Dedicated to Professor Şerban Strătilă on the occasion of his 70^{th} birthday

ON ADDITIVE PRESERVERS OF SEMI-BROWDER OPERATORS

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In this paper we provide a complete description of additive surjective continuous maps in the algebra of all bounded linear operators acting on a complex separable infinite-dimensional Hilbert space, preserving semi-Browder's operators in both directions.

AMS 2010 Subject Classification: 47B49, 47L99, 47A55, 47B37.

Key words: additive preservers, semi-Browder operators.

1. INTRODUCTION

Let H be a separable complex infinite dimensional Hilbert space. The algebra of all bounded linear operators acting on H is denoted by $\mathcal{L}(H)$. For an operator $T \in \mathcal{L}(H)$, write T^* for its adjoint, $\mathrm{N}(T)$ for its kernel and $\mathrm{R}(T)$ for its range. The *ascent* $\mathrm{a}(T)$ and *descent* $\mathrm{d}(T)$ of $T \in \mathcal{L}(H)$ are defined by

$$\mathrm{a}(T) = \min\{n \geq 0\colon \mathrm{N}(T^n) = \mathrm{N}(T^{n+1})\}$$

$$\mathrm{d}(T) = \min\{n \geq 0\colon \mathrm{R}(T^n) = \mathrm{R}(T^{n+1})\},$$

where the minimum over the empty set is taken to be infinite, see [8, 6]. The set of all operators of finite ascent (resp. descent) will be denoted by $\mathcal{A}(H)$ (resp. $\mathcal{D}(H)$).

An operator $T \in \mathcal{L}(H)$ is called *upper* (resp. *lower*) *semi-Fredholm* if $\mathrm{R}(T)$ is closed and $\dim \mathrm{N}(T)$ (resp. $\operatorname{codim} \mathrm{R}(T)$) is finite. For such operators T the *index* is defined by

$$\mathrm{ind}(T) = \dim \mathrm{N}(T) - \operatorname{codim} \mathrm{R}(T) \in \mathbb{Z} \cup \{\pm\infty\},$$

and if the index is finite, T is said to be *Fredholm*.

Let us introduce the following subsets :

(i) $\mathcal{F}_+(H)$ the set of upper semi-Fredholm operators,
(ii) $\mathcal{F}_-(H)$ the set of lower semi-Fredholm operators,
(iii) $\mathcal{F}_\pm(H) := \mathcal{F}_+(H) \cup \mathcal{F}_-(H)$ the set of *semi-Fredholm* operators,

(iv) $\mathcal{F}(H) := \mathcal{F}_+(H) \cap \mathcal{F}_-(H)$ the set of Fredholm operators,
(v) $\mathcal{B}_+(H) := \mathcal{F}_+(H) \cap \mathcal{A}(H)$ the set of *upper semi-Browder* operators,
(vi) $\mathcal{B}_-(H) := \mathcal{F}_-(H) \cap \mathcal{D}(H)$ the set of *lower semi-Browder* operators,
(vii) $\mathcal{B}_\pm(H) := \mathcal{B}_+(H) \cup \mathcal{B}_-(H)$ the set of *semi-Browder* operators,
(viii) $\mathcal{B}(H) := \mathcal{B}_+(H) \cap \mathcal{B}_-(H)$ the set of *Browder* operators.

We refer to [6] for more information about semi-Fredholm, Fredholm, semi-Browder and Browder operators.

Let $\mathcal{S}$ denote any of the subsets ((i))–(viii). A surjective additive map $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ is said to *preserve* $\mathcal{S}$ in both directions if $T \in \mathcal{S} \Leftrightarrow \Phi(T) \in \mathcal{S}$.

In [4, 5], the authors studied linear maps on $\mathcal{L}(H)$ preserving semi-Fredholm operators or Fredholm operators in both directions. Observe that the problem makes sense only in the infinite dimensional case. In fact, every complex matrix is Fredholm, and consequently, every map preserves such subsets. Also, it should be mentioned that these subsets are invariant under finite rank perturbations. This constrains to search information on these maps in the Calkin algebra. More precisely, it is shown that such maps preserve the ideal of compact operators in both directions and their induced maps on the Calkin algebra are Jordan automorphism.

