring elements of a ground set such that each set is as balanced as possible, namely it has approximately the same number of elements in each color. The Beck-Fiala theorem, a major theorem in discrepancy theory gives a bound for the imbalance in function of the times each element appears across all sets. Working essentially on the AA-Conjecture (see Conjecture 7.1.3 in [1]), Nik Weaver showed that KS has a positive solution if and only if his Conjecture (KS_r) is true for some $r \geq 2$ (see [16]):

Conjecture 7.1. (KS_r). *There exist universal constants $N \geq 2$ and $\epsilon > 0$ such that the following holds: Let $v_1, ..., v_n \in C^k$ satisfy $||v_i|| \leq 1$ for all $i \leq n$, and suppose*

$$\sum_i |\langle u, v_i \rangle|^2 \leq N$$

for every unit vector $u \in C^k$. Then there exists a partition $\sigma_1, \ldots, \sigma_r$ of $\{1, ..., n\}$ such that

$$\sum_{i \in \sigma_j} |\langle u, v_i \rangle|^2 \leq N - \epsilon$$

for every unit vector $u \in C^k$ and all $j \leq r$.

Marcus, Spielman and Srivastava have proven in their recent article [11] a generalization of KS_r by showing that if all vectors have sufficiently small norm then an appropriately low discrepancy partition must exist. Their result, given below, implies KS_r for $r = 2$ and hence shows that projections with "small diagonal" are paveable, and actually paved into two blocks:

Corollary 7.2. *Let $v_1, ..., v_n \in C^k$ be column vectors satisfying $\sum_i v_i v_i^* = I$ and $||v_i||^2 \leq \epsilon$ for all $i \leq n$. Then there exists a partition σ_1, σ_2 of $\{1, ..., n\}$ such that for $j \in \{1, 2\}$,*

$$||\sum_{i \in \sigma_j} v_i v_i^*|| \leq \frac{(1 + \sqrt{2\epsilon})^2}{2}.$$

Translations in various approaches already follow, let us cite a very recent one into operator algebras language, by Akemann-Weaver [5]:

Theorem 7.3. *For any finite-dimensional (say n is the dimension) projection matrix P whose diagonal entries are at most ϵ (a positive number), there is a subset σ of $\{1, ..., n\}$ such that $||PP_\sigma P|| \leq \frac{1}{2} + o(\epsilon)$ and $||PP_{\{1,..,n\}\setminus\sigma} P|| \leq \frac{1}{2} + o(\epsilon)$ where $o(\epsilon) = \sqrt{2\epsilon} + \epsilon$.*

Note that, in the Marcus-Spielman-Srivastava result, the estimate depends only on the size of the vectors and not on the number of vectors or the dimension of the space. Consequently, their result can be generalized to infinite-dimensional Hilbert spaces (again for a straightforward proof, see [5]).

THEOREM 7.4. *For any projection P of $B(H)$ whose diagonal entries with respect to a fixed basis are at most ϵ (a positive number), there is a subset σ of N such that $||PP_\sigma P|| \leq \frac{1}{2} + o(\epsilon)$ and $||PP_{N\setminus\sigma}P|| \leq \frac{1}{2} + o(\epsilon)$ where $o(\epsilon) = \sqrt{2\epsilon} + \epsilon$.*

Discrepancy turned the paving problem into a "sharing" problem.

8. PAVING OR SHARING THE ROAD?

So, although pure state extension from a continuous masa to the full algebra are never unique, they are always unique from a discrete masa. Of course, questions remain! Who is going to get us an operator theoretic proof? We cannot pave all matrices into 2 blocks [13], can we pave into 3? What would the ϵ approximation be? Chuck already adapted himself to the new situation and changed his vocabulary from "paving" to "sharing" and asks us: How likely is a random sharing to be a "good sharing"?

Sharing the solution of a long-standing problem is a nice birthday present Tsoutsou, I wish all of us many more!

Acknowledgments. The author wishes to thank the Istanbul Center for Mathematical Sciences for their hospitality.

REFERENCES

[1] C. Akemann and J. Anderson, *Lyapunov Theorems for Operator Algebras*. Memoirs Amer. Math. Soc. **94** (1991).

[2] C. Akemann, J. Anderson and B. Tanbay, *The Kadison-Singer Problem for the Direct Sum of Matrix Algebras*. Positivity **16** (2012), 53–66.

[3] C. Akemann, J. Anderson and B. Tanbay, *Weak Paveability and the Kadison-Singer Problem*. J. Operator Theory **71** (2014), *1*, 101–106.

[4] C. Akeman, B. Tanbay and A. Ülger, *A Note on the Kadison-Singer Problem*. J. Operator Theory **63** (2010), *2*, 101–112.

[5] C. Akemann and N. Weaver, *A Lyapunov-type Theorem from Kadison-Singer*. arXiv:1308.5276v1, (2013).

[6] J. Anderson, *Extensions, Restrictions, and Representations of States on C^*-algebras*. Trans. Amer. Math. Soc. **249** (1979), 303–329 .

[7] J. Bourgain and L. Tzafriri, *On a Problem of Kadison and Singer*. J. Reine Angew. Math. **420** (1991), 1–43.

[8] P. Casazza and D. Edidin, *Equivalents of the Kadison-Singer Problem*. Contemp. Math. **435** (2007), 123–142.

[9] P. Dirac, *Quantum Mechanics*. Oxford Univ. Press 1947.

[10] R. Kadison and I. Singer, *Extensions of Pure States*. Amer. J. Math. **81** (1959), 383–400.

[11] A. Marcus, D. Spielman and N. Srivastava, *Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem*. arXiv:1306.3969, (2013).

[12] S. Popa, *A II_1-factor Approach to the Kadison-Singer Problem*. arXiv:1303.1424, (2013).

[13] B. Tanbay, *Extensions of Pure States on Algebras of Operators*. Dissertation, Univ. of California at Berkeley, 1989.

[14] B. Tanbay, *Pure State Extensions and Compressibility of the l_1-algebra.* Proc. Amer. Soc. **113** (1991), 707–713.

[15] B. Tanbay, *Approximating the Averaging Operator*. Tr. J. of Math. **16** (1992), 85–94.

[16] N. Weaver, *The Kadison-Singer Problem in Discrepancy Theory*, Discrete Math. **278** (2004), 227–239.

Received 24 October 2013

Boğaziçi University,
Department of Mathematics,
34342 Istanbul,
Turkey
tanbay@boun.edu.tr