Recently, in [3], the authors considered the linear preserver problem that is trivial in the finite dimension case, but the related subsets are not stable under finite rank perturbations. Indeed, they prove that a surjective additive continuous map $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ preserves in both directions $\mathcal{B}$, if and only if Φ possesses one of the following two forms :

$$\Phi(S) = cASA^{-1} \text{ for all } S \in \mathcal{L}(H), \tag{1.1}$$

or

$$\Phi(S) = cAS^*A^{-1} \text{ for all } S \in \mathcal{L}(H), \tag{1.2}$$

where $A : H \to H$ is an invertible bounded linear, or conjugate linear, operator and c is a non-zero complex number. They establish also that Φ preserves in both directions $\mathcal{B}_+$, or $\mathcal{B}_-$, if and only if it satisfies (1.1).

The purpose of this paper is to extend these results to the case of semi-Browder operators. More precisely, the main results of this paper are the following theorems:

THEOREM 1.1. *Let H be a separable infinite-dimensional Hilbert space, and let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map. Then the following assertions are equivalent:*
(i) *Φ preserves in both directions $\mathcal{A} \cup \mathcal{D}$;*
(ii) *there exists an invertible bounded linear, or conjugate linear, operator $A : H \to H$ and a non-zero complex number c such that $\Phi(S) = cASA^{-1}$ for all $S \in \mathcal{L}(H)$.*

THEOREM 1.2. *Let H be a separable infinite-dimensional Hilbert space, and let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map. Then the following assertions are equivalent:*

(i) *Φ preserves in both directions $\mathcal{B}_\pm$;*

(ii) *there exists an invertible bounded linear, or conjugate linear, operator $A : H \to H$ and a non-zero complex number c such that either $\Phi(S) = cASA^{-1}$ for all $S \in \mathcal{L}(H)$, or $\Phi(S) = cAS^*A^{-1}$ for all $S \in \mathcal{L}(H)$.*

2. PROOF OF MAIN RESULT

Recall that the *hyper-range* and the *hyper-kernel* of an operator $T \in \mathcal{L}(X)$ are respectively the T-invariant subspaces $\mathcal{R}^\infty(T) := \bigcap_n \mathrm{R}(T^n)$ and $\mathcal{N}^\infty(T) := \bigcup_n \mathrm{N}(T^n)$. Notice that if T has finite ascent then $T_{|\mathcal{R}^\infty(T)}$ is bijective, see [3].

Let $x, y \in H$ be nonzero. We denote, as usual, by $x \otimes y$ the rank one operator given by $(x \otimes y)(z) = \langle z, y\rangle x$ for all $x \in H$.

The following theorem that is established in [3] will play a crucial role in this work.

THEOREM 2.1. *Let $T \in \mathcal{L}(H)$ be an operator with finite ascent p. Then $T|\mathcal{R}^\infty(T)$ is bijective. Moreover, if $x, y \in H$, then $T + x \otimes y$ has infinite ascent if and only if the following assertions hold:*

(i) *$x = a + b_0$ where $a \in \mathrm{N}(T^p)$ and $b_0 \in \mathcal{R}^\infty(T)$;*

(ii) *$\langle b_1, y\rangle = -1$ and $\langle b_i, y\rangle = 0$ for all $i \geq 2$, where $b_i = (T|\mathcal{R}^\infty(T))^{-i} b_0$;*

(iii) *$\langle T^i a, y\rangle = 0$ for all $i \geq 0$.*

Moreover, in this case $\{b_i\}_{i \geq 1}$ is a linearly independent set.

LEMMA 2.2. *Let $T \in \mathcal{B}_\pm(H)$ and $F \in \mathcal{L}(H)$ be a rank one operator. Then there exists a non zero complex number α such that $T + \lambda F \in \mathcal{B}_\pm(H)$ for all $\lambda \in \mathbb{C} \setminus \{\alpha\}$.*

Proof. Let $F = x \otimes y$ where $x, y \in H$. Since $\mathcal{B}_\pm(H)$ is invariant under the passage to the adjoint, we can suppose that T has finite ascent. If there exists $\alpha \in \mathbb{C}$ such that $\mathrm{a}(T + \alpha x \otimes y) = \infty$, then it follows by Theorem 2.1 that $x = a + b_0$, where $a \in \mathcal{N}^\infty(T)$ and $b_0 \in \mathcal{R}^\infty(T)$, and $\alpha < (T|\mathcal{R}^\infty(T))^{-1} b_0, y >= -1$. Moreover, the decomposition $x = a + b_0$ is unique because $\mathcal{N}^\infty(T) \cap \mathcal{R}^\infty(T) = \{0\}$. This shows that $T + \lambda F$ has finite ascent, and so belongs to $\mathcal{B}_\pm(H)$, for all complex number $\lambda \neq \alpha$. $\square$

Recall that an operator $T \in \mathcal{L}(H)$ is said to be *Weyl* if it is Fredholm of index zero.

LEMMA 2.3. *Let $T \in \mathcal{L}(X)$ be a Weyl operator. The following assertions are equivalent:*

(i) *T has finite ascent,*
(ii) *T has finite descent,*
(iii) *$T \in \mathcal{B}_{\pm}(H)$.*

Proof. Clearly it suffices to establish (i) $\Leftrightarrow$ (ii). Let n be a positive integer. It follows that T^n is Fredholm and

$$0 = n\,\mathrm{ind}(T) = \mathrm{ind}(T^n) = \dim \mathrm{N}(T^n) - \mathrm{codim}\,\mathrm{R}(T^n).$$

Hence, $\dim \mathrm{N}(T^n) = \dim \mathrm{N}(T^{n+1})$ if and only if $\dim \mathrm{R}(T^n) = \dim \mathrm{R}(T^{n+1})$, which establishes the equivalence (i) $\Leftrightarrow$ (ii). □

Using the previous Lemma, we can get easily the following result.

LEMMA 2.4. *Let $T \in \mathcal{L}(H)$ be a semi-Fredholm operator and let $F \in \mathcal{L}(H)$ be a finite rank operator.*

(i) *If* $\mathrm{ind}(T) \leq 0$, *then $T + F \in \mathcal{B}_{\pm}(H)$ if and only if* $\mathrm{a}(T + F) < \infty$*;*
(ii) *If* $\mathrm{ind}(T) \geq 0$, *then $T + F \in \mathcal{B}_{\pm}(H)$ if and only if* $\mathrm{d}(T + F) < \infty$.

Let $T \in \mathcal{L}(X)$ be a semi-Fredholm operator. It is well known that T is upper (resp. lower) semi-Browder if and only if T has finite ascent (resp. descent), see [1]. Consequently,

$$\mathcal{B}_{+}(H) = \mathcal{F}_{\pm}(H) \cap \mathcal{A}(H) \quad \text{and} \quad \mathcal{B}_{-}(H) = \mathcal{F}_{\pm}(H) \cap \mathcal{D}(H). \tag{2.1}$$

PROPOSITION 2.5. *Let T be a bounded operator on H. Then the following assertions are equivalent:*

(i) *T is a semi-Browder operator;*
(ii) *for every $S \in \mathcal{L}(H)$ there exists $\varepsilon_0 > 0$ such that $T + \varepsilon S \in \mathcal{A}(H) \cup \mathcal{D}(H)$ for all $\varepsilon < \varepsilon_0$.*

Proof. (i) $\Rightarrow$ (ii) follows from the openness of $\mathcal{B}_{\pm}(H)$.

(ii) $\Rightarrow$ (i). By (2.1), it suffices to establish that T is semi-Fredholm. Suppose the contrary. Then either $R(T)$ is not closed or $\dim \mathrm{N}(T) = \infty = \mathrm{codim}\,\overline{R(T)}$. In both cases, for every finite-codimensional subspace $H_0 \subset H$ neither of the restrictions $T_{|H_0}$, $T^*_{|H_0}$ is bounded below. So for each $\varepsilon > 0$ there exist $x, y \in H_0$ such that $\|x\| = 1 = \|y\|$, $\|Tx\| < \varepsilon$ and $\|T^*y\| < \varepsilon$. Hence, we can find inductively an orthonormal system $\{x_{n,k}, y_{n,k} : n, k \in \mathbb{N}\}$ such that

$$y_{m,l} \perp \{x_{n,k}, Tx_{n,k}, T^2x_{n,k}\}, \|Tx_{n,k}\| < \frac{1}{4^n 2^k} \text{ and } \|T^*y_{n,k}\| < \frac{1}{4^n 2^k}$$

for all $n, k, m, l \in \mathbb{N}$. Let $L_1 = \mathrm{Span}\{x_{n,k}, Tx_{n,k} : n, k \in \mathbb{N}\}$ and $L_2 = \mathrm{Span}\{y_{n,k}, T^*y_{n,k} : n, k \in \mathbb{N}\}$. By construction we have $L_1 \perp L_2$. Define

$S \in \mathcal{L}(H)$ by $S|(L_1 \oplus L_2)^\perp = 0$,

$$\begin{cases} Sx_{n,1} = -2^n T x_{n,1} & \text{for all } n \geq 1 \\ Sx_{n,k} = -2^n T x_{n,k} + 2^{-(n+k)} x_{n,k-1} & \text{for all } n \geq 1 \text{ and } k \geq 2 \\ S|(L_1 \ominus \mathrm{Span}\{x_{n,k} : n, k \in \mathbb{N}\}) = 0, \end{cases}$$

and $S|L_2 = V^*$ where $V \in \mathcal{L}(L_2)$ is given by

$$\begin{cases} Vy_{n,1} = -2^n T^* y_{n,1} & \text{for all } n \geq 1 \\ Vy_{n,k} = -2^n T^* y_{n,k} + 2^{-(n+k)} y_{n,k-1} & \text{for all } n \geq 1 \text{ and } k \geq 2 \\ V|(L_2 \ominus \mathrm{Span}\{y_{n,k} : n, k \in \mathbb{N}\}) = 0. \end{cases}$$

Note that V and S are bounded because

$$\sum_{n,k} \| Vy_{n,k} \| \leq 2 \sum_{n,k} 2^{-(n+k)} \text{ and } \sum_{n,k} \| Sx_{n,k} \| \leq 2 \sum_{n,k} 2^{-(n+k)} < \infty.$$

Moreover, for each n, we have $\mathrm{a}(T + 2^{-n}S) = \mathrm{a}(T^* + 2^{-n}S^*) = \infty$, *i.e.*, $\mathrm{d}(T + 2^{-n}S) = \infty$, a contradiction. □

For a subset $\Gamma \subseteq \mathcal{L}(H)$, we write $\mathrm{Int}(\Gamma)$ for its interior.

Corollary 2.6. *We have* $\mathcal{B}_\pm(H) = \mathrm{Int}(\mathcal{A}(H) \cup \mathcal{D}(H))$.

Proof. Since $\mathcal{B}_\pm(H)$ is an open subset contained in $\mathcal{A}(H) \cup \mathcal{D}(H)$, it suffices to show that $\mathrm{Int}(\mathcal{A}(H) \cup \mathcal{D}(H)) \subseteq \mathcal{B}_\pm(H)$. Let $T \notin \mathcal{B}_\pm(H)$. Then, using Proposition 2.5, there exists $S \in \mathcal{L}(H)$ and a sequence (ε_n) that converges to zero and for which $\mathrm{a}(T + \varepsilon_n S) = \mathrm{d}(T + \varepsilon_n S) = \infty$. This implies that $T \notin \mathrm{Int}(\mathcal{A}(H) \cup \mathcal{D}(H))$. □

Lemma 2.7. *Let* $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ *be a surjective additive continuous map. If* ϕ *preserves in both directions* $\mathcal{B}_\pm$, *then* Φ *is injective and preserves the set of rank one operators in both directions.*

Proof. Suppose on the contrary that there exists $T \neq 0$ such that $\Phi(T) = 0$. Then, by ([3], Theorem 2.15), there exists an invertible operator $S \in \mathcal{L}(H)$ such that $\mathrm{a}(S + T) = \infty$. Hence $\Phi(S + \lambda T) = \Phi(S) \in \mathcal{B}_\pm(H)$, and so $S + \lambda T \in \mathcal{B}_\pm(H)$, for all $\lambda \in \mathbb{R}$. The continuity of the index implies that $\mathrm{ind}(S + \lambda T) = \mathrm{ind}(S)$ for all $\lambda \in \mathbb{R}$. Consequently, $S + T$ is Weyl with infinite ascent. Thus $S + T \notin \mathcal{B}_\pm(H)$, a contradiction.

Now suppose that there exists a rank one operator F such that $\dim \mathrm{R}(\Phi(F)) \geq 2$. Then it follows by Lemma ([3], Theorem 2.15) that there exists an invertible operator $R \in \mathcal{L}(H)$ satisfying

$$\mathrm{a}(R + \Phi(F)) = \mathrm{a}(R - \Phi(F)) = \infty. \tag{2.2}$$

Write $R = \Phi(S)$ where $S \in \mathcal{B}_\pm(H)$. By Lemma 2.2, we get that $S + \lambda F \in \mathcal{B}_\pm(H)$ for all $\lambda \in \mathbb{C} \setminus \{\alpha\}$ where α is non zero complex number. Hence,

$R + \lambda\Phi(F) \in \mathcal{B}_{\pm}(H)$ for all $\lambda \in \mathbb{R} \setminus \{\alpha\}$. But, the continuity of the index implies that

$$\operatorname{ind}(R + \lambda\Phi(F)) = \operatorname{ind}(R) = 0 \text{ for all } \lambda \in \mathbb{R} \setminus \{\alpha\}.$$

Consequently, $R + \lambda\Phi(F)$ has finite ascent for all $\lambda \in \mathbb{R} \setminus \{\alpha\}$, which contradicts (2.2). Thus, $\dim \mathrm{R}(\Phi(T)) \leq 1$. Since Φ is bijective and Φ^{-1} satisfies the same properties as Φ, we obtain that Φ preserves the set of rank one operators in both directions. This completes the proof. □

Let τ be a field automorphism of $\mathbb{C}$. An additive map $A : H \to H$ will be called τ-semi linear if $A(\lambda x) = \tau(\lambda)Ax$ holds for all $\lambda \in \mathbb{C}$ and $x \in H$. Notice that if A is bounded, then so is τ, and consequently, τ is either the identity or the complex conjugation, see [2].

Moreover, in this case, the adjoint operator $A' : H' \to H'$ defined by the equation $\langle x, A'y' \rangle = \tau(\langle Ax, y' \rangle)$ for all $x \in H, y' \in H'$, is again τ-semi linear.

Note that we do not identify H with its dual H'. Let $J : H \to H'$ be the natural conjugate linear mapping defined by $\langle u, Jx \rangle = \langle u, x \rangle \quad (x, u \in H)$.

For $A \in \mathcal{L}(H)$, let $A^* : H \to H$ be the Hilbert space adjoint. We have $A^* = J^{-1}A'J$.

Lemma 2.8. *Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map. If Φ preserves $\mathcal{B}_{\pm}$ in both directions, then:*

either there exist continuous bijective mappings $A, B : H \to H$, either both linear or both conjugate linear, such that

(2.3) $\quad \Phi(F) = AFB$ *for all finite rank operators* $F \in \mathcal{L}(H)$,

or there exist continuous bijective mappings $C, D : H \to H$, either both linear or both conjugate linear, such that

(2.4) $\quad \Phi(F) = CF^*D$ *for all finite rank operators* $F \in \mathcal{L}(H)$.

Proof. From the previous Lemma 2.7 and ([7], Theorem 3.3), there exists a ring automorphism τ of $\mathbb{C}$, and either τ-semi linear bijective maps $A : H \to H$ and $E : H' \to H'$, such that

(2.5) $\quad \Phi(x \otimes f) = Ax \otimes Ef$ for all $x \in H$ and $f \in H'$,

or τ-semi linear bijective maps $R : H \to H'$ and $G : H' \to H$ such that

(2.6) $\quad \Phi(x \otimes f) = Gf \otimes Rx$ for all $x \in H$ and $f \in H'$.

Since Φ is continuous, then so are τ, A, E, R and G.

In the first case set $B = E'$. It is easy to verify that $\Phi(F) = AFB$ for all rank one operators and, by additivity of Φ, for all finite rank operators $F \in \mathcal{L}(H)$.

In the second case set $C = GJ$ and $D = J^{-1}R'$. Again it is easy to verify that $\Phi(F) = CF^*D$ for all finite rank operators $F \in \mathcal{L}(H)$. $\square$

If we replace Φ by $\Psi : \mathcal{L}(H) \to \mathcal{L}(H)$ defined by $\Psi(T) = A^{-1}\Phi(T)A$ in the first case (by $\Psi(T) = C^{-1}\Phi(T)C$ in the second case, respectively), we can assume that A (resp. C) is the identity mapping. Note that in this case B (resp. D) is a linear mapping.

The following lemma which is necessary for proving the main theorem can be proved in similar way as ([3], Lemma 3.11).

LEMMA 2.9. *Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map that preserves $\mathcal{B}_\pm$, in both directions. Then:*

(i) *if $B \in \mathcal{L}(H)$ is an invertible operator and $\Phi(F) = FB$ for all finite rank operator F, then there exists a non-zero $c \in \mathbb{C}$ such that $\Phi(S) = cS$ for all $S \in \mathcal{L}(H)$;*

(ii) *if $D \in \mathcal{L}(H)$ is an invertible operator and $\Phi(F) = F^*D$ for all finite rank operator F, then there exists a non-zero $c \in \mathbb{C}$ such that $\Phi(S) = cS^*$ for all $S \in \mathcal{L}(H)$.*

Proof of Theorem 1.2. Suppose that Φ preserves $\mathcal{B}_\pm$ in both directions, then by Lemma 2.8, Φ possesses one of the two forms (2.3) and (2.4). Clearly, if $\Phi(F) = AFB$ for all rank one operator $F \in \mathcal{L}(H)$, then the map $\Phi_2(.) = A^{-1}\Phi(.)A$ preserves $\mathcal{B}_\pm$, and $\Phi_2(F) = FBA$. Hence, by Lemma 2.9 (i), there is a nonzero complex c such that $\Phi_2(S) = cS$, and so $\Phi(S) = cASA^{-1}$, for all $S \in \mathcal{L}(H)$. In similar way, we show that if Φ satisfies (2.4) then $\Phi(T) = c'CS^*C^{-1}$ for all $S \in \mathcal{L}(H)$, where c' is a non zero complex number. The converse implication is obvious. $\square$

Proof of Theorem 1.1. If Φ preserves $A \cup \mathcal{D}$ in both directions then it preserves $\mathcal{B}_\pm$ in both directions, and hence Φ possesses one of the two forms in Theorem 1.2.

Let e_n, $n \geq 1$, be an arbitrary orthonormal basis of H, and consider the bounded operator $T \in \mathcal{L}(H)$ given by $Te_n = n^{-1}e_{n+1}$. Clearly, T is an injective quasi-nilpotent operator with $\mathrm{a}(T^*) = \infty$. Consequently, $\mathrm{d}(T) = \mathrm{d}(T^*) = \infty$, $S \in \mathcal{A} \cup \mathcal{D}$ and $S^* \notin \mathcal{A} \cup \mathcal{D}$. This shows that Φ can not take the second form. The converse implication is obvious. $\square$

We end this section by the following two remarks. Recall that an additive map Φ between two algebras is called *unital* if $\Phi(I) = I$.

Remark 2.10. Let $\mathcal{R}$ be any one of the subsets $\{\mathcal{A} \cup \mathcal{D}, \mathcal{B}_\pm\}$, and define the corresponding spectrum by

$$\sigma_{\mathcal{R}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{R}\}.$$

Using Theorem 1.2, the form of unital continuous additive maps $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ such that $\sigma_{\mathcal{R}}(\Phi(T)) = \sigma_{\mathcal{R}}(T)$ can be easily determined.

Remark 2.11. Theorem 1.2 can be without any change formulated for additive mappings $\Phi : \mathcal{L}(H) \to \mathcal{L}(K)$ preserving any of the classes $\mathcal{A} \cup \mathcal{D}$ or $\mathcal{B}_{\pm}$, where H, K are separable infinite-dimensional Hilbert spaces.

CONJECTURE 2.12. *Does Theorem 1.1 remain true if we omit the continuity of* Φ *?*

Acknowledgments. The first and the third authors was partially supported by I+D MEC project no. MTM 2010-17687 (Spain). The second author was supported by grants 201/09/0473 of GA CR and IAA100190903 of GA AV.

REFERENCES

[1] H. Heuser, *Über Operatoren Mit Endlichen Defecten.* Inaug. Diss. Tübingen, 1956.

[2] M. Kuczma, *An Introduction To The Theory Of Functional Equations And Inequalities.* Państwowe Wydawnictwo Naukowe, Warszawa, 1985.

[3] M. Mbekhta, V. Müller and M. Oudghiri, *Additive preservers of the ascent, descent and related subsets.* To appear in Operator Theory, 2012.

[4] M. Mbekhta and P. Šemrl, *Linear maps preserving semi-Fredholm operators and generalized invertibility.* Linear Multilinear Algebra **57** (2009), *1*, 55–64.

[5] M. Mbekhta, *Linear maps preserving the set of Fredholm operators.* Proc. Amer. Math. Soc. **135** (2007), *11*, 3613–3619.

[6] V. Müller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras.* Operator Theory: Advances and Applications, **139**. Second Ed. Birkhäuser Verlag, Basel, 2007.

[7] M. Omladič and P. Šemrl, *Additive mappings preserving operators of rank one.* Linear Algebra Appl. **182** (1993), 239–256.

[8] A.E. Taylor and D. Lay, *Introduction To Functional Analysis.* second ed. John Wiley & Sons, New York, Chichester, Brisbane, 1980.

Received 1 April 2013

